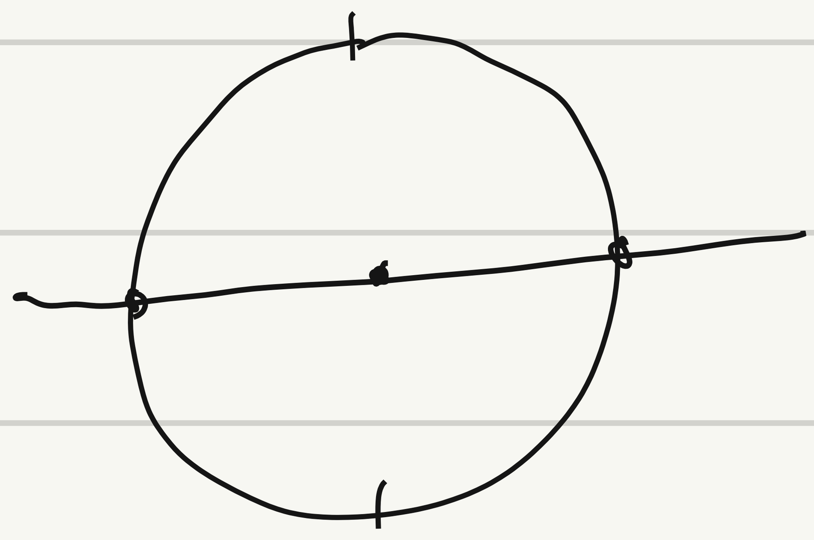
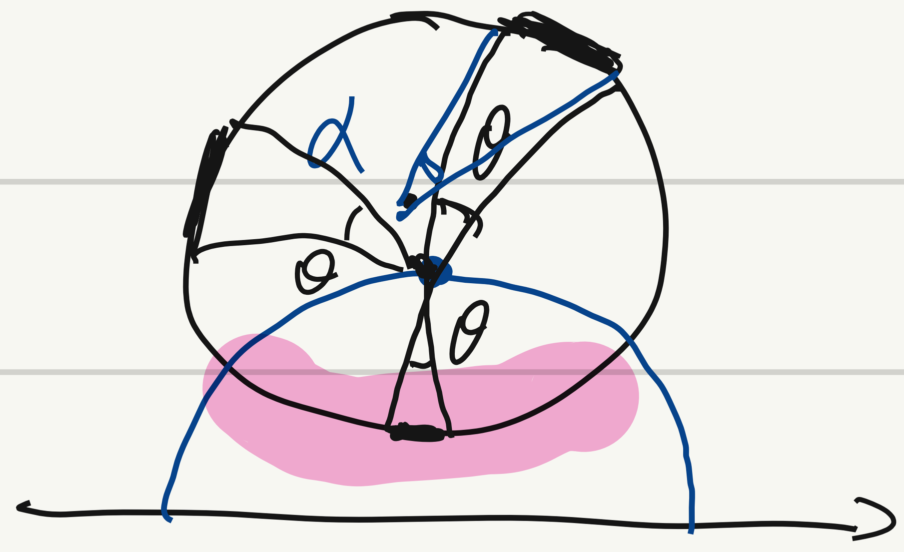


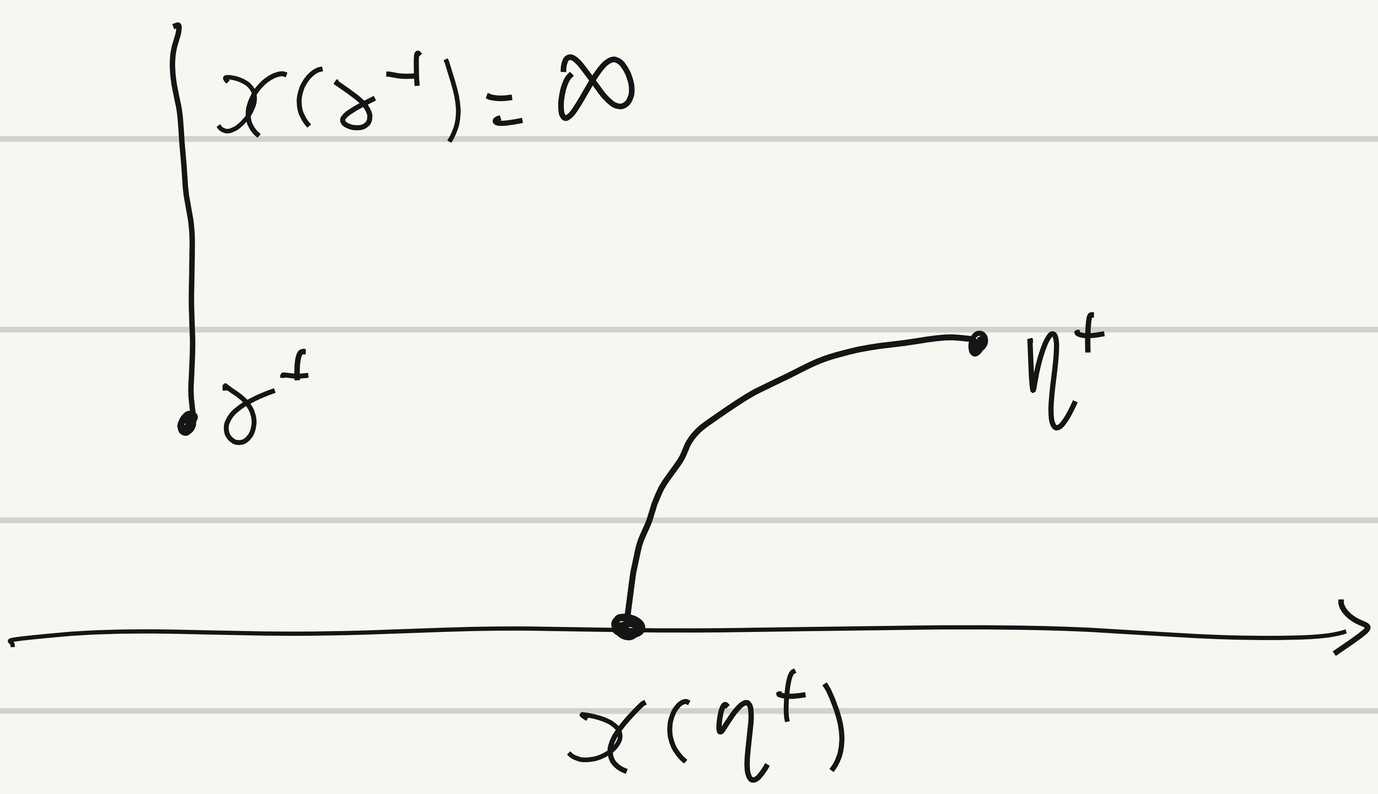


half geod / ray

$\partial\mathbb{H} = \{\gamma^+ \mid \gamma^+ \text{ ray in } \mathbb{H}\}/\sim \qquad \gamma^+ \sim \eta^+ \iff \exists M>0\ \forall t\ d_{\mathbb{H}}(\gamma^+(t), \eta^+(t)) \le M.$

<u>Prop</u>: $\partial\mathbb{H} \cong \hat{\mathbb{R}} = \mathbb{R} \cup \{\infty\} \cong S^1$

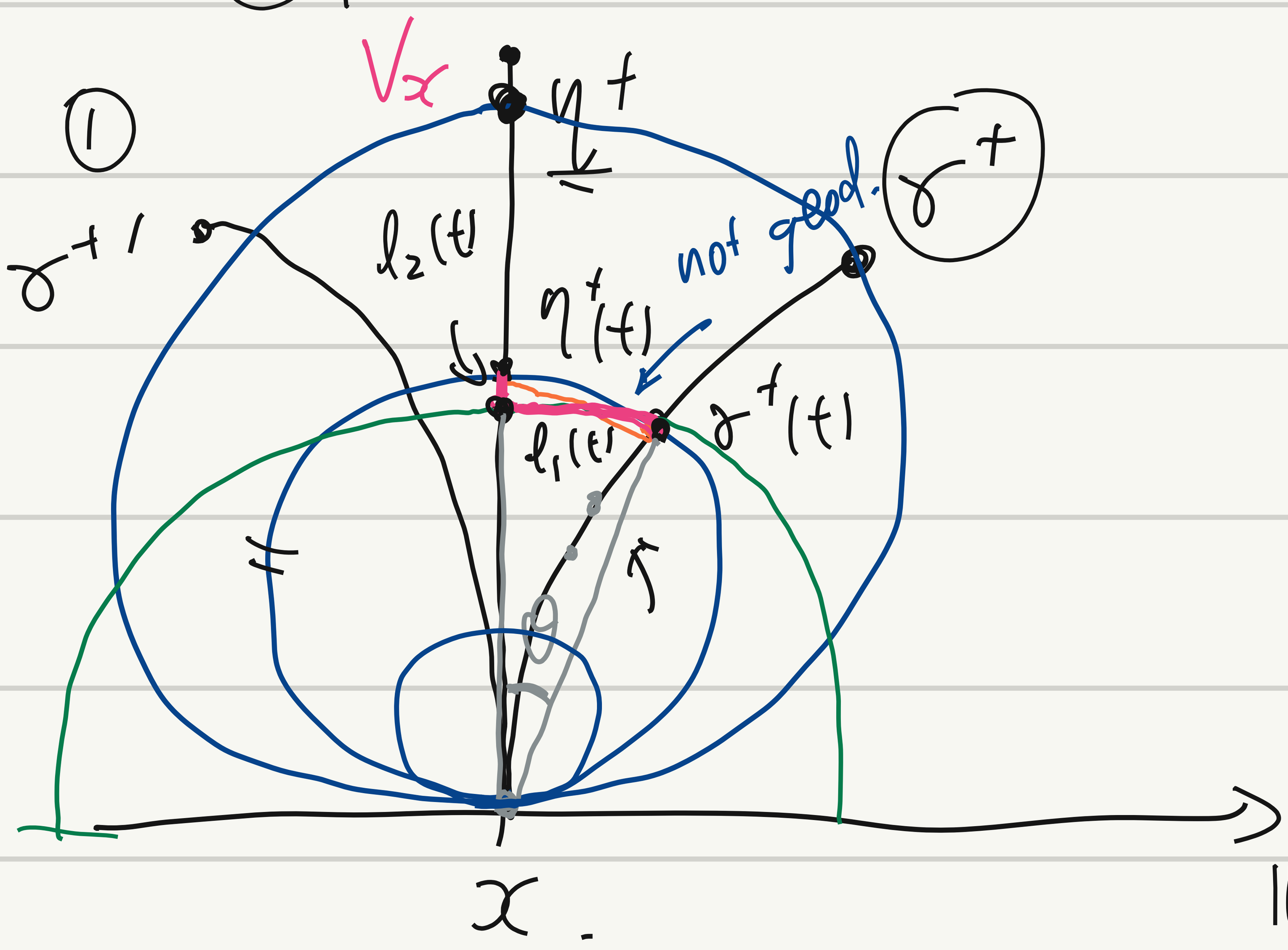
γ^+ ray. $\quad x(\gamma^+) :=$ end point of γ^+.



<u>Prop</u>: $\gamma^+ \sim \eta^+$ iff $x(\gamma^+) = x(\eta^+)$ $\quad \exists M$

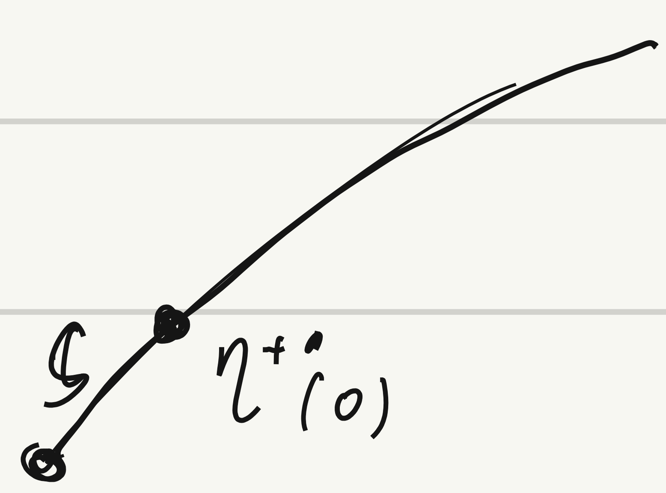
<u>Proof</u>: ① $x(\gamma^+) = x(\eta^+) \Rightarrow \gamma^+ \sim \eta^+ \quad \cdot\ d_{\mathbb{H}}(\gamma^+(t), \eta^+(t)) < M \quad \forall t.$

② $x(\gamma^+) \ne x(\eta^+) \Rightarrow \gamma^+ \not\sim \eta^+ \quad \cdot\ d_{\mathbb{H}}(\gamma^+(t), \eta^+(t)) \to \infty,\ t \to \infty$



$\forall \gamma^+ \quad x(\gamma^+) = x$

$\gamma^+ \sim \eta^+$



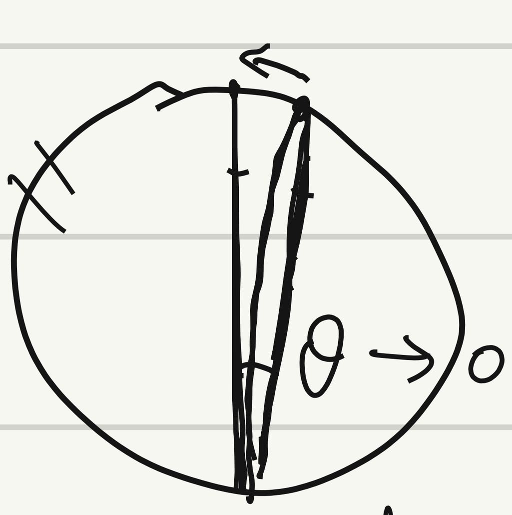
$|\eta^+(t) - \eta^{+\prime}(t)| = \delta$

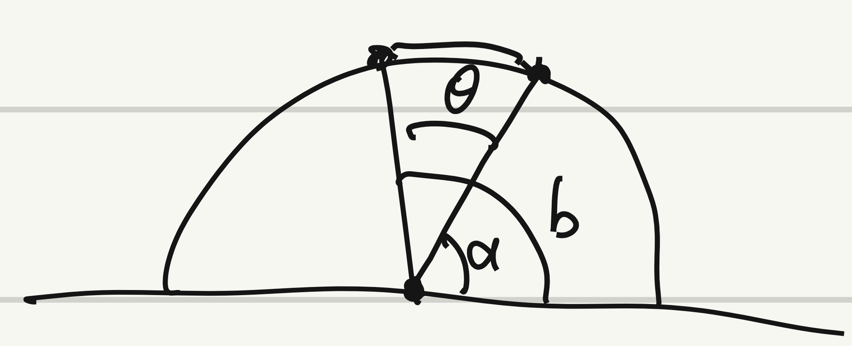
$\eta^+ \sim \eta^{+\prime}$

$\theta \to 0$, as $t \to \infty \Rightarrow l_1(t) \to 0, \quad (t \to \infty)$

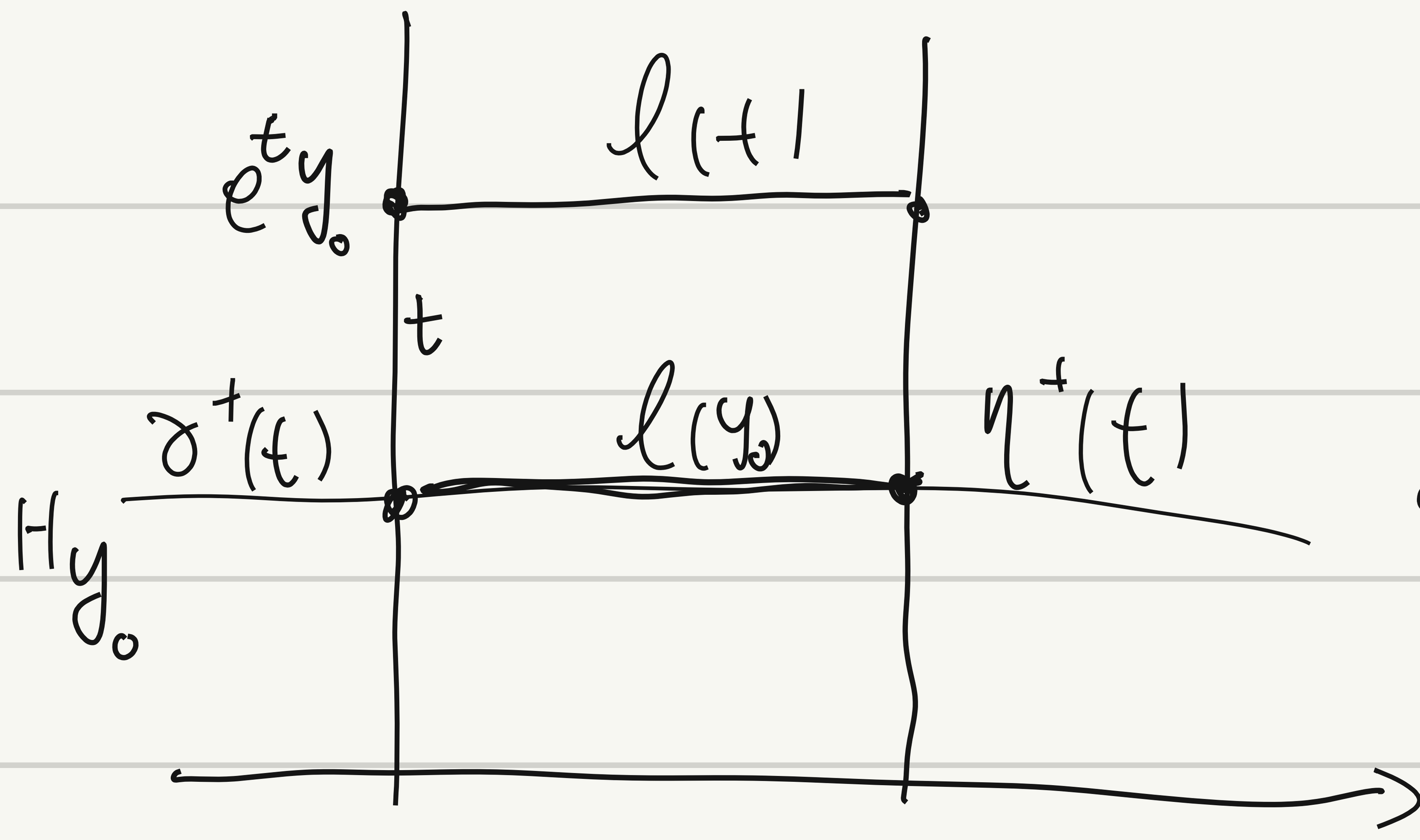
$l_2(t)$

$\theta \to 0$



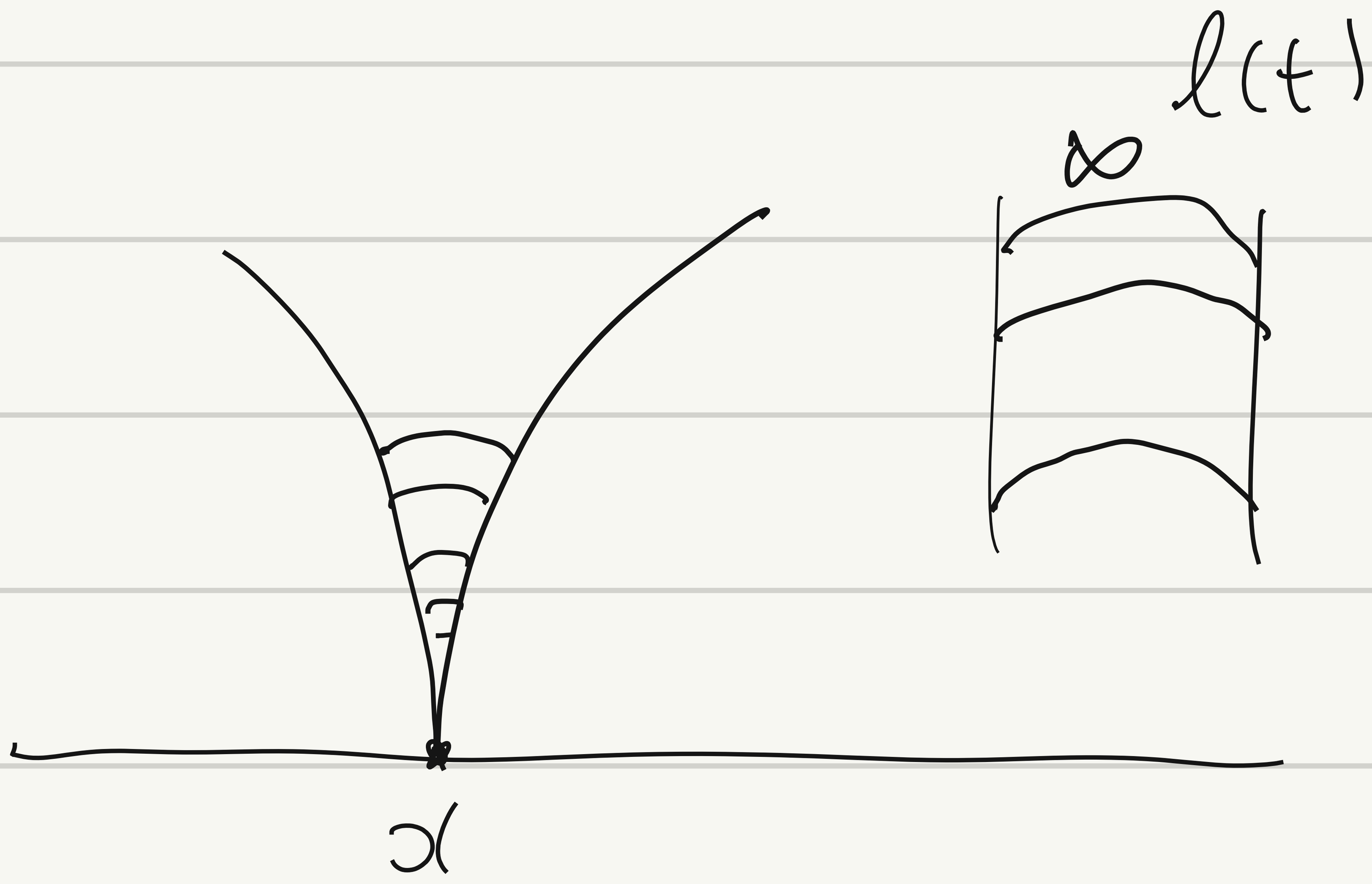


$d_{\mathbb{H}}(\eta^+(t), \gamma^+(t)) \le l_1(t) + l_2(t) \to 0 \qquad t \to \infty.$



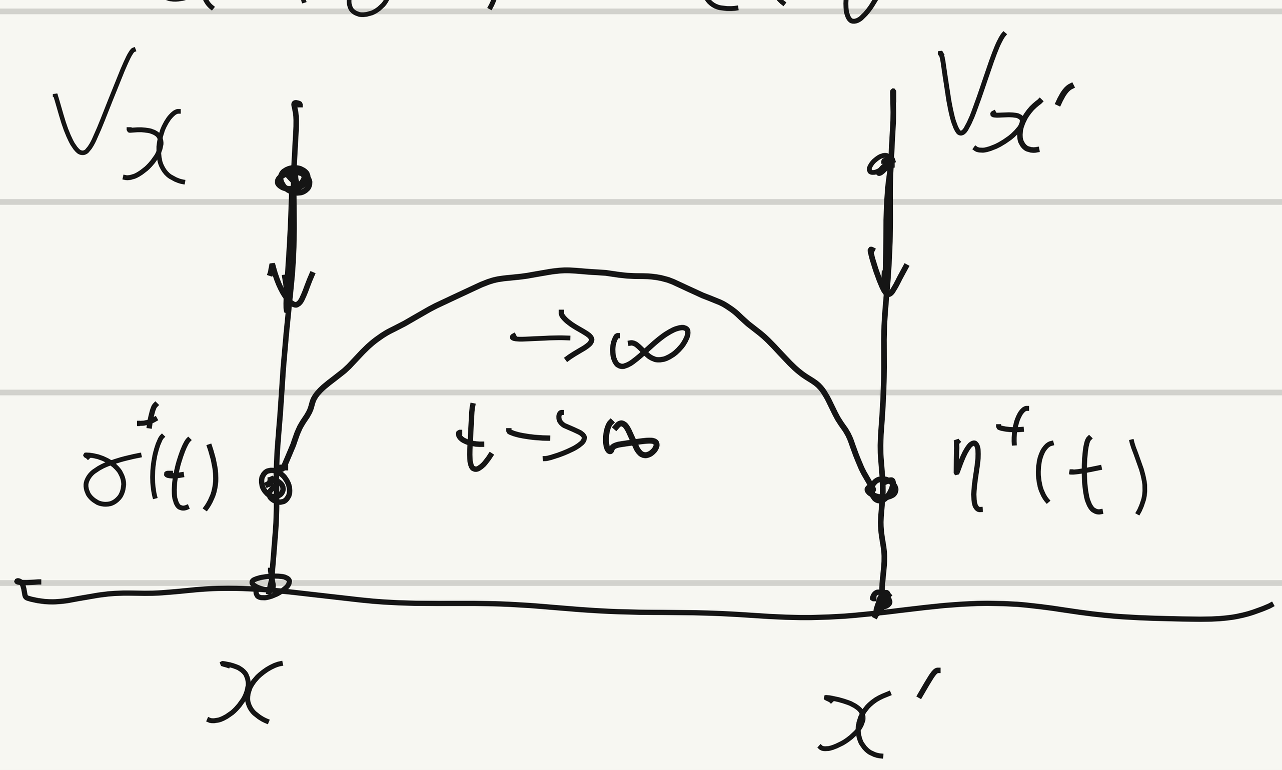
$\ell(t) = \ell(e^t y_0) \to 0, \quad t \to \infty$

$$\frac{\ell(y_0)}{\underline{e^t y_0}} \to 0, \quad t \to \infty$$



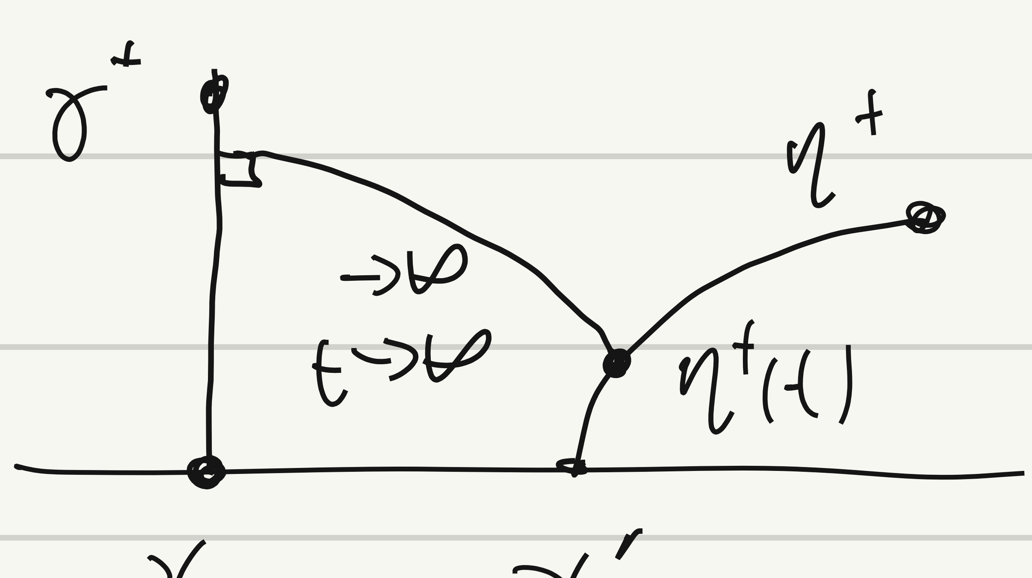
$\Rightarrow \left(x(\gamma^+) = x(\eta^+) \Rightarrow \gamma^+ \sim \eta^+ \right)$

② $x(\gamma^+) \neq x(\eta^+) \Rightarrow \gamma^+ \not\sim \eta^+$



(*) $d_{\mathbb{H}}(\gamma^+(t), \eta^+(t)) < \underline{d_{\mathbb{H}}(\eta^+(t), V_x)}$

$\to \infty, \quad t \to \infty$



$\Rightarrow$ (*) holds in this case.

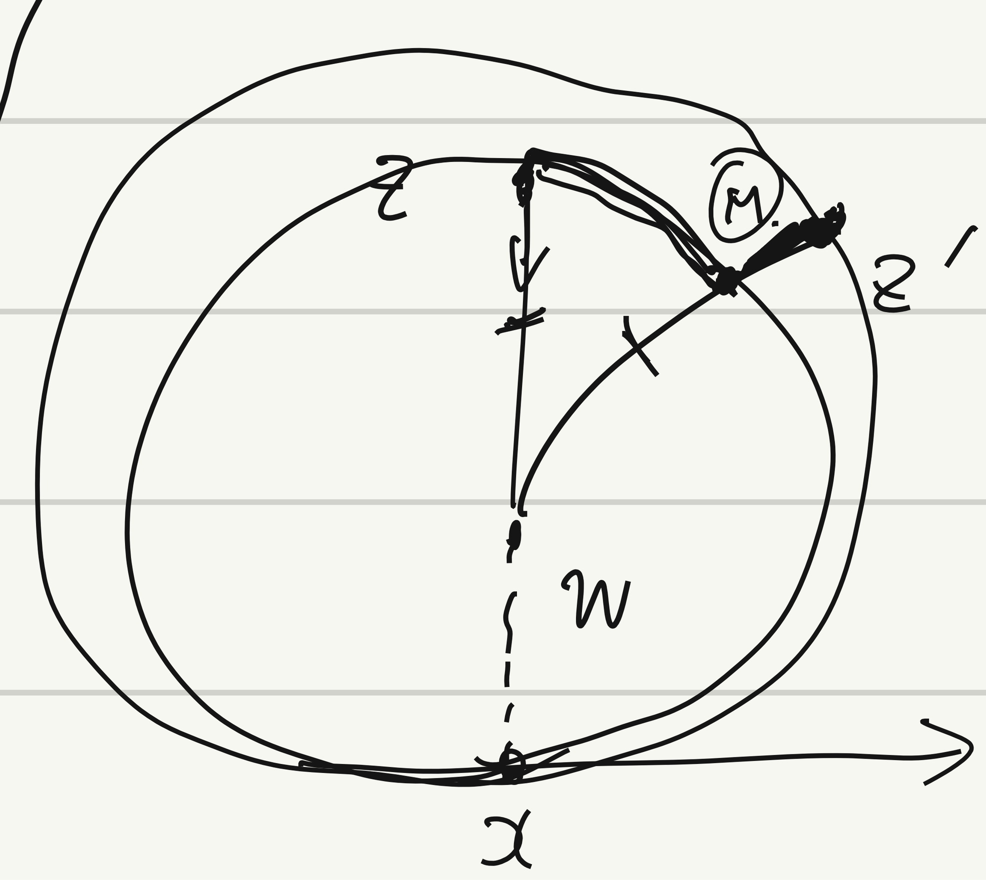
$\left(x(\gamma^+) \neq x(\eta^+) \Rightarrow \gamma^+ \not\sim \eta^+ \right)$

Rmk:

$d_{\mathbb{H}}(\gamma^+(t), \eta^+(t)) \to 0$

$t \to \infty.$

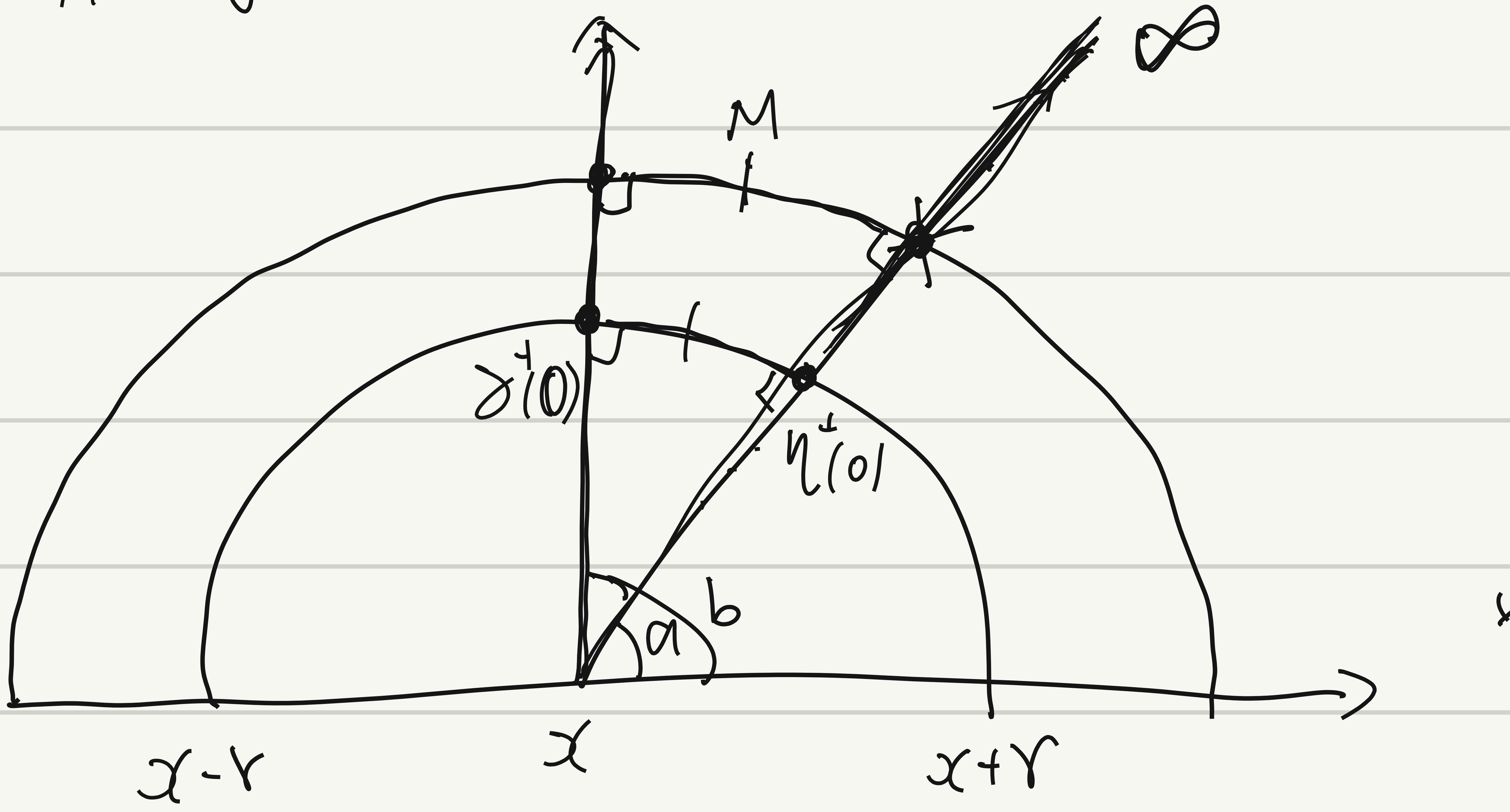
$d_{\mathbb{H}}(\gamma^+(t), \eta^+(t)) \to M, \quad t \to \infty$

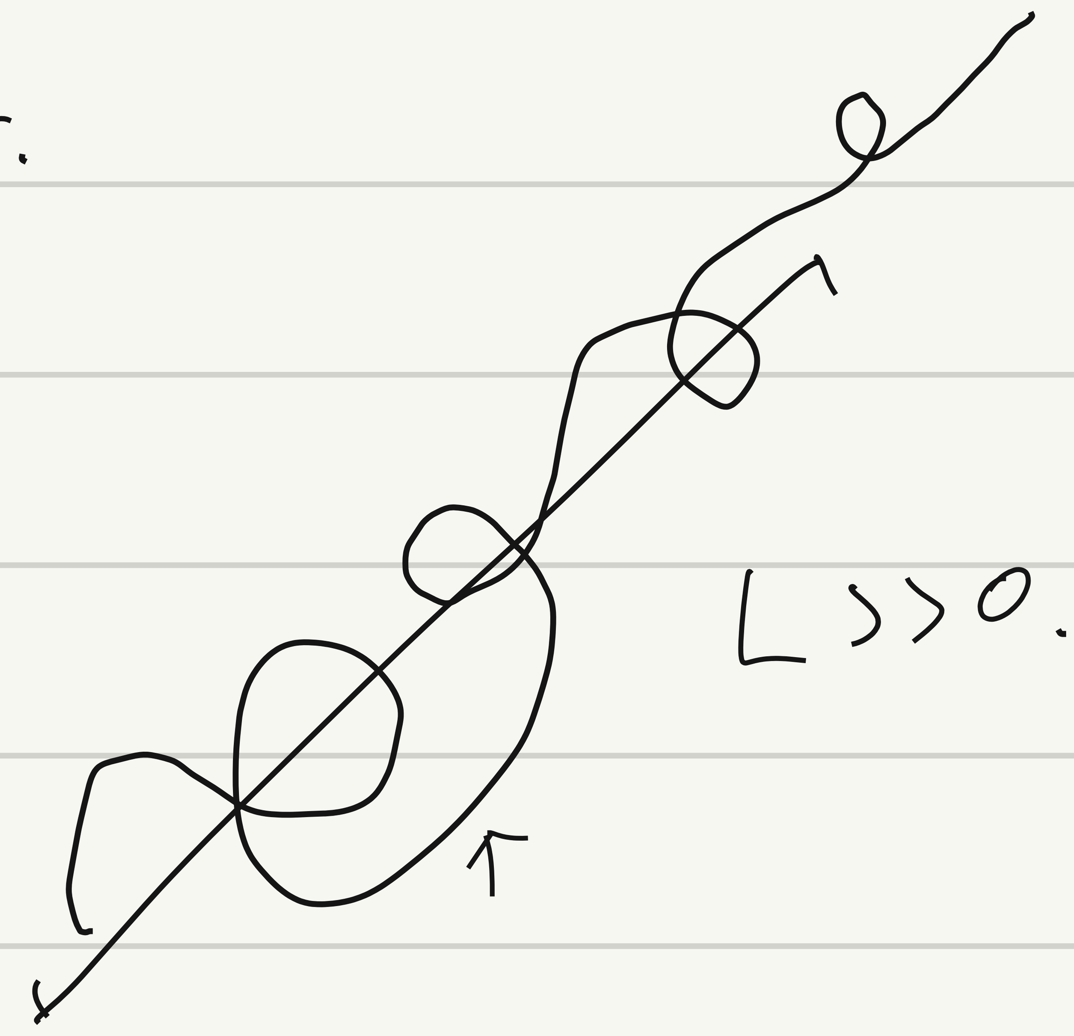


$B_w(z, z') = \underline{d_{\mathbb{H}}(z, w) - d_{\mathbb{H}}(z', w)}$

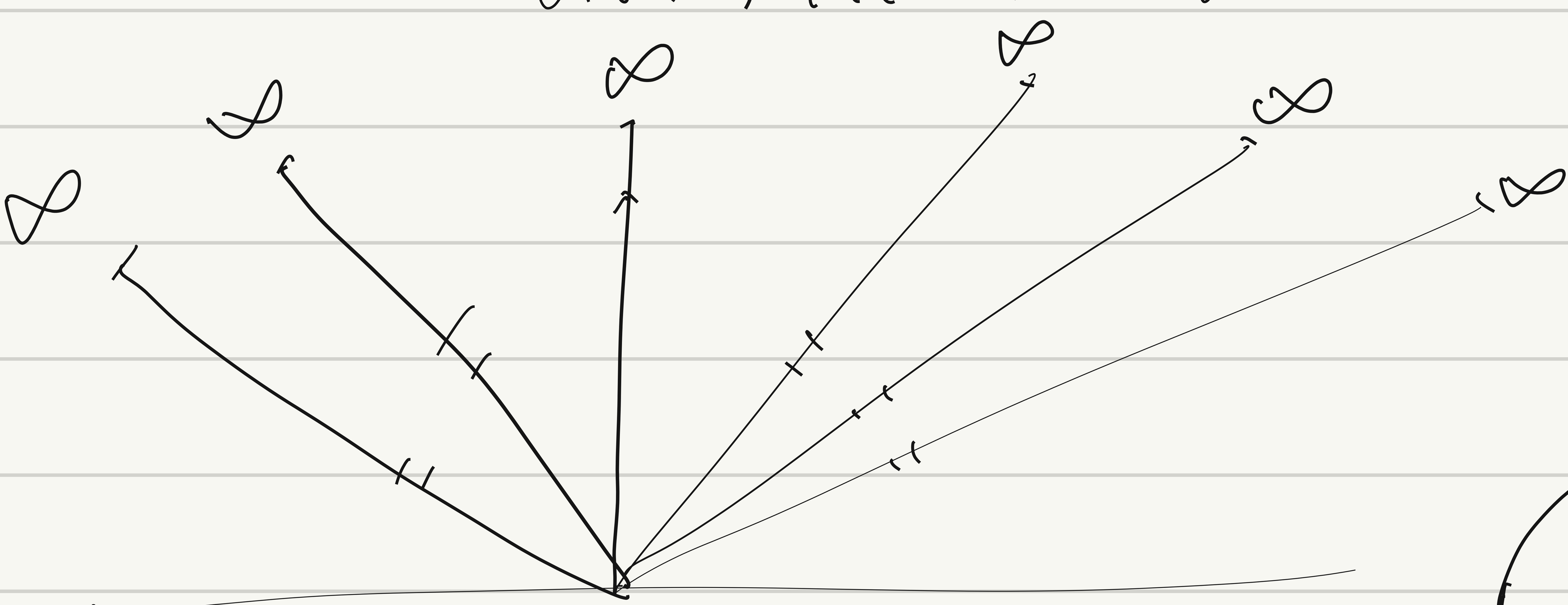
$H_x(z) = \{ z' \mid B_w(z, z') \xrightarrow[w \to x]{} 0 \}$

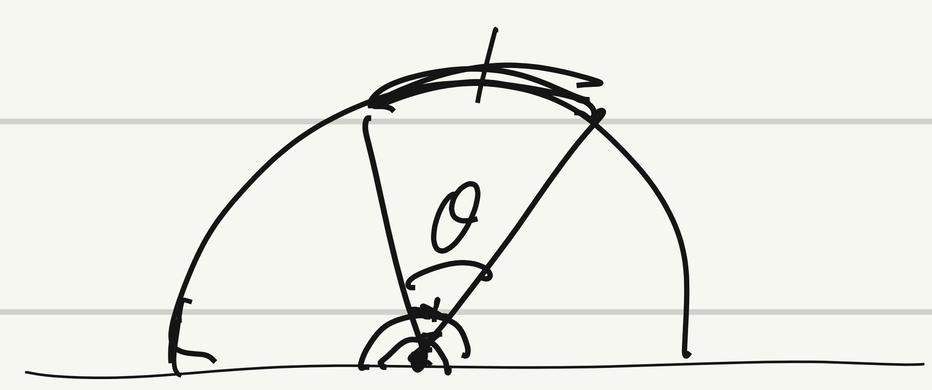
- Hypercycles bounded to its center.

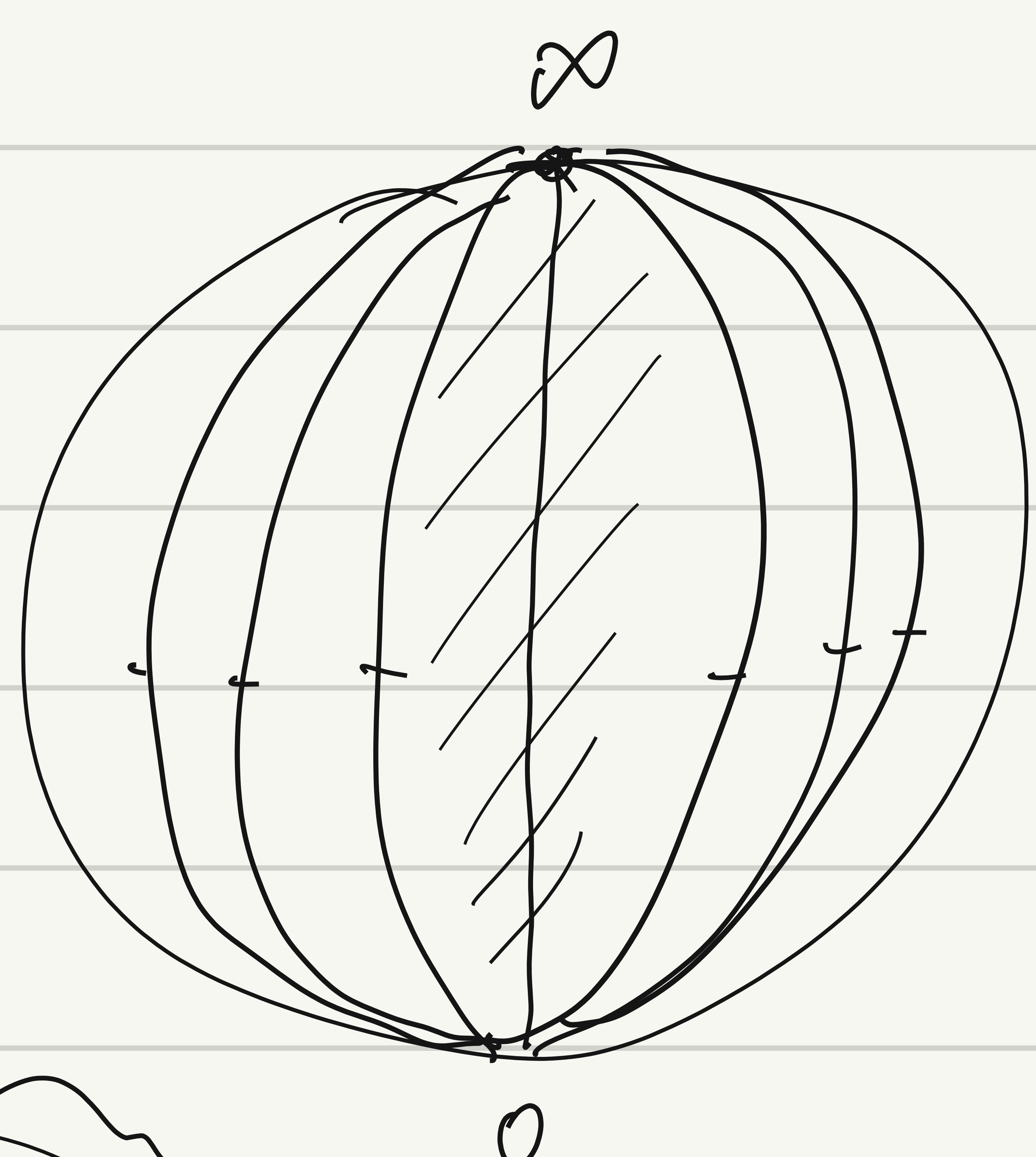


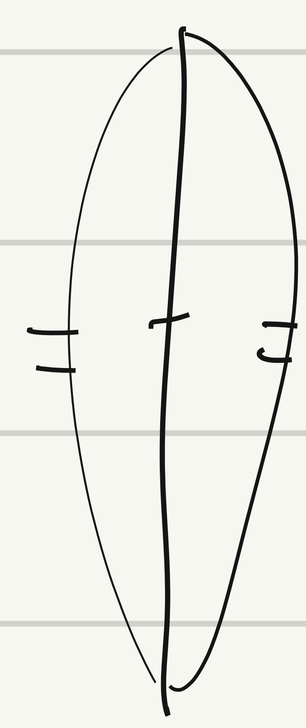


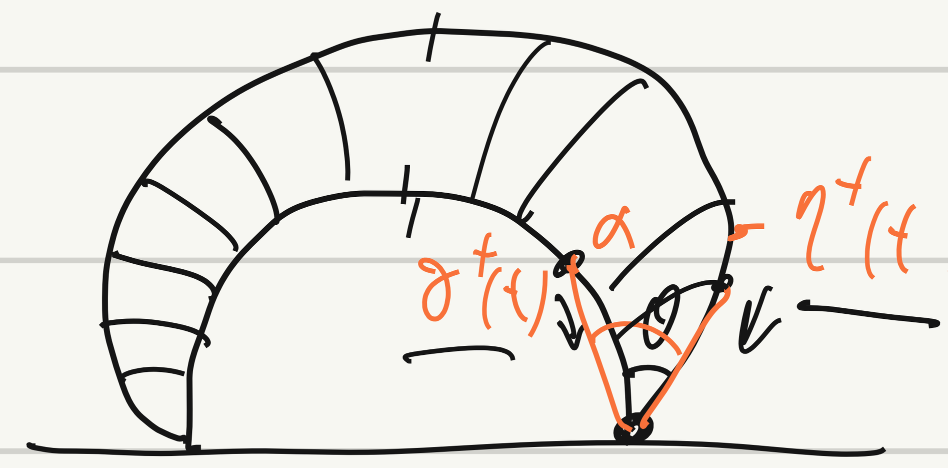
$$d(\gamma^+(t), \eta^+(t)) = M \quad \forall t.$$





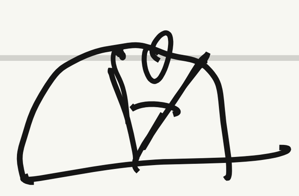


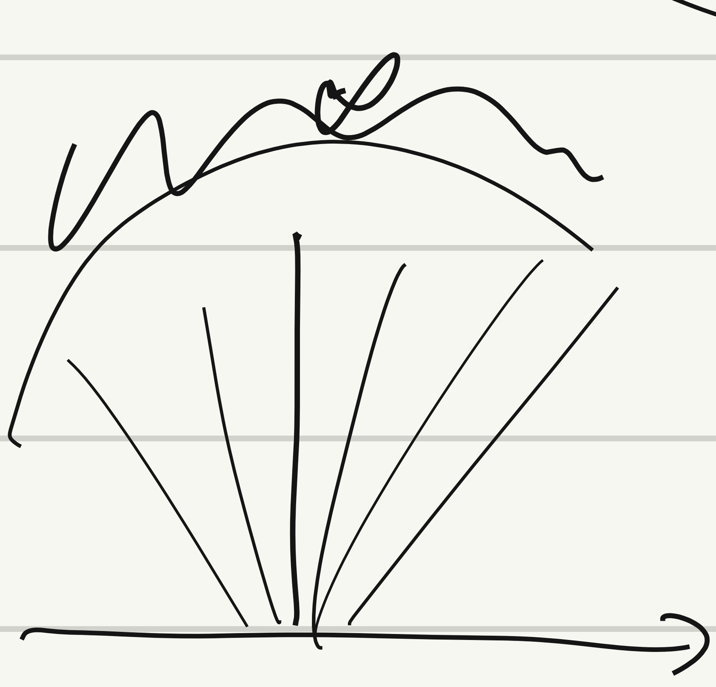




$\alpha(t) \to 0.$

$d_{\mathbb{H}}(\gamma^+(t), \eta^+(t)) < M$



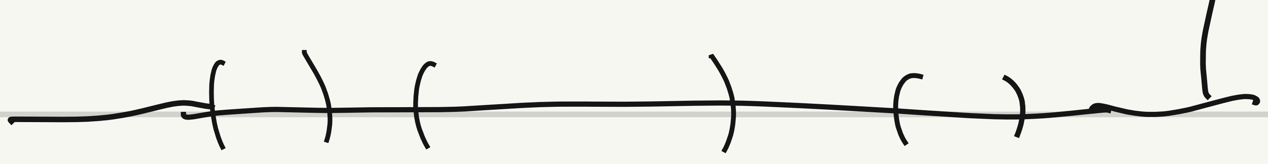


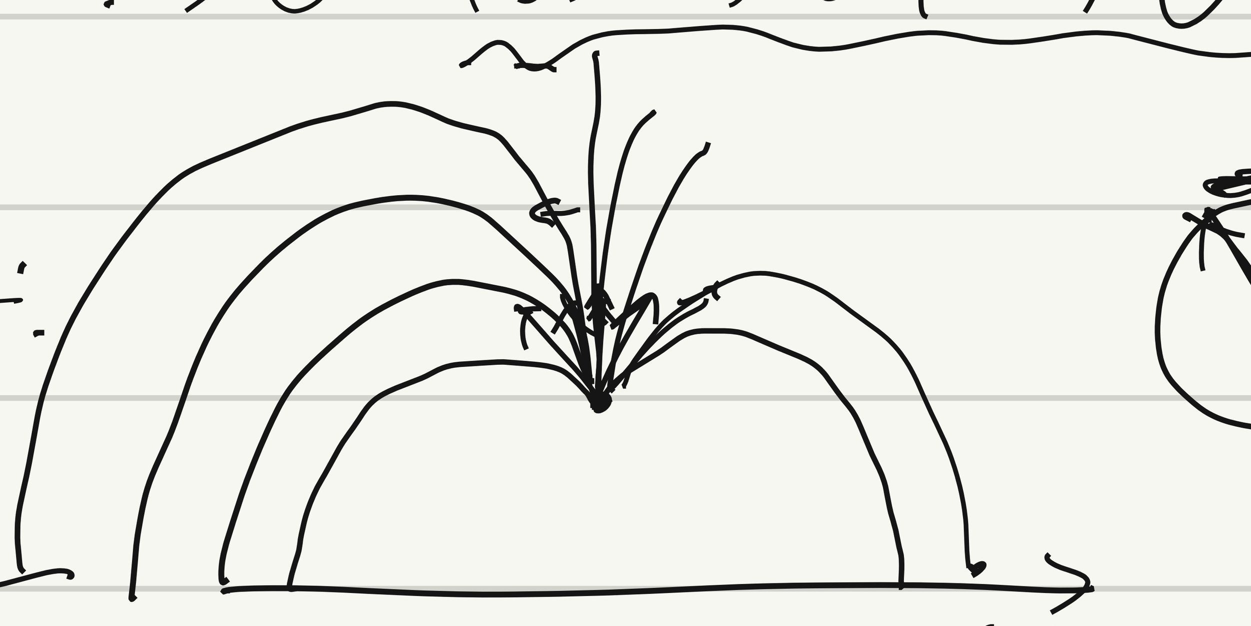
$\mathbb{H} \cup \partial\mathbb{H}$

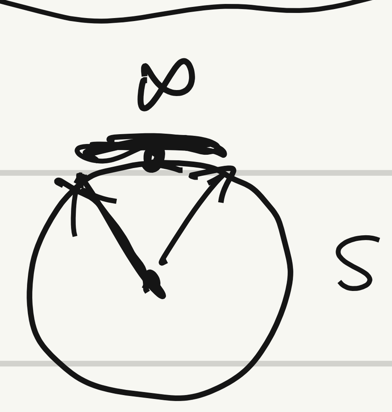
$\partial\mathbb{H} \cong \mathbb{R} \cup \{\infty\}$ → ① (a, b).

→ ② $(b, \infty) \cup \{\infty\} \cup (\infty, a)$.

$\partial\mathbb{H} \cong S^1$ homeo.



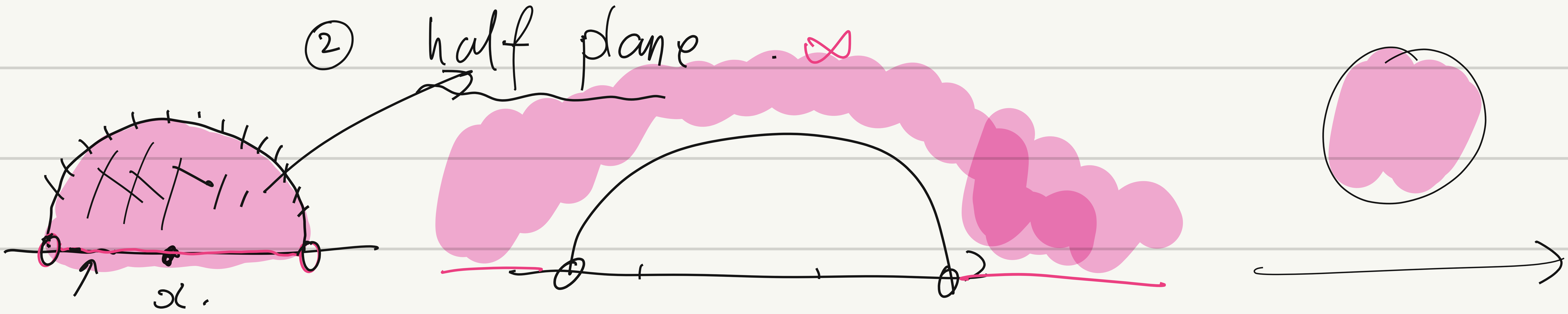




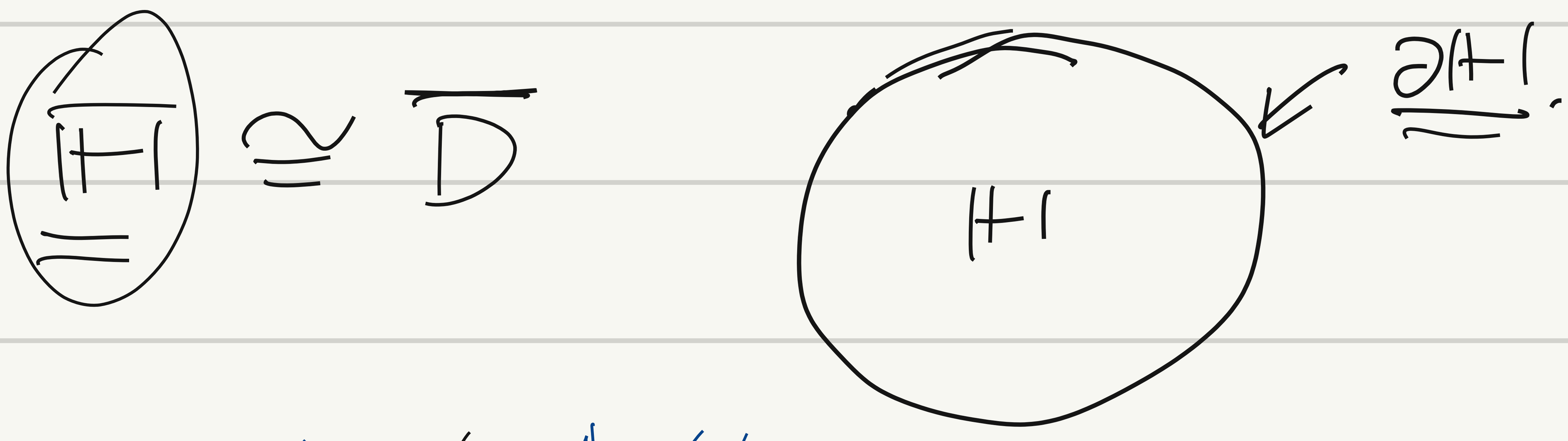
$\mathbb{H} \cup \partial\mathbb{H} = \overline{\mathbb{H}}$

basis. ① open disks. (basis in $\mathbb{H}$)

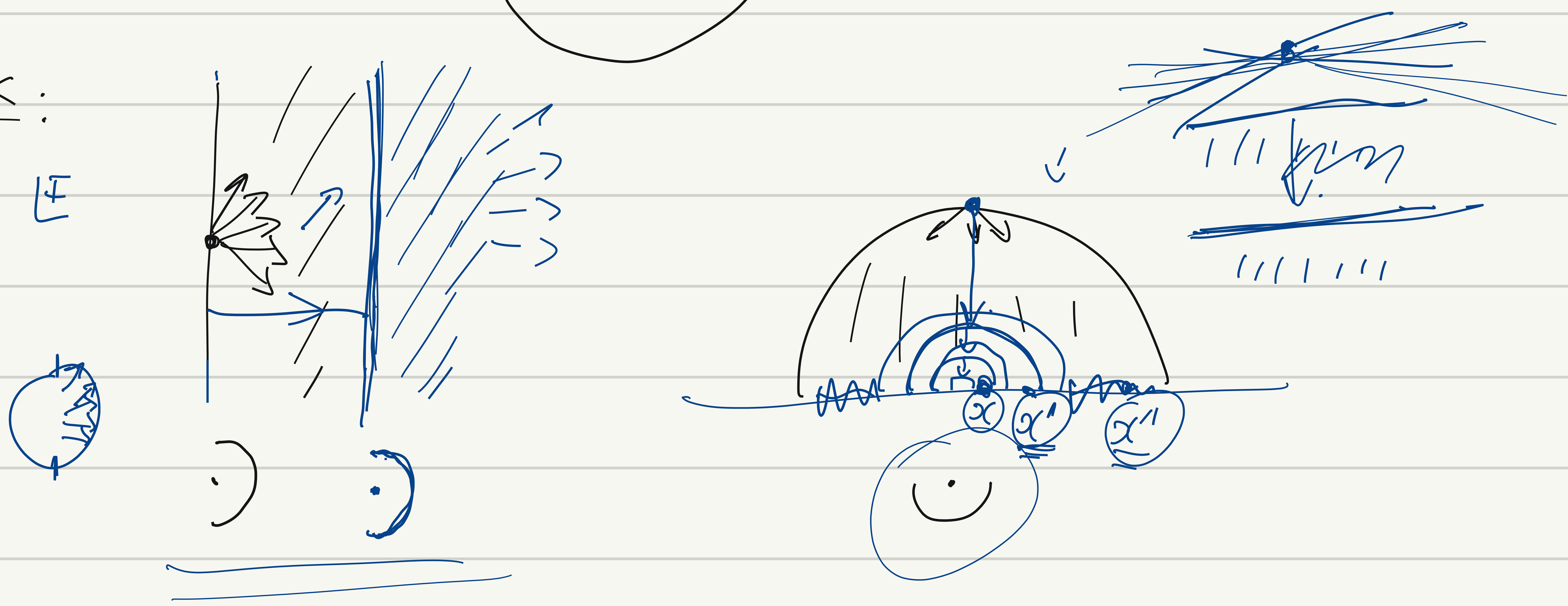
② half plane.



$\overline{\mathbb{H}} \cong \overline{D}$



Rmk:



II. Isometry group of $\mathbb{H}$:

1. **Def**: $f: \mathbb{H} \to \mathbb{H}$ is an isometry if $\forall w, z \in \mathbb{H}$ we have. $d_{\mathbb{H}}(f(w), f(z)) = d_{\mathbb{H}}(w, z)$.

~~$f_* ds_z = dw_s$~~

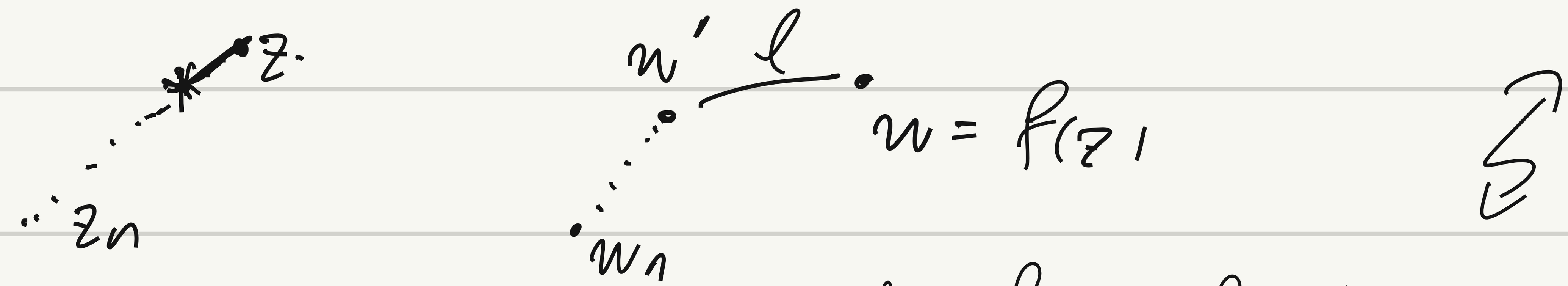
Ob: Identity map: $z \mapsto z$ is an isometry.

Some immediate consequence of the def.

Let $f: \mathbb{H} \to \mathbb{H}$ be an isometry.

Prop: f continuous.

Proof: Otherwise.

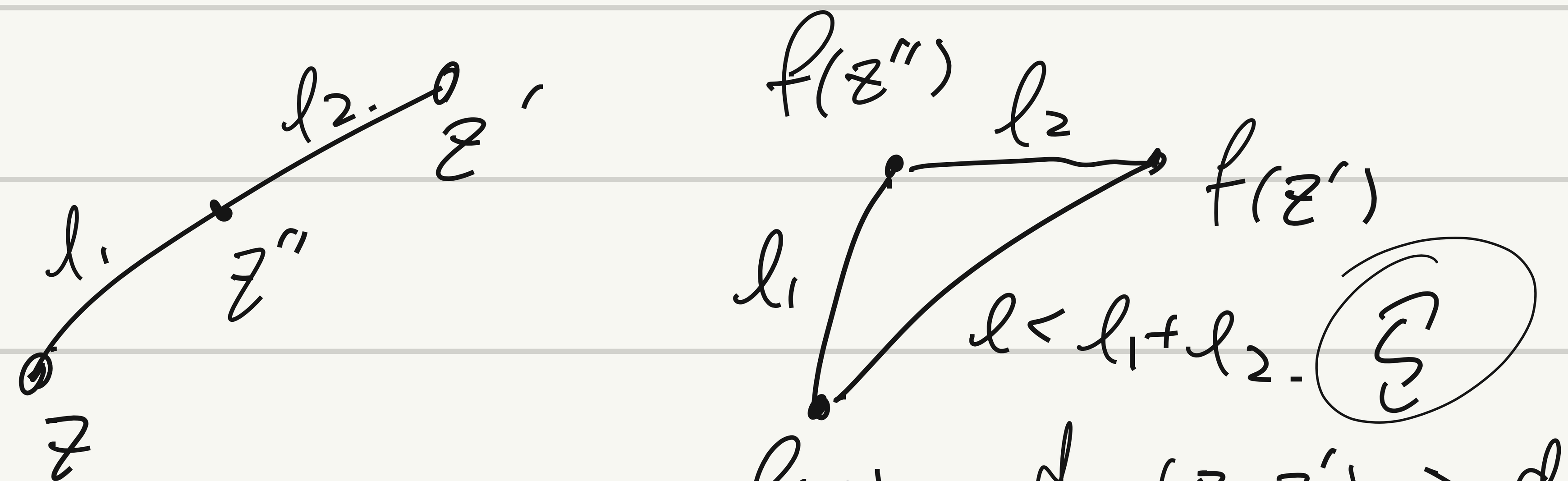


$d_{\mathbb{H}}(z_n, z) \to 0$ $\quad$ $w_n = f(z_n)$ $\quad$ $d_{\mathbb{H}}(f(z_n), f(z)) \to \ell$. ∎

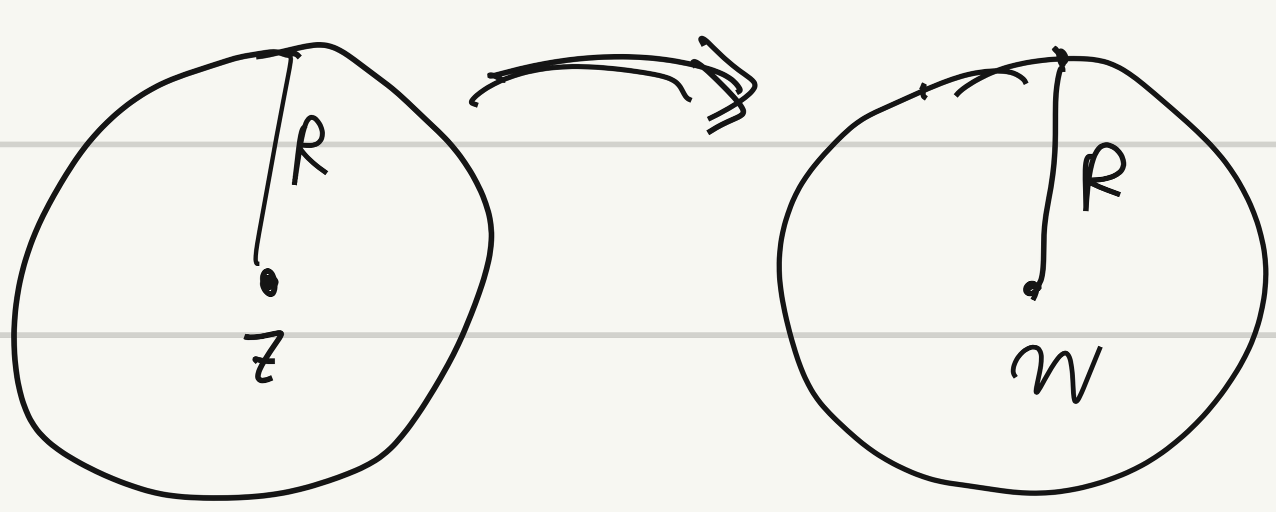
Prop: f sends ① geod's to geod's. $\Rightarrow$ extend. f to a map on $\overline{\mathbb{H}}$.

② circles to circles.
concentric. $\quad$ concentric.

Proof:



$d_{\mathbb{H}}(z, z') > d_{\mathbb{H}}(f(z), f(z'))$.

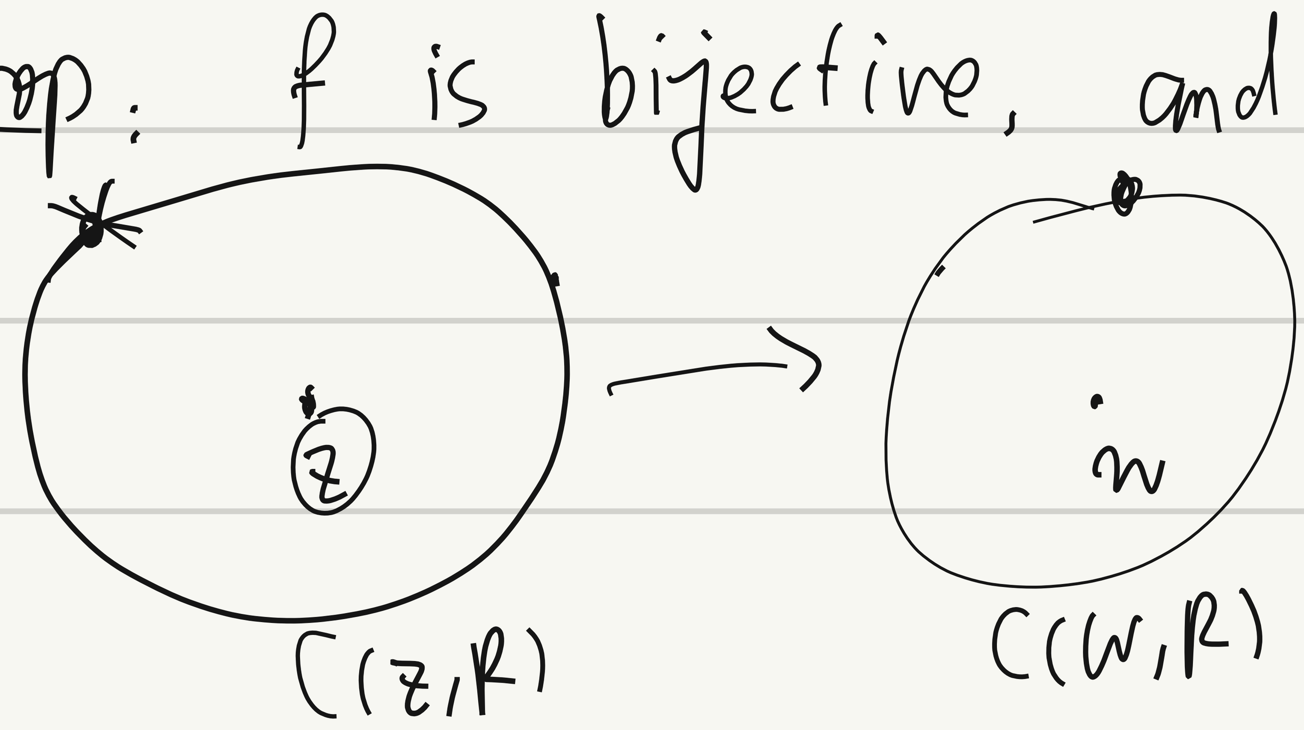


$f \quad C(z, R) \longrightarrow C(w, R)$, $\quad w = f(z)$. ∎

circles $\leftrightarrow$ horocycles hypercycles

Prop: f sends horocycles to horocycles
hypercycles to hypercycles.

Prop: f is bijective, and $f^{-1}: \mathbb{H} \to \mathbb{H}$ is also an isometry.



Prop, $f_1: \mathbb{H} \to \mathbb{H}$ isometries.
$f_2: \mathbb{H} \to \mathbb{H}$

then $f_2 \circ f_1: \mathbb{H} \to \mathbb{H}$ is also an isometry.

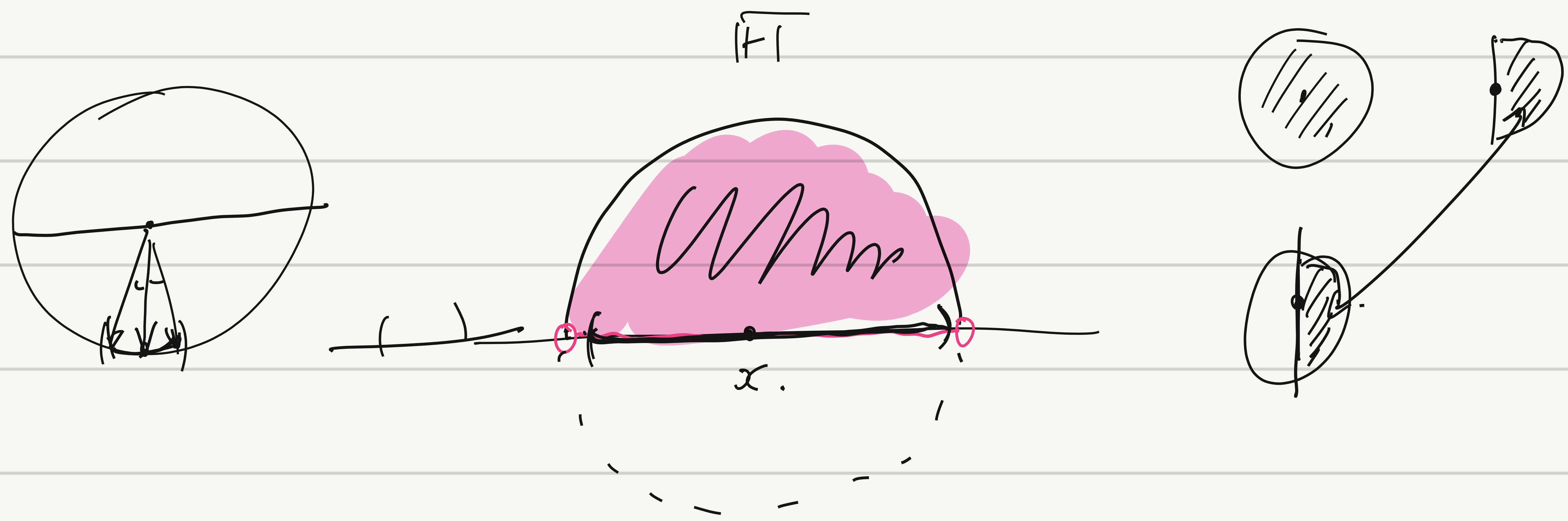
$$d_{\mathbb{H}}((f_2\circ f_1)(z), (f_2\circ f_1)(w)) = d_{\mathbb{H}}(f_1(z), f_1(w)) = d_{\mathbb{H}}(z,w)$$

$f_2(f_1(z))$ $f_2(f_1(w))$ ↑ f_2 isometry. ↑ f_1 isometry

Prop: $\underline{\mathrm{Isom}}(\mathbb{H}) := \{ f: \mathbb{H} \to \mathbb{H} \mid \text{isometry} \}$ is a group.

$$f_3\circ(f_2\circ f_1) = (f_3\circ f_2)\circ f_1.$$

Isometry $\qquad \mathrm{Isom}(\mathbb{H}) :=$ group of isometries of $\mathbb{H}$.



- id

$\mathrm{Isom}(\mathbb{H})$ is a group. - f $\exists f^{-1}$

- $(f_1\circ f_2)\circ f_3 = f_1\circ(f_2\circ f_3)$.

Prop: $\forall$ K, K' subset of $\mathbb{H}$, $d_{\mathbb{H}}(K,K') = d_{\mathbb{H}}(f(K), f(K'))$.

2. First example.:

$t\in\mathbb{R}$ $\quad T_t(z) = z+t \qquad \mathbb{C}\to\mathbb{C}$

$\lambda\in\mathbb{R}_{>0}$ $\quad \Phi_\lambda(z) = \lambda z \qquad \mathbb{C}\to\mathbb{C}$

- Prop: T_t, Φ_λ presere. $\mathbb{H}$ $\quad (T_t(\mathbb{H})\subseteq\mathbb{H} \quad \Phi_\lambda(\mathbb{H})\subseteq\mathbb{H})$

  Proof: $\forall z$ $\mathrm{Im}(z)>0$, $\cdot\ \mathrm{Im}\,T_t(z)>0 \qquad \mathrm{Im}\,T_t(z) = \mathrm{Im}(z) > 0$

  $\cdot\ \mathrm{Im}\,\Phi_\lambda(z) > 0. \qquad \mathrm{Im}\,\Phi_\lambda(z) = \lambda\,\mathrm{Im}(z) > 0$

- Prop: T_t, Φ_λ are isometries.

  Proof: $\forall w, z\in\mathbb{H}$ $\quad \left|\dfrac{T_t(w)-T_t(z)}{\overline{T_t(w)}-T_t(z)}\right| = \left|\dfrac{w-z}{\bar{w}-z}\right|$ ✓

$$\left|\frac{\phi_\lambda(w)-\phi_\lambda(z)}{\overline{\phi_\lambda(w)}-\phi_\lambda(z)}\right| = \frac{\lambda}{\lambda}\left|\frac{w-z}{\bar{w}-z}\right| = \left|\frac{w-z}{\bar{w}-z}\right| \qquad \blacksquare$$

$$d_{\mathbb{H}}(\phi_\lambda(w), \phi_\lambda(z)) = d_{\mathbb{H}}(w, z).$$

Rmk: $\gamma : [a,b] \to \mathbb{H}$ $\qquad \eta = T_t \circ \gamma : [a,b] \to \mathbb{H}$.

$$\|\dot\eta(t)\|_{\mathbb{E}} = \|\dot\gamma(t)\|_{\mathbb{E}} \Longrightarrow \|\dot\eta(t)\|_{\mathbb{H}} = \|\dot\gamma(t)\|_{\mathbb{H}}$$

$+ \ \mathrm{Im}(\eta(t)) = \mathrm{Im}(\gamma(t))$.

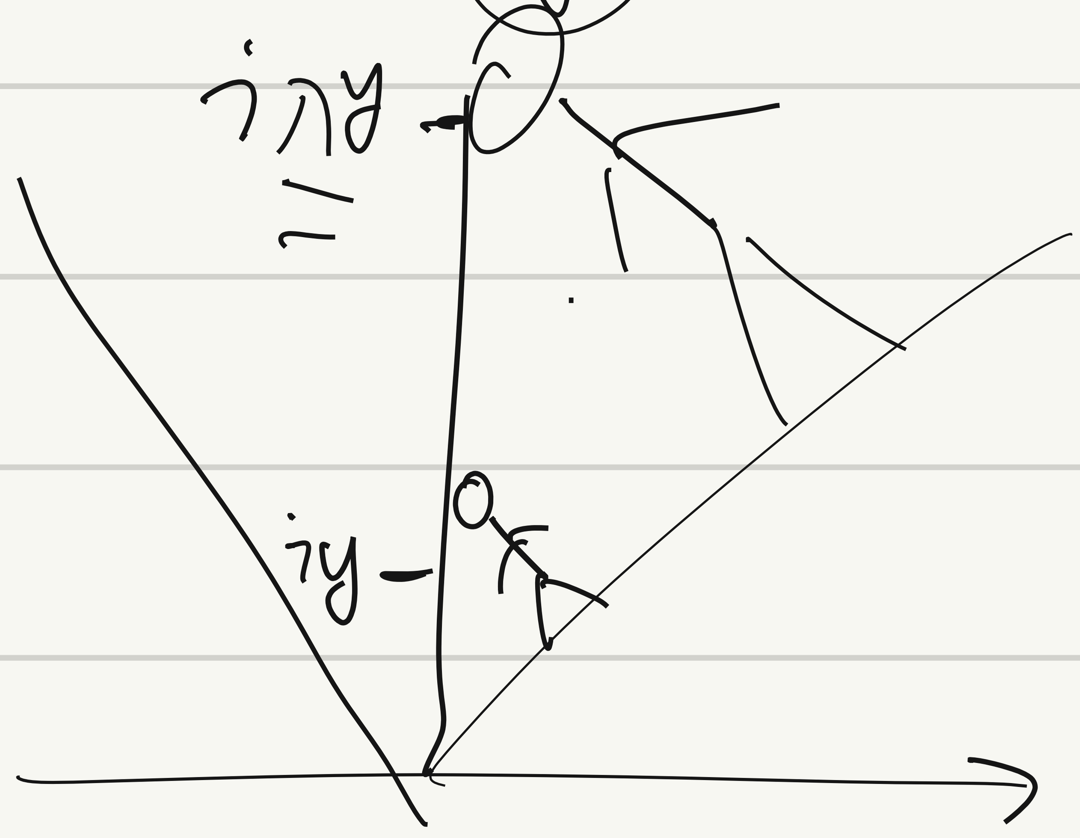
$$\eta = \phi_\lambda \circ \gamma : [a,b] \to \mathbb{H}.$$

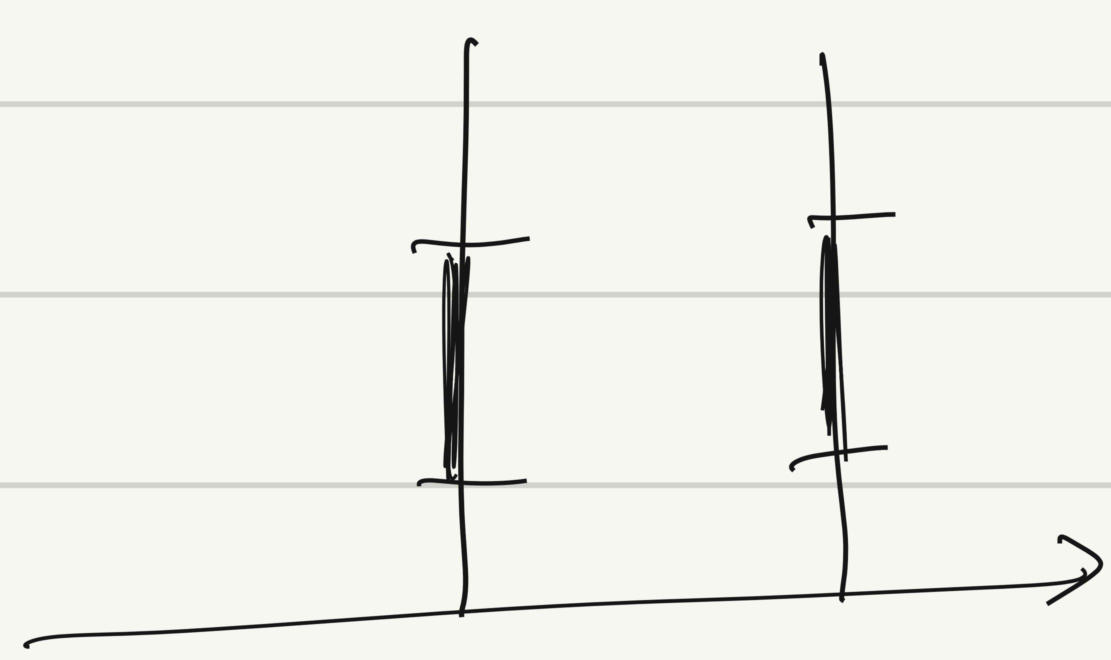
$$\dot\eta(t) = \lambda\dot\gamma(t)$$

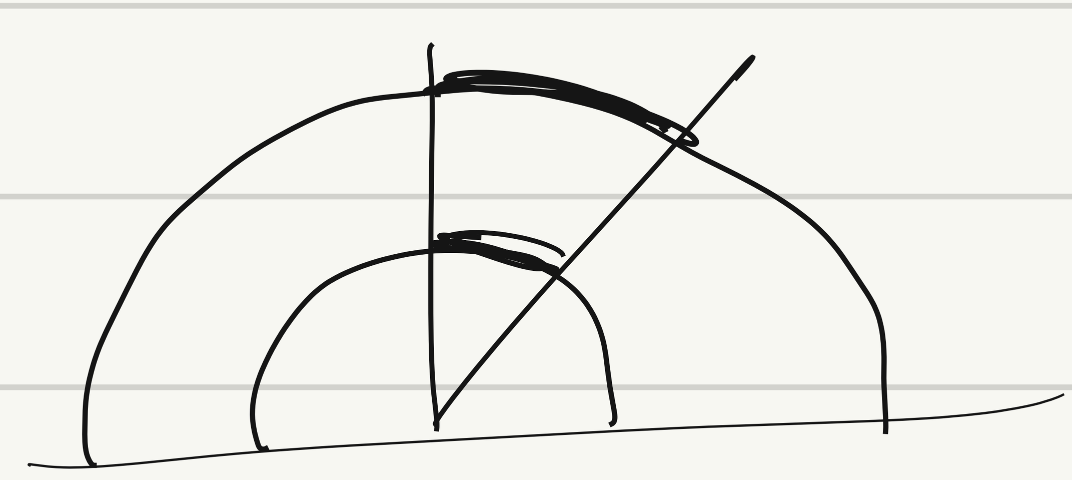
$$\mathrm{Im}(\eta(t)) = \lambda\, \mathrm{Im}(\gamma(t))$$

$$\|\dot\eta(t)\|_{\mathbb{H}} = \frac{\|\dot\eta(t)\|_{\mathbb{E}}}{\mathrm{Im}\,\eta(t)} = \frac{\cancel{\lambda}\,\|\dot\gamma(t)\|_{\mathbb{E}}}{\cancel{\lambda}\,\mathrm{Im}(\gamma(t))} \qquad \blacksquare$$

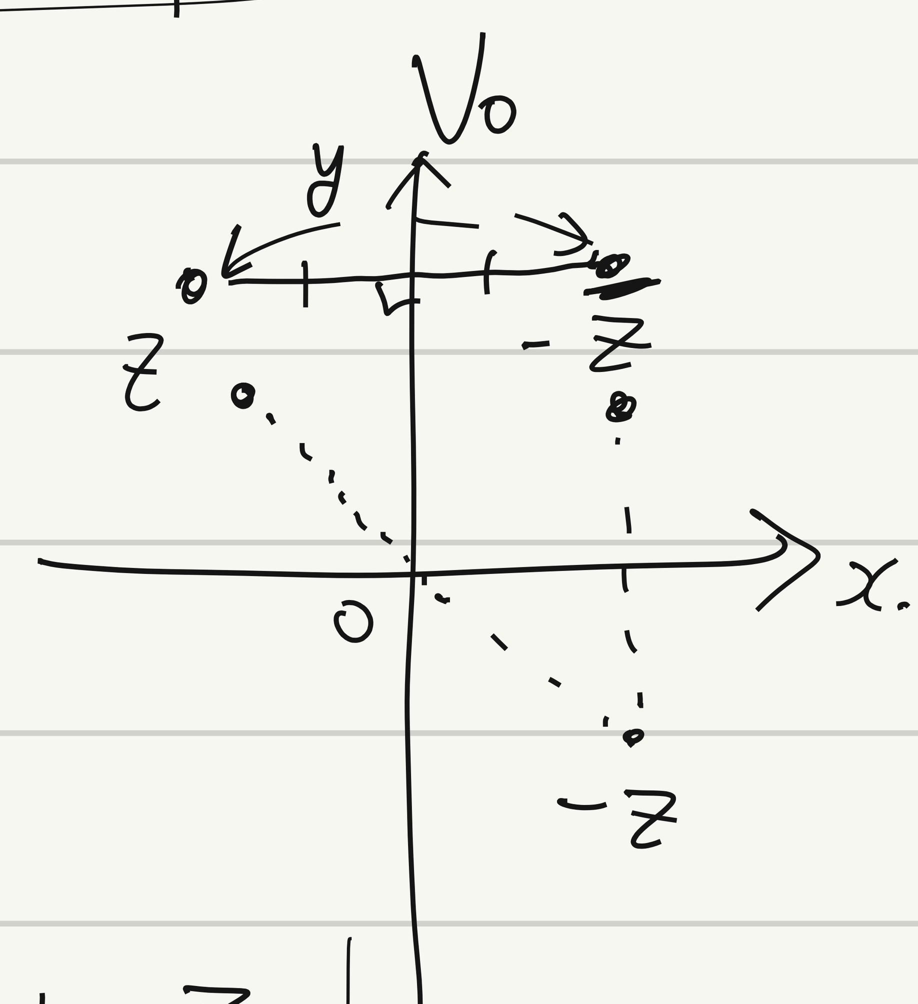
$$ds = \frac{\sqrt{dx^2+dy^2}}{y}$$







3. Reflection:

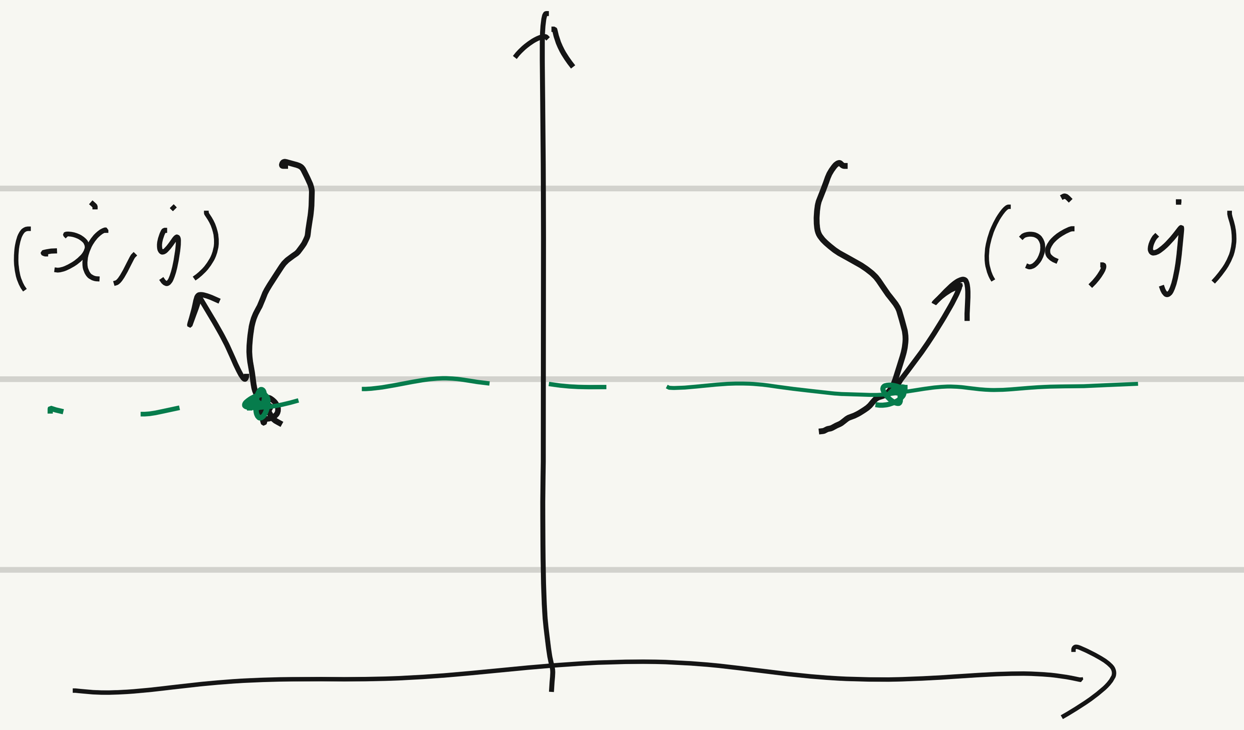


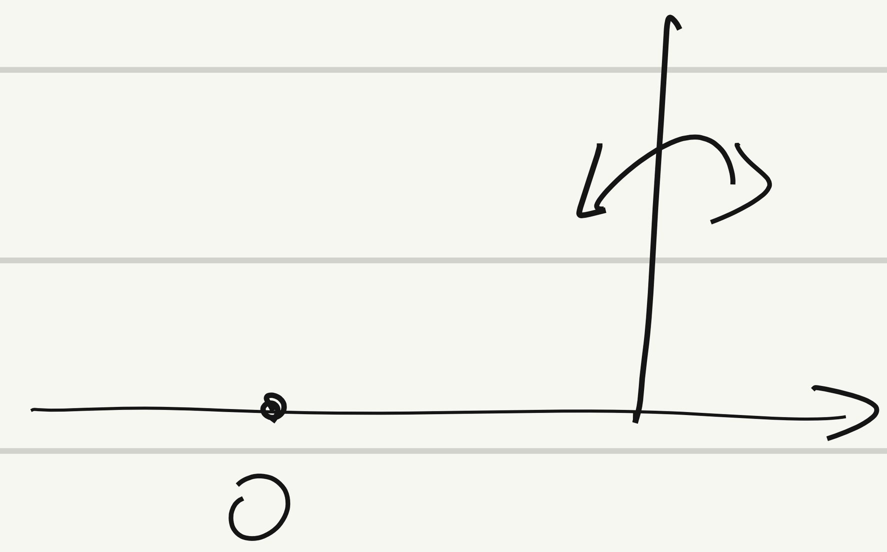
- Consider $\tau_0(z) = -\bar{z} : \mathbb{C} \to \mathbb{C}$

Prop: $\tau_0 : \mathbb{H} \to \mathbb{H}$ is an isometry.

$x + iy \mapsto -x + iy$.

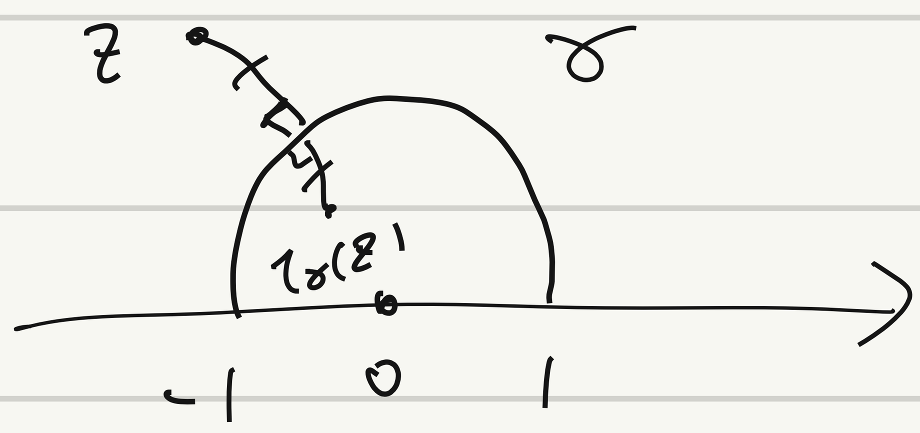
Proof:
$$\left|\frac{\tau_0(w)-\tau_0(z)}{\tau_0(\bar{w})-\tau_0(z)}\right| = \left|\frac{\bar{w}-\bar{z}}{w-\bar{z}}\right| = \left|\frac{w-z}{\bar{w}-z}\right|. \qquad \blacksquare$$





• Reflections along other geodesic ?

(1)



Let $\tau_\gamma : \mathbb{H} \longrightarrow \mathbb{H}$ s.t.

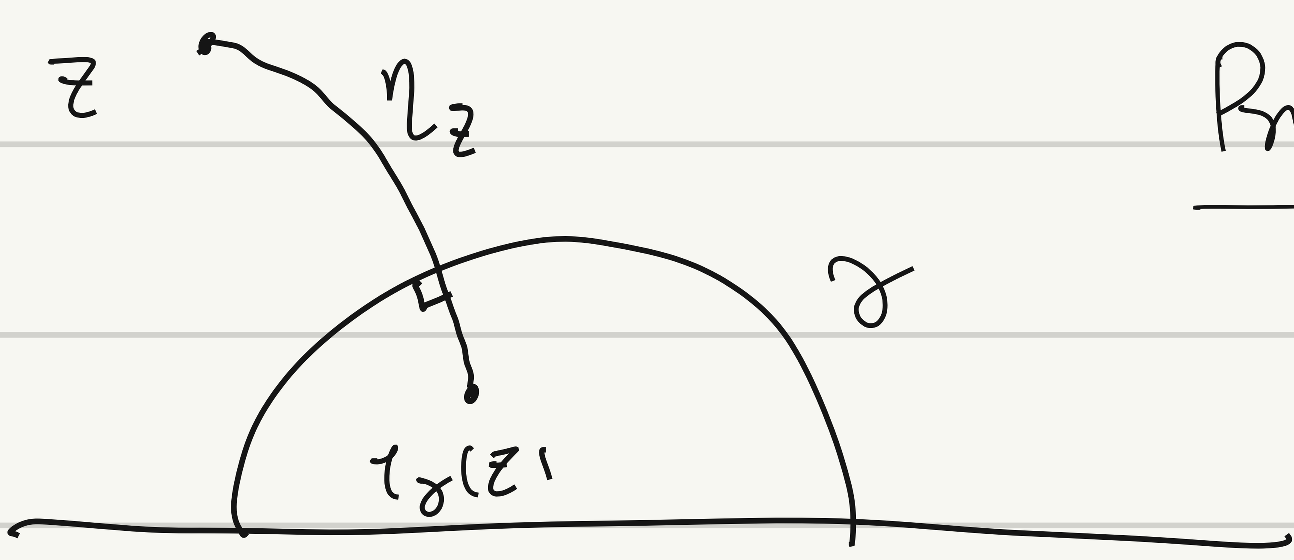
① $\tau_\gamma(z) = z \quad \forall z \in \gamma$

② τ_γ is an isometry.

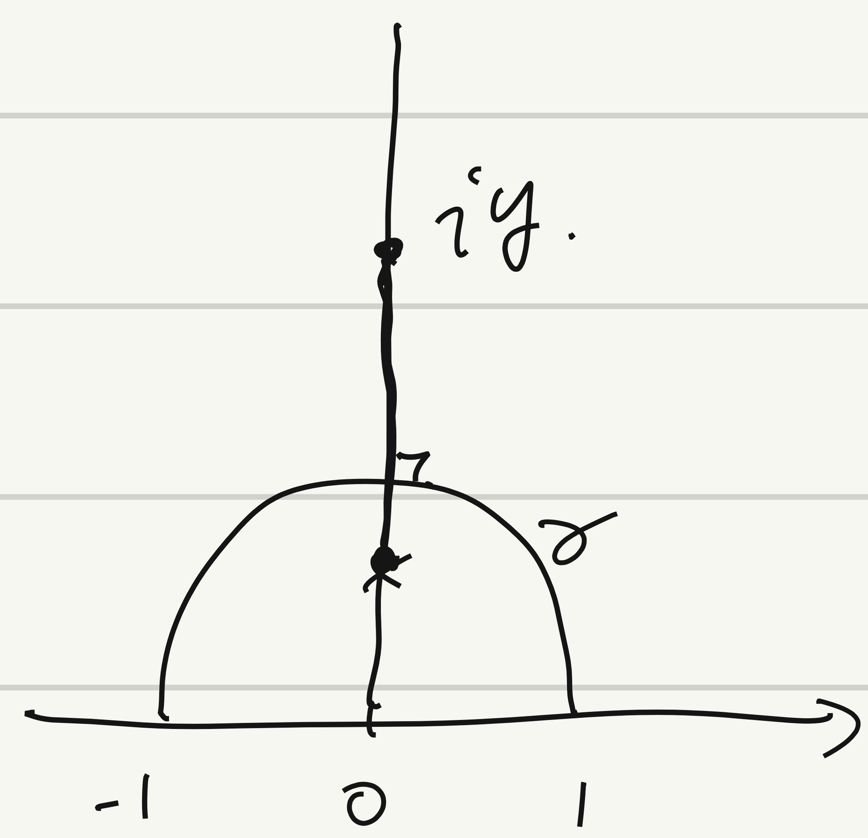
Ob: $d_{\mathbb{H}}(\underline{z}, \underline{\gamma}) = d_{\mathbb{H}}(\tau_\gamma(z), \gamma)$

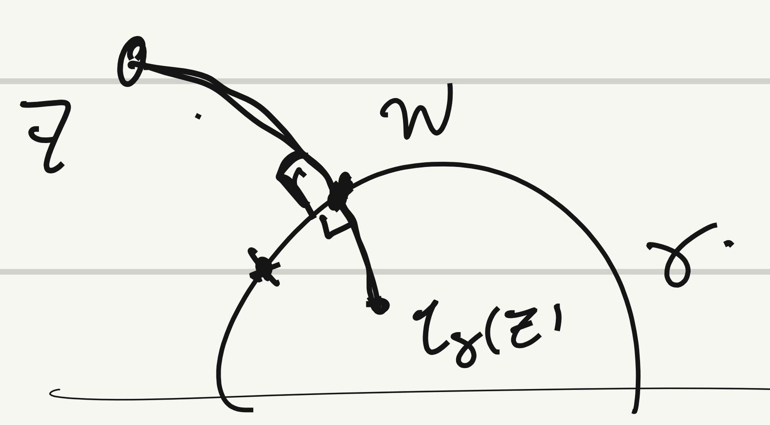
Cor: $\forall z \notin \gamma$, η_z connecting z and $\tau_\gamma(z)$

then $\eta_z \perp \gamma$.



Proof. ①

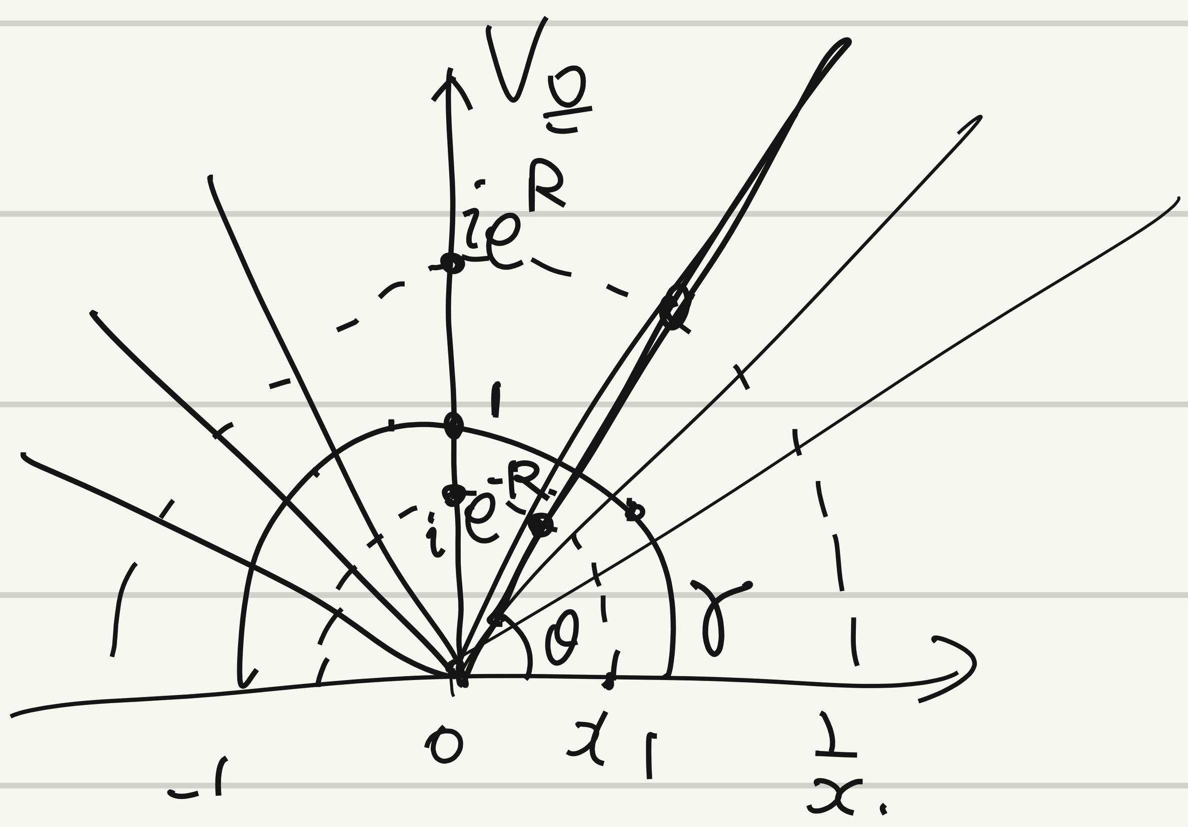




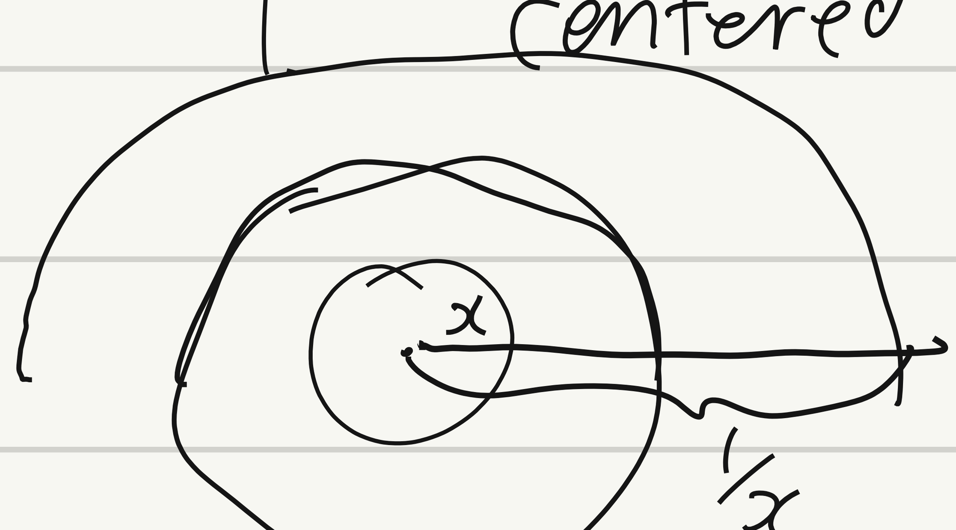
$d_{\mathbb{H}}(z,w) = d_{\mathbb{H}}(z,\gamma)$

$\forall w \in \gamma \quad \tau_\gamma(w) = w.$

Cor: $\forall \eta$, $\eta \perp \gamma$, $\underline{\tau_\gamma(\eta) = \eta}$.
subsets of $\mathbb{H}$



Cor: τ_γ preserve all hypercycles centered at V_0.



$|\bar{w}\bar{z}| = |w||z| = |w\bar{z}|$

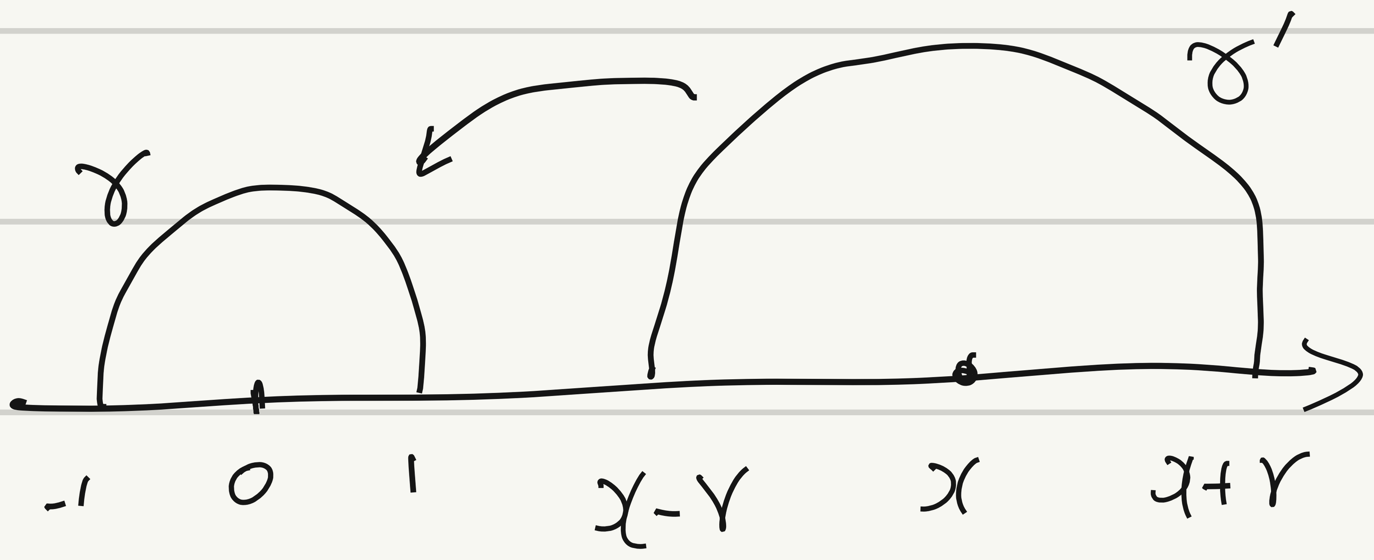
$$\boxed{\tau_\gamma(z) = \frac{1}{\bar{z}}}$$

$$\left|\frac{\frac{1}{\bar{w}} - \frac{1}{\bar{z}}}{\frac{1}{w} - \frac{1}{\bar{z}}}\right| = \left|\frac{\bar{z} - \bar{w}}{\bar{z} - w}\right| = \left|\frac{w - z}{\bar{w} - z}\right|$$

$\tau_\gamma : \mathbb{H} \longrightarrow \mathbb{H}$

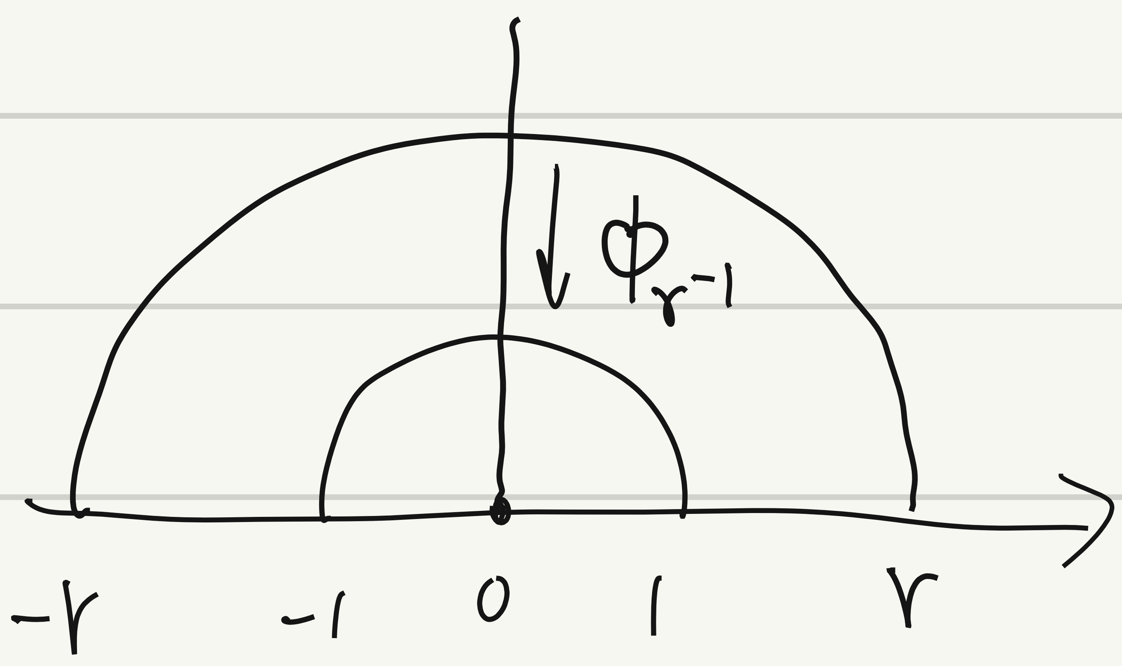
$z = re^{i\theta} \longmapsto r^{-1}e^{i\theta} = \frac{1}{\bar{z}}$

(2) Let γ' be a circular geod of Euclidean center x,
radius r.



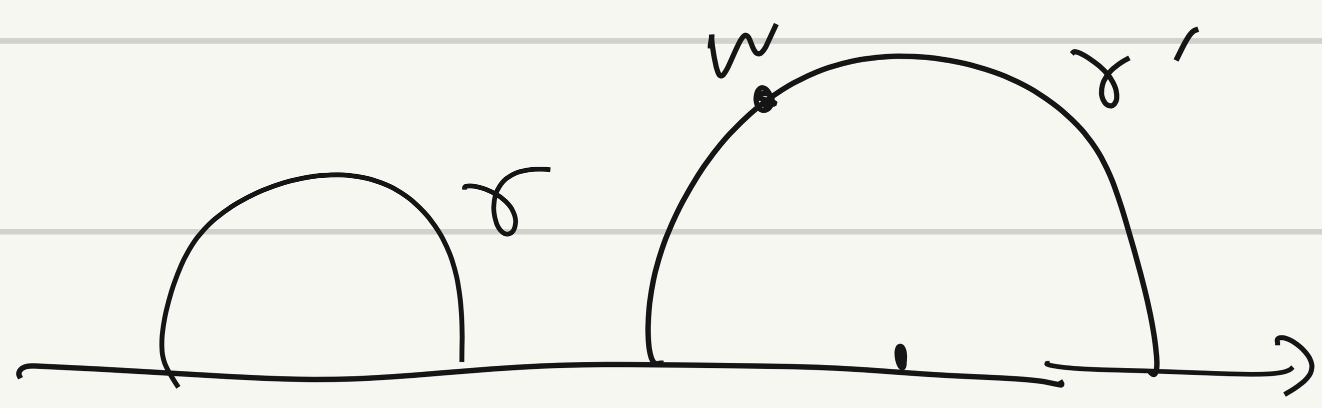
$\downarrow T_{-x}$.

$f = \phi_{r^{-1}} \circ T_{-x}$



Prop: $I_{\gamma'} = f^{-1} \circ I_\gamma \circ f$

reflection w.r.t. γ'



$I_{\gamma'}(w) = w$

isometry.

$f(z) = \phi_{r^{-1}}(T_{-x}(z))$

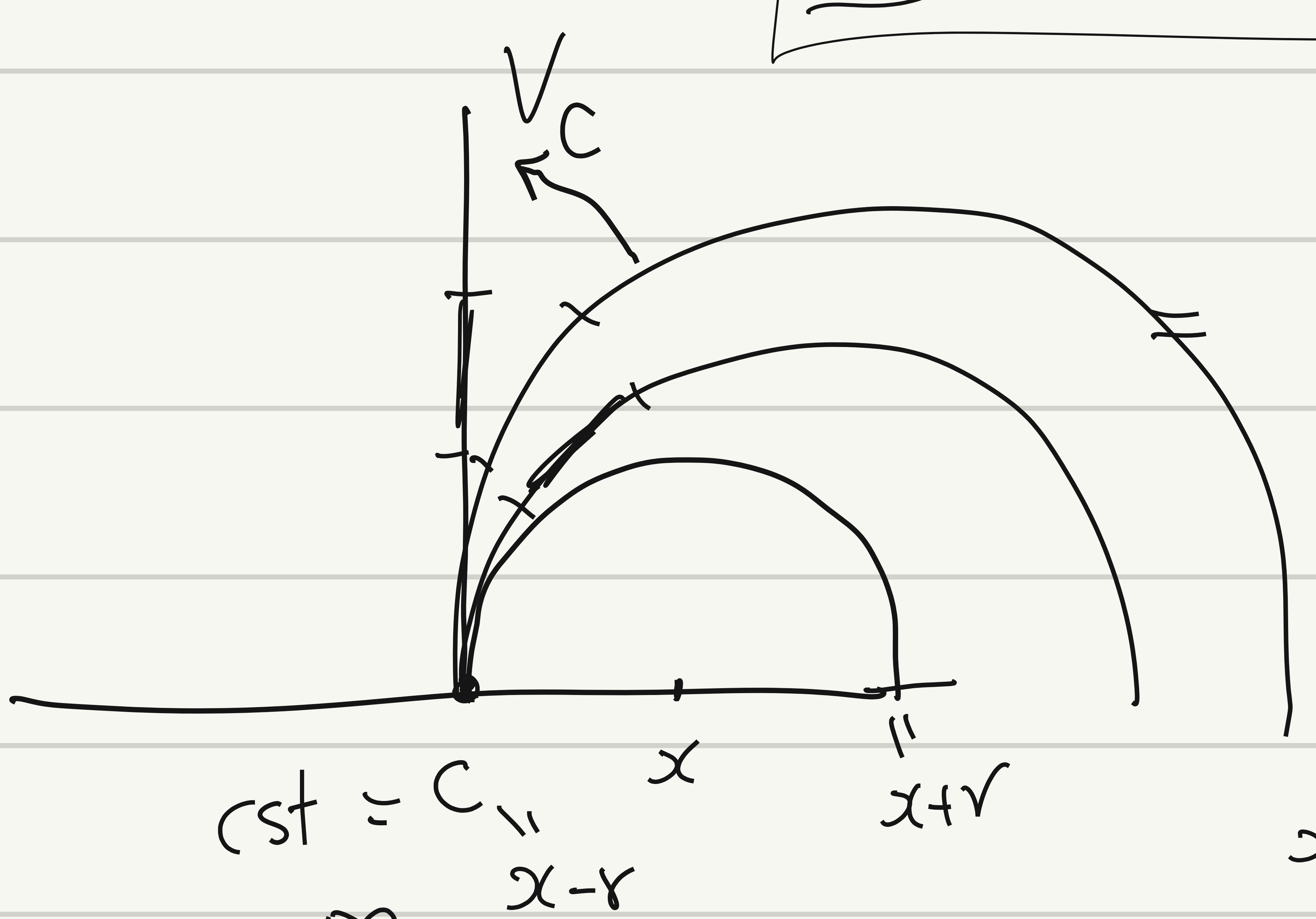
$= \frac{z-x}{r}$

$I_{\gamma'}: \mathbb{H} \xrightarrow{f} \mathbb{H} \xrightarrow{I_\gamma} \mathbb{H} \xrightarrow{f^{-1}} \mathbb{H}$

$z \longmapsto \frac{z-x}{r} \longmapsto \frac{1}{\frac{\bar z - x}{r}} = \frac{r}{\bar z - x} \longmapsto r\frac{r}{\bar z - x} + x.$

$f^{-1}(w) = rw + x$

$$\boxed{I_{\gamma'}(z) = \frac{x\bar z + r^2 - x^2}{\bar z - x}}$$



$= \frac{x\bar z + (r-x)(r+x)}{\bar z - x.} = \frac{\bar z + (-C)\frac{r+x}{x}}{\frac{1}{x}\bar z - 1}$

2

0

$\lim_{x\to\infty} \frac{r+x}{x} = 2$

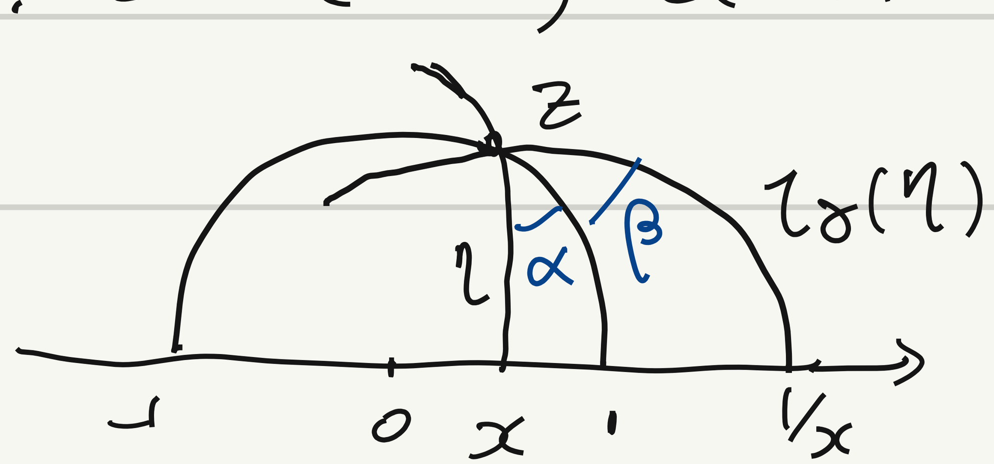
$\to -\bar z + 2C$

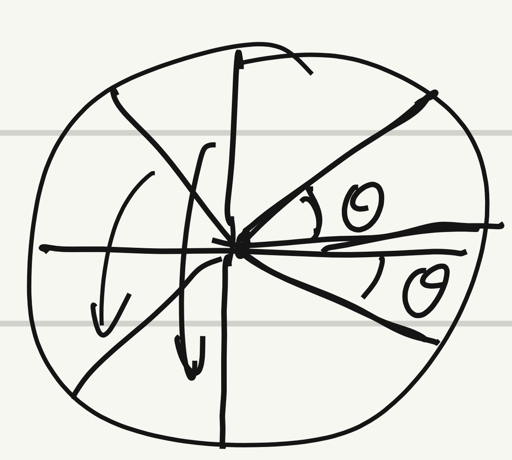
reflection along

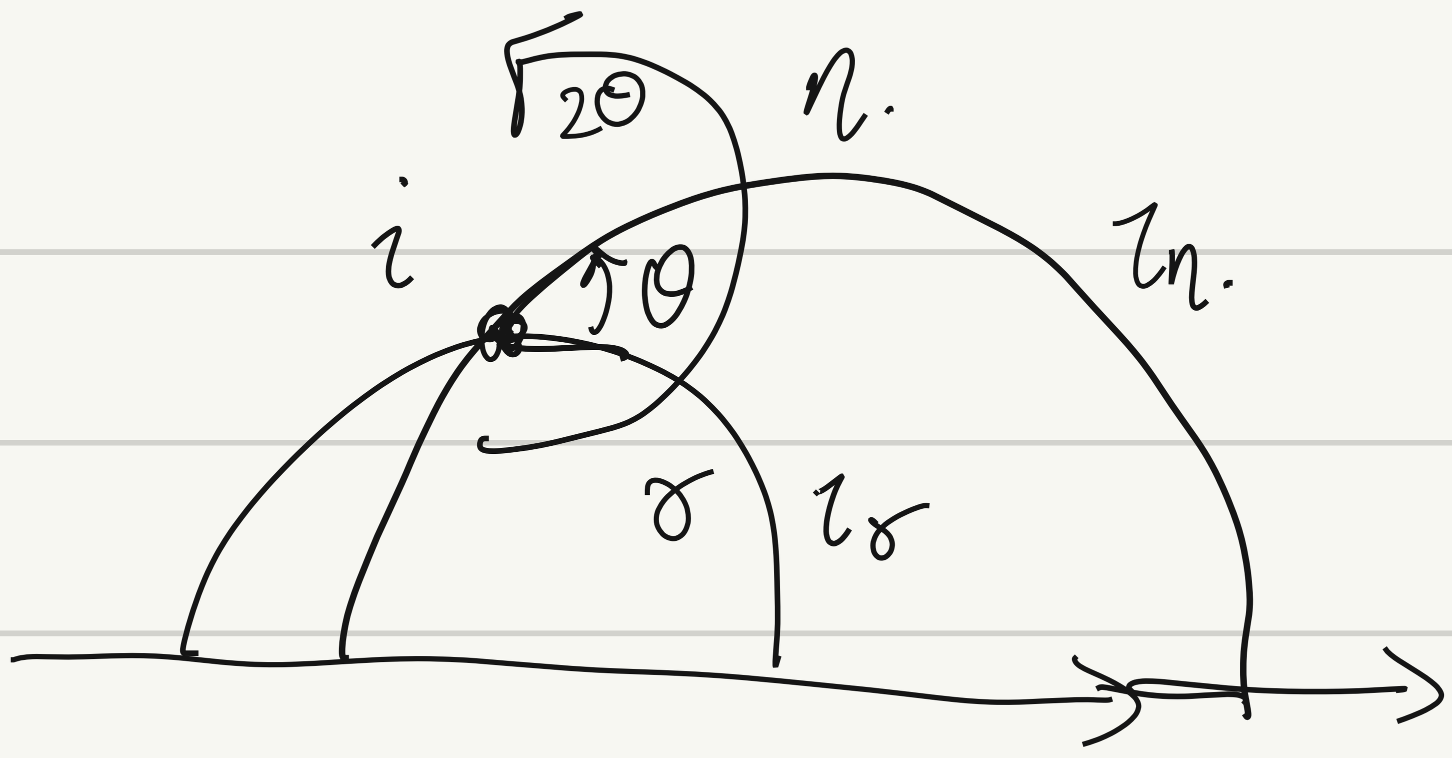
$V_c = V_{x-r}.$

Let: $z = e^{i\theta}$, $x \in \mathbb{R}$.

Prop: $\alpha = \beta$





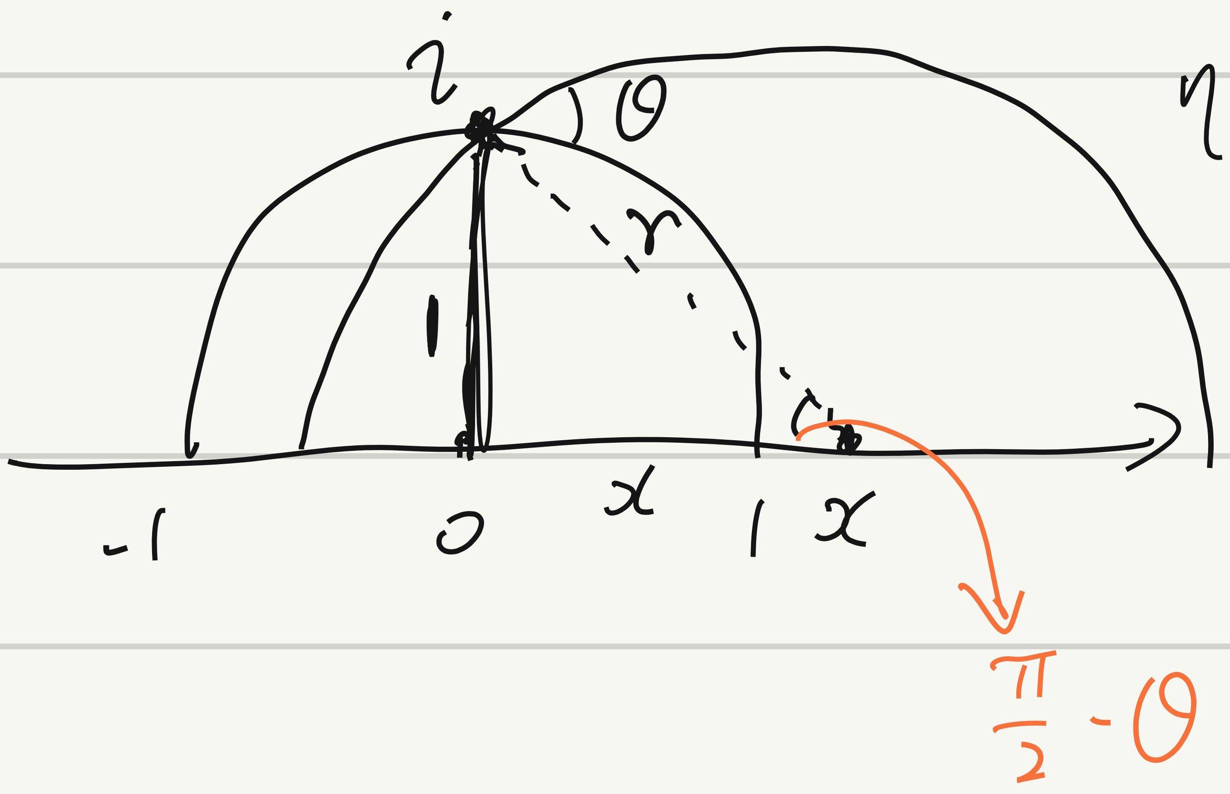


Prop: $I_\eta \circ I_\gamma = \rho_\theta$

① isometry

② rotate $\mathbb{H}$! around i for angle 2θ.

Prop: $$\rho_\theta(z) = \frac{\cos\theta\, z + \sin\theta}{-\sin\theta\, z + \cos\theta}.$$



$r = \sqrt{x^2+1}$

$\sin\theta = \frac{x}{\sqrt{x^2+1}}$

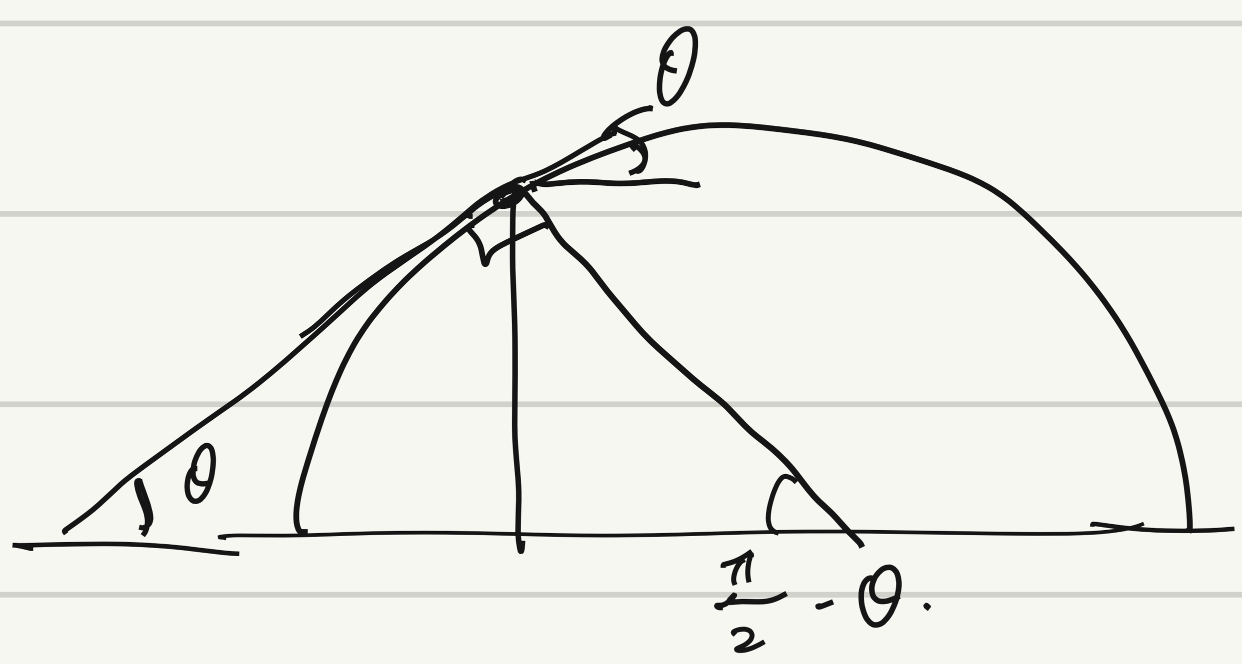
$\cos\theta = \frac{1}{\sqrt{x^2+1}}$

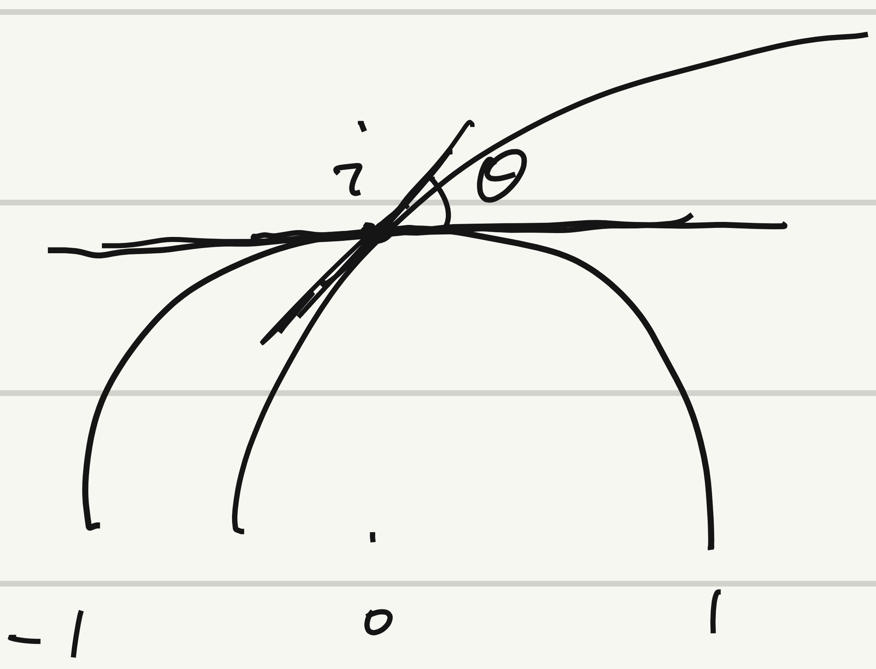
$$I_\eta = \frac{x\bar{z} + r^2 - x^2}{\bar{z} - x}.$$

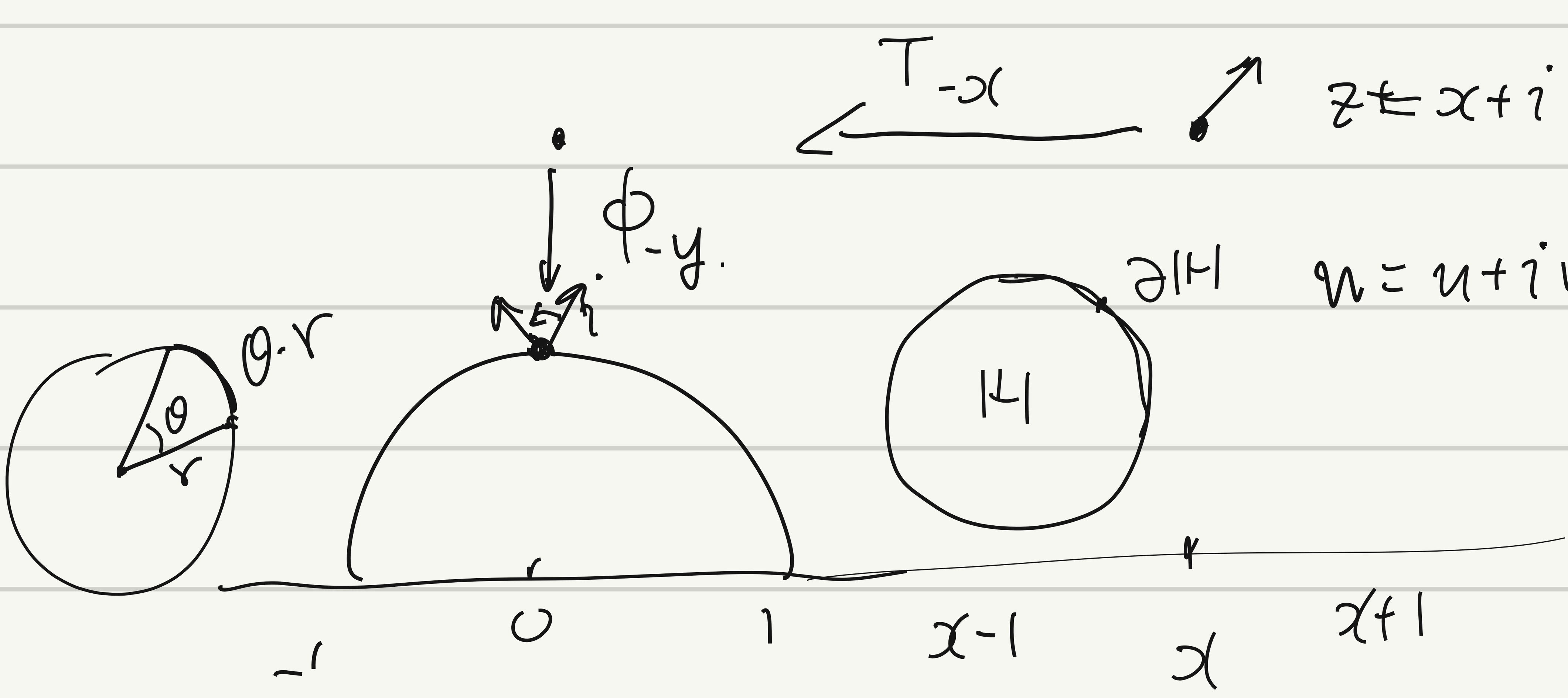
$$I_\eta \circ I_\gamma(z) = \frac{x\left(\overline{\tfrac{1}{\bar{z}}}\right) + r^2 - x^2}{\left(\overline{\tfrac{1}{\bar{z}}}\right) - x}$$

$$= \frac{x z^{-1} + 1}{z^{-1} - x} = \frac{z + x}{-xz + 1}$$

$$= \frac{\cos\theta\, z + \sin\theta}{-\sin\theta\, z + \cos\theta}. \quad \boxtimes$$







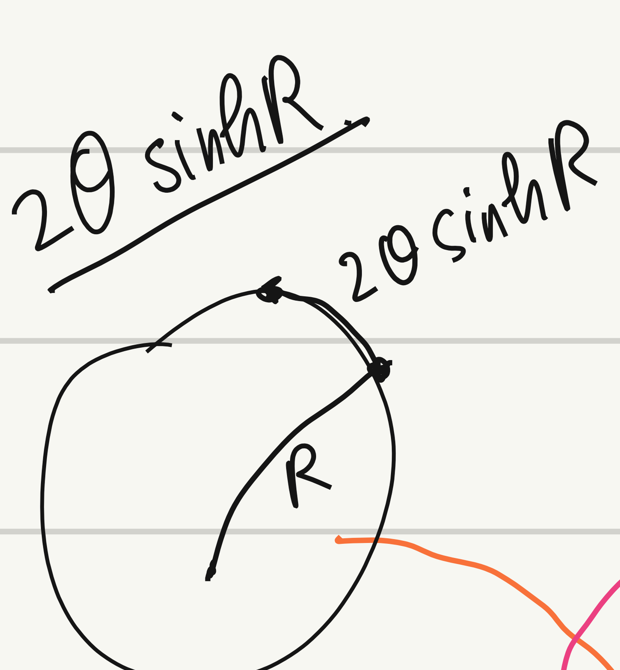
$$z \xmapsto{(T_x \circ \phi_y)^{-1}} i \xmapsto{T_u \circ \phi_v} w.$$

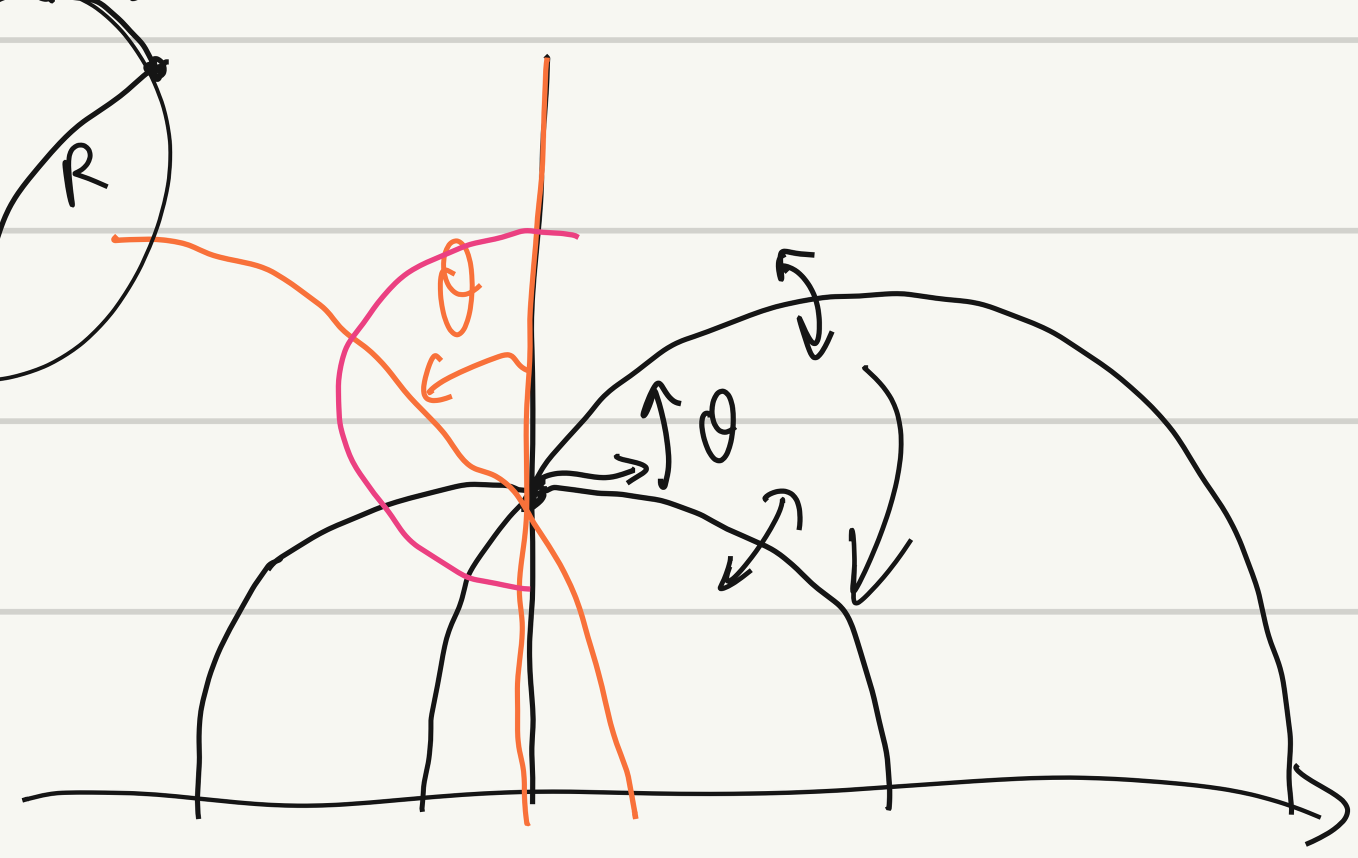
$w = T_u \circ \phi_v(i)$

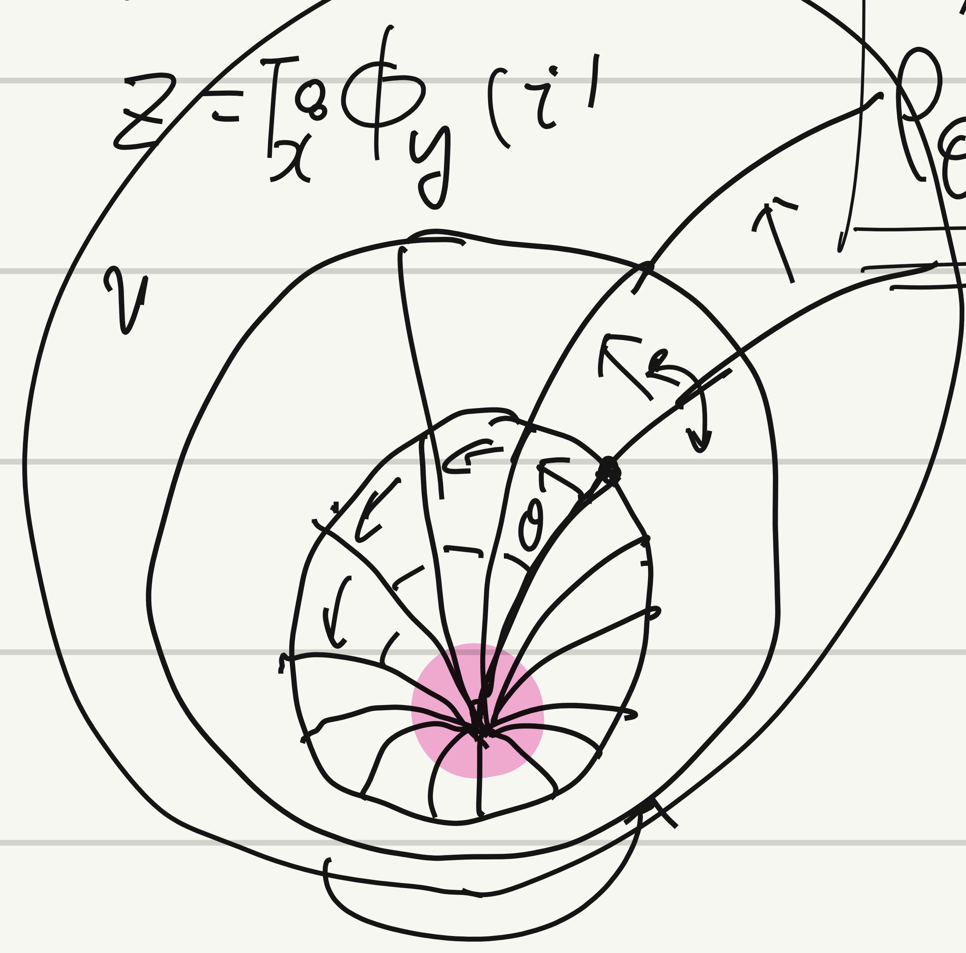
$z = T_x \circ \phi_y(i)$

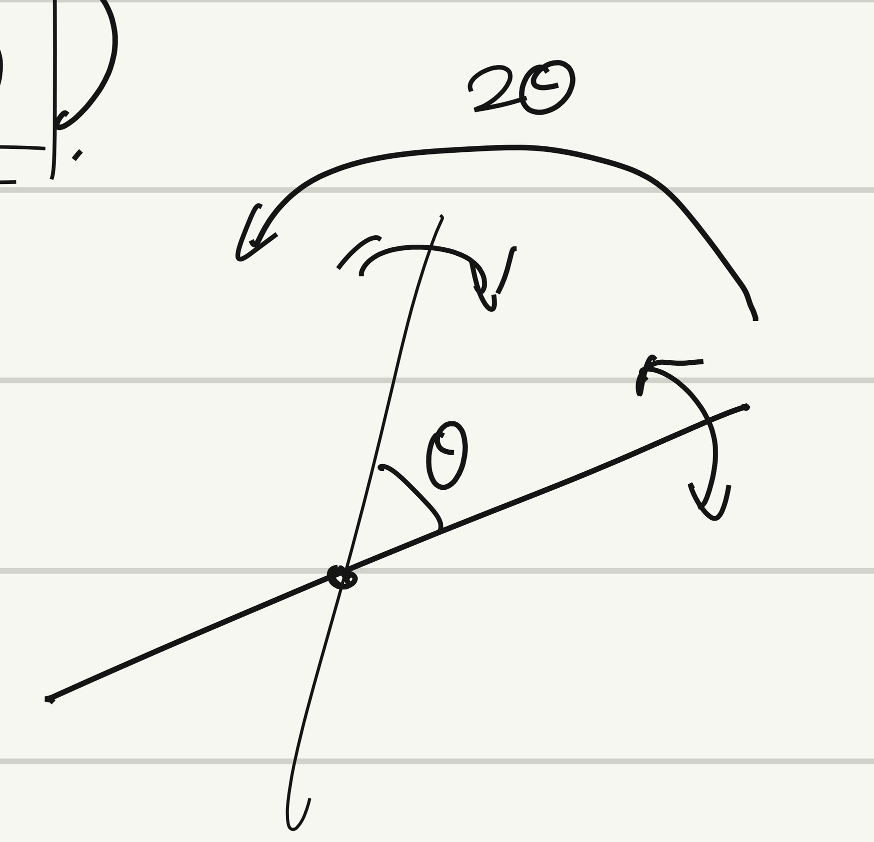
$\boxed{\begin{matrix} T_x \\ \phi_\lambda \\ \rho_\theta \end{matrix}} \longleftrightarrow SL(2,\mathbb{R})$

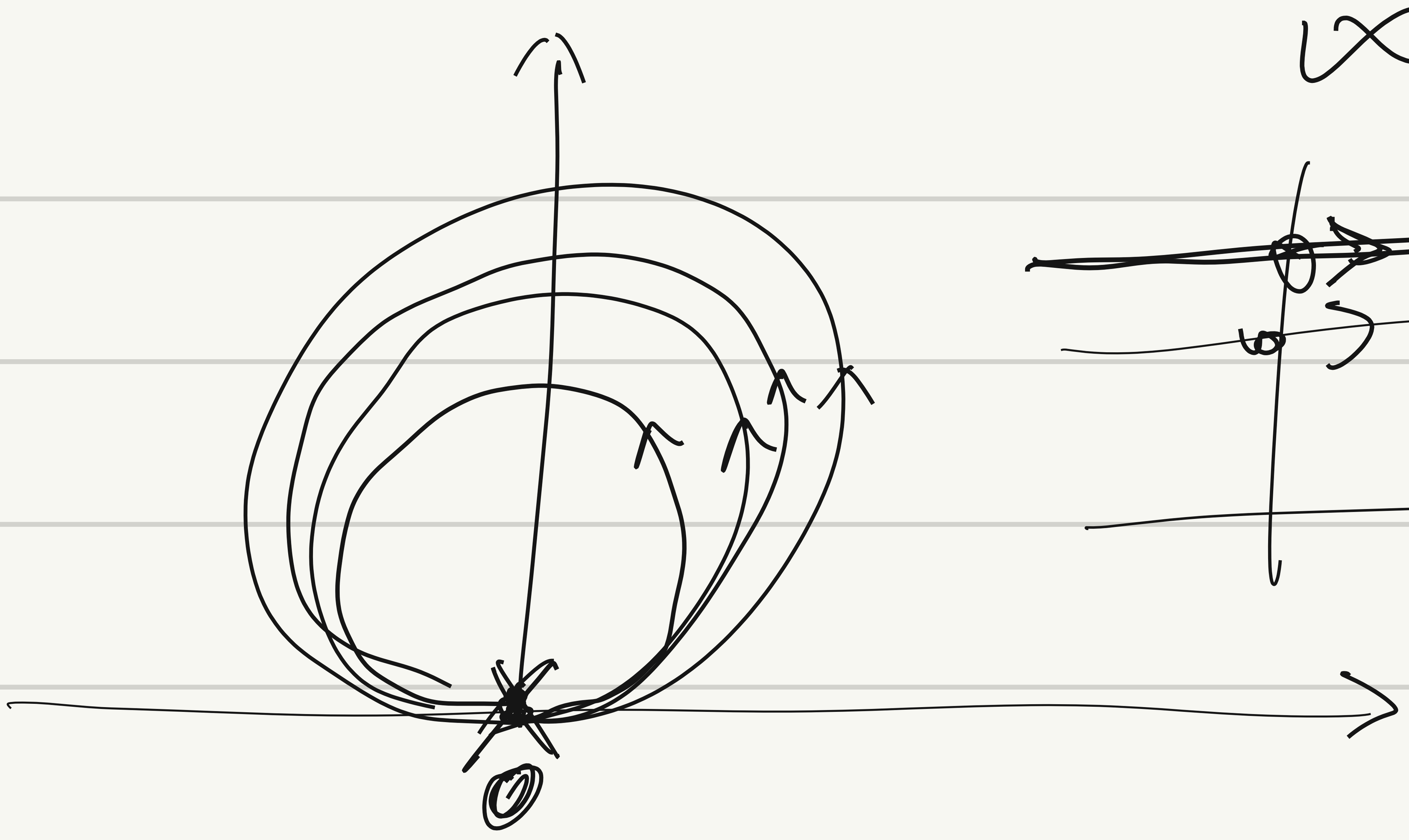
$\phi_{-y} \circ T_{-x}(z) = i$

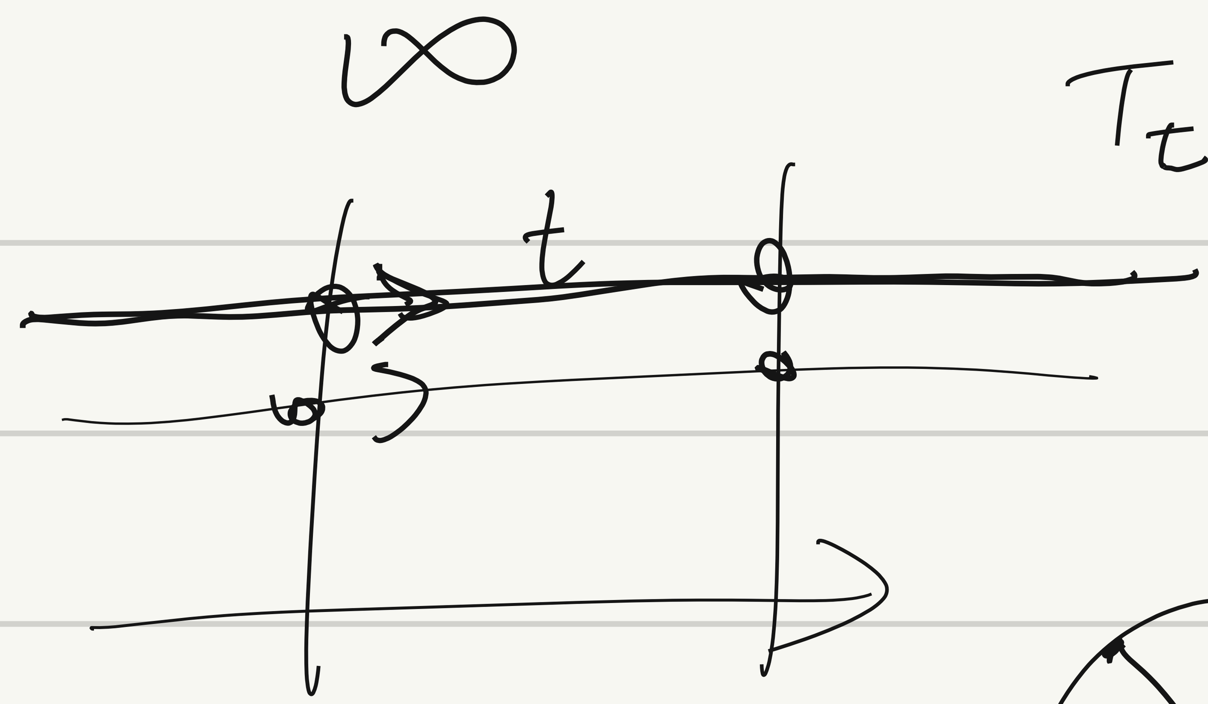




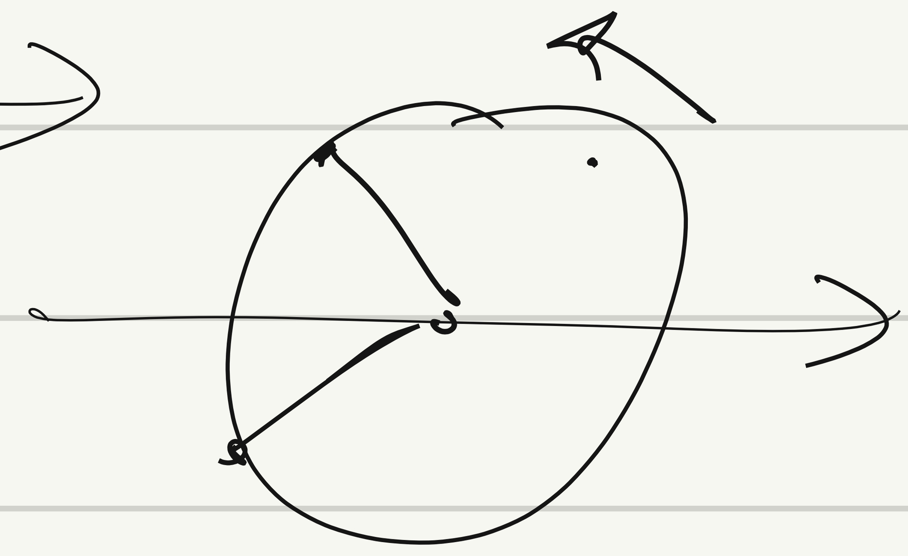


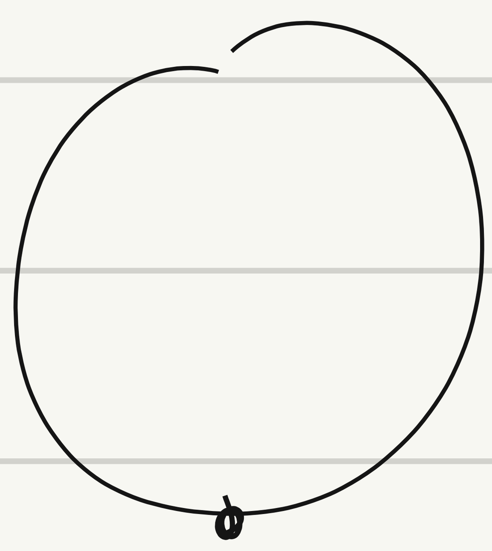


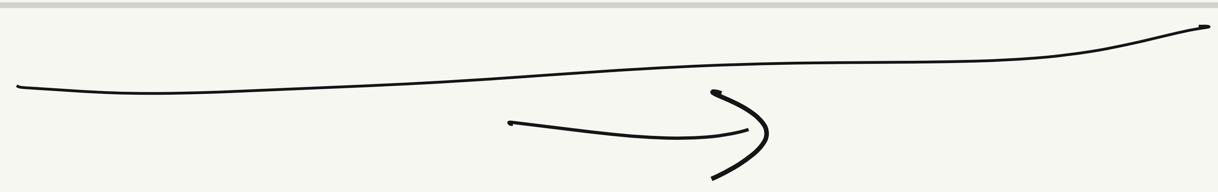


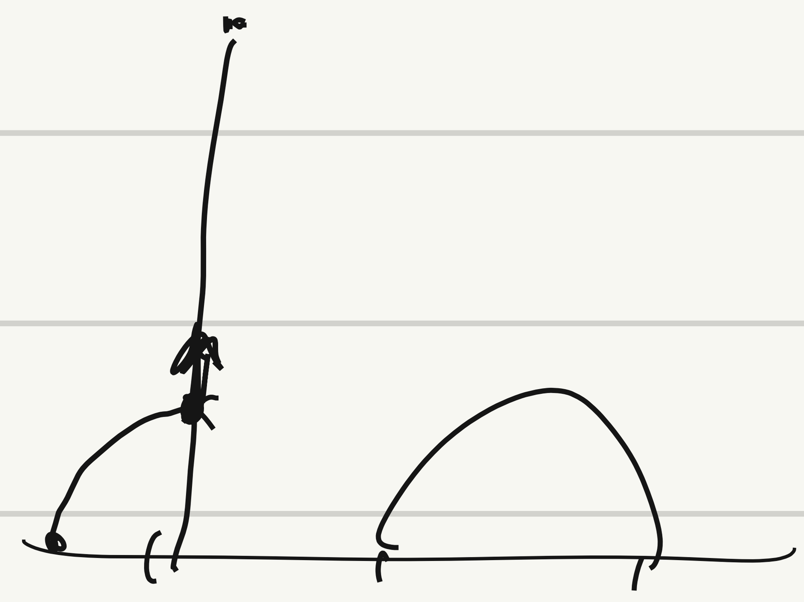


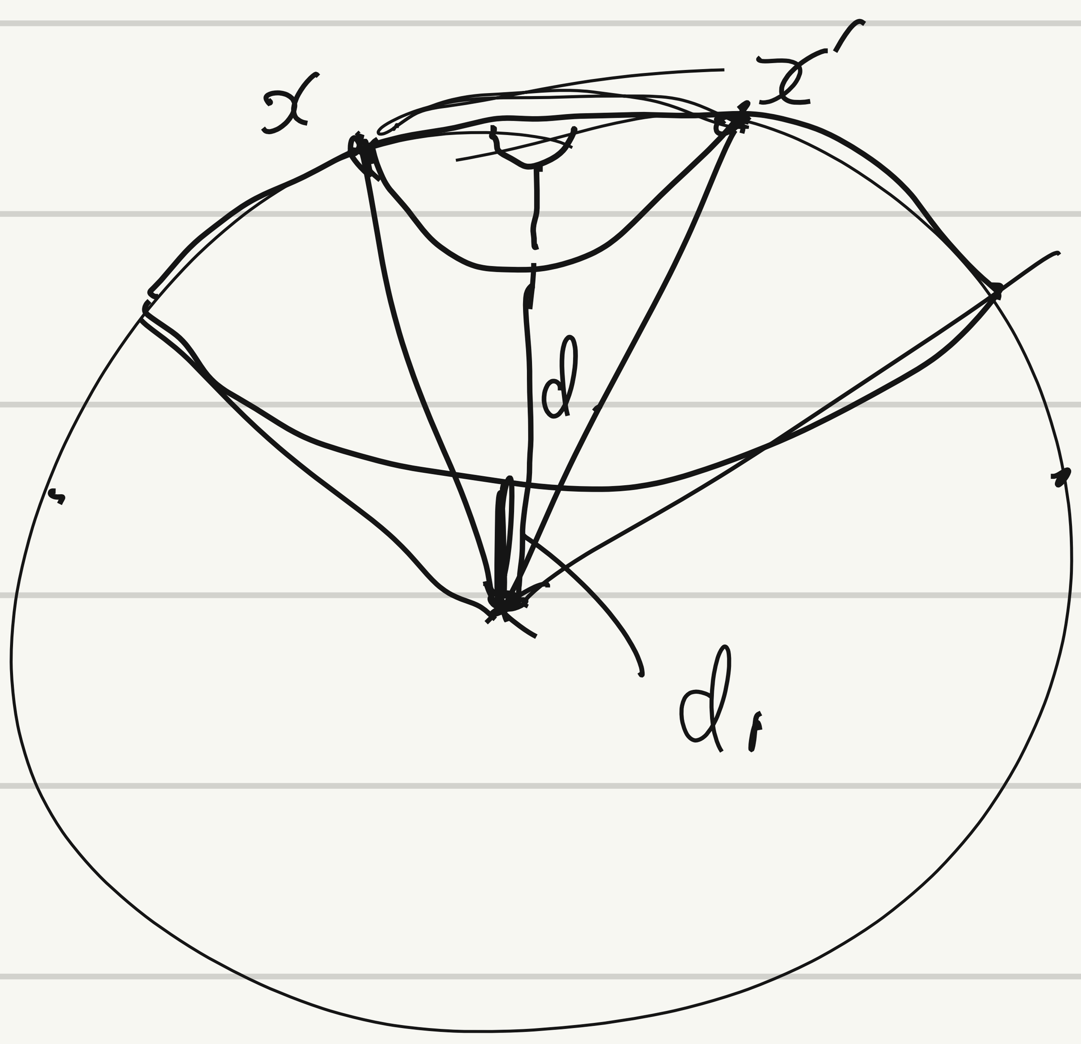
T_t.





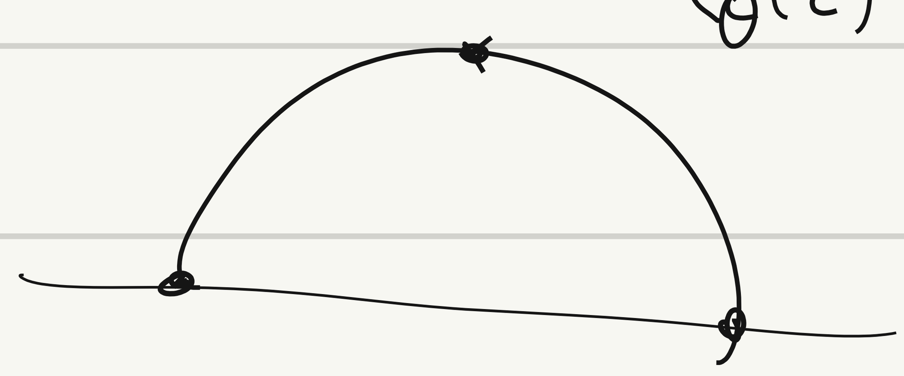






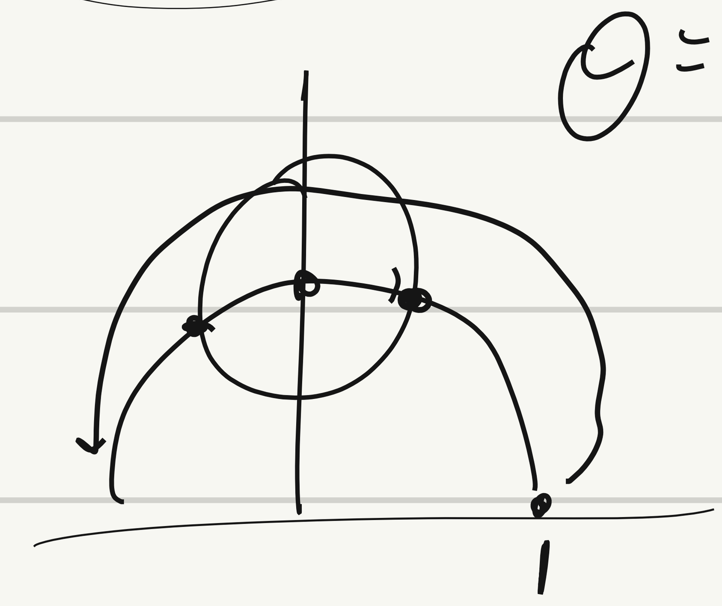
$d_{\partial\mathbb{H}}(x, x') = e^{-d}$.

$e^{-d_1} > e^{-d_2}$



$\rho_\theta(z) = \dfrac{\cos\theta\, z + \sin\theta}{-\sin\theta\, z + \cos\theta}$.

$\theta = \frac{\pi}{2}$



$\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ $\circlearrowright \mathbb{E}$ rotation angle θ.

$\circlearrowright \mathbb{H}$ rotation angle 2θ.

Möbius. Transformation

$\frac{1}{-z}$.