Introduction to hyperbolic surfaces

Exercises VII

For $n > 0$ integer, we call a polygon an n-gon, if it has n vertices. We denote it by P_n.

For any n and g non-negative integers, we denote by $S_{g,n}$ an oriented topological surface of genus g with n boundary components. If there is no boundary components, we will simply denote the surface by S_g.

We denote by Möb($\mathbb{H}$) the group of Möbius transformations on $\mathbb{H}$.

1. For a surface S, we denote by $\chi(S)$ its Euler characteristic.

   a) (Easy) Compute $\chi(P_n)$.

   b) (Easy) Consider the construction of a flat torus S_1 by gluing opposite sides of P_4. Compute $\chi(S_1)$.

   (Hint: Check how many vertices and edges left after the identification of the sides of P_4. Then use the formula $V - E + F$ to compute $\chi(S_1)$.)

   c) (Easy) Consider the construction of a genus g surface S_g using a $4g$-gon. Compute $\chi(S_g)$.

   d) (Normal) Let P_{4g} be the polygon used to construct S_g. By cutting out a triangle in the interior, we create a surface which is topological $S_{0,2}$ a sphere with two holes. Using the same gluing pattern as in c), we get the surface $S_{g,1}$. Compute $\chi(S_{0,2})$ and $\chi(S_{g,1})$.

   e) (Normal) By cutting out n disjoint triangles from P_{4g} and keep the same gluing pattern as in c), we construct surfaces $S_{0,n+1}$ and $S_{g,n}$ before and after the gluing. Compute $\chi(S_{0,n+1})$ and $\chi(S_{g,n})$.

   f) (Hard) We consider gluing surfaces along boundary to get new surfaces.

      i. Using e), for $n > 1$, show that $\chi(S_{g,n}) = \chi(S_{g+1,n-2})$.

      ii. Using e), for $n_1 > 0$ and $n_2 > 0$, show that $\chi(S_{g_1,n_1}) + \chi(S_{g_2,n_1}) = \chi(S_{g_1+g_2,n_1+n_2-2})$.

      iii. Check both equalities still hold when either g, g_1 or g_2 is 0.

   g) (Easy) Compute the number of pair of pants in a pants decomposition of S_g.

   h) (Easy) Compute the number of curves used in a pair of pants decomposition of S_g.

   i) (Easy) Based on the answers of the question g) and h), guess the answers of the same questions for $S_{g,n}$ with $n > 0$. Check if it is correct.

2. a) (Easy) For any fours pairwise distinct points x_1, x_2, x_3 and x_4 in $\partial\mathbb{H}$, we define the cross ratio to be

   $$\mathbb{B}(x_1, x_2; x_3, x_4) = \frac{(x_1 - x_4)(x_2 - x_3)}{(x_1 - x_3)(x_2 - x_4)}.$$

   Show that $\mathbb{B}$ is invariant under Möbius transformations, i.e. for any $f \in$ Möb($\mathbb{H}$) we have:

   $$\mathbb{B}(x_1, x_2; x_3, x_4) = \mathbb{B}(f(x_1), f(x_2); f(x_3), f(x_4)).$$

   (Hint: Check T_t, ϕ_λ and ρ_θ for x_1, x_2, x_4 and x_4 finite. Then show that it can be extended to the case where one of them is ∞.)

b) (Easy) Let η denote the geodesic ending at x_1 and x_2, and η' denote the geodesic ending at x_3 and x_4. Show that

   i. η intersects η' if and only if $\mathbb{B}(x_1, x_2; x_3, x_4) < 0$;

   ii. η and η' are disjoint if and only if $\mathbb{B}(x_1, x_2; x_3, x_4) > 0$.

   (Hint: Use an isometry to send η to V_0.)

c) (Hard) Let $f \in \text{Möb}(\mathbb{H})$ and γ be a geodesic ending at x and x'. Show that f is hyperbolic if

$$\mathbb{B}(x, f(x'); x', f(x)) < 0$$

   (Hint: Show that the translation distance $l(f)$ of f is not 0. Check first the cyclic order of end points of γ and $f(\gamma)$ on $\partial\mathbb{H}$. Then observe how the half planes with boundary γ moved by f. Use the distance between γ and $f(\gamma)$ to find a lower bound for $l(f)$.)

3. Consider the flat torus $S_1 = \mathbb{R}^2/\mathbb{Z}^2$. Let $l(x, y)$ be the line passing $(0, 0)$ and (x, y). The projection of $l(x, y)$ to S_1 is a simple geodesic, denoted by $\gamma(x, y)$. Moreover, $\gamma(x, y)$ is closed if and only if $(x, y) \in \mathbb{Z}^2$. Let (p, q) and (r, s) be two distinct points in $\mathbb{Z}^2$. Compute the intersection number between $\gamma(p, q)$ and $\gamma(r, s)$.