Introduction to hyperbolic surfaces

Exercises I

We consider points and paths in the upper half plane $\mathbb{H}$. We use $l_{\mathbb{H}}$ and $l_{\mathbb{E}}$ as notations for the hyperbolic length and the Euclidean length respectively.

1. (Easy) Let I denote the horizontal segment connecting i and $2+i$. Let $y>0$, and γ_y denote the path which is the union of the following three Euclidean segments:
   - the vertical segment connecting i and iy,
   - the horizontal segment connecting iy and $2+iy$,
   - the vertical segment connecting $2+i$ and $2+iy$.

   a) Find a parametrization of I and a parametrization of γ_y.

   b) Compute $l_{\mathbb{H}}(I)$ and $l_{\mathbb{H}}(\gamma_y)$.

   c) Find $y_0>0$, such that γ_{y_0} is the shortest among all γ_y's for $y>0$.

2. (Normal) Let I denote the horizontal segment connecting i and $2+i$ as above. Let $y>0$, and η_y denote the path which is the union of the following two segments:
   - the Euclidean segment connecting i and $1+iy$,
   - the Euclidean segment connecting $1+iy$ and $2+i$.

   a) Find a parametrization of η_y.

   b) Compute $l_{\mathbb{H}}(\eta_y)$.

   c) Compare $l_{\mathbb{H}}(\eta_y)$ for $y=2$ and $l_{\mathbb{H}}(I)$.

3. (Hard) Let N be a positive integer. Let I_N denote the horizontal segment connecting $-N+i$ and $N+i$.

   a) Compute $l_{\mathbb{H}}(I_N)$.

   b) Describe the geodesic γ_N connecting $-N+i$ and $N+i$, and compute $l_{\mathbb{H}}(\gamma_N)$.

   c) Find a function $f:\mathbb{N}_+\to\mathbb{R}$, such that

$$\lim_{N\to+\infty}\frac{f(N)}{l_{\mathbb{H}}(\gamma_N)}=1$$

4. (Normal) Let w and z be two points in $\mathbb{H}$. Let $\gamma:[a,b]\to\mathbb{H}$ be a regular path connecting w and z.

   a) Show that for any $y>0$, if for all $t\in[a,b]$, we have $\operatorname{Im}\gamma(t)\le y$ (i.e γ is entirely below the horizontal line H_y), then we have

$$l_{\mathbb{H}}(\gamma)\ge\frac{l_{\mathbb{E}}(\gamma)}{y}$$

b) Let $v = \operatorname{Im} w$. Show that for any $y > v$, if there exists a $t \in [a, b]$, such that $\operatorname{Im} \gamma(t) > y$ (i.e. γ crosses H_y), we have

$$l_{\mathbb{H}}(\gamma) \geq \left| \log \frac{y}{v} \right| .$$

c) Use a) and b) to show that $\mathrm{d}_{\mathbb{H}}(w, z) = 0$ if and only if $w = z$.