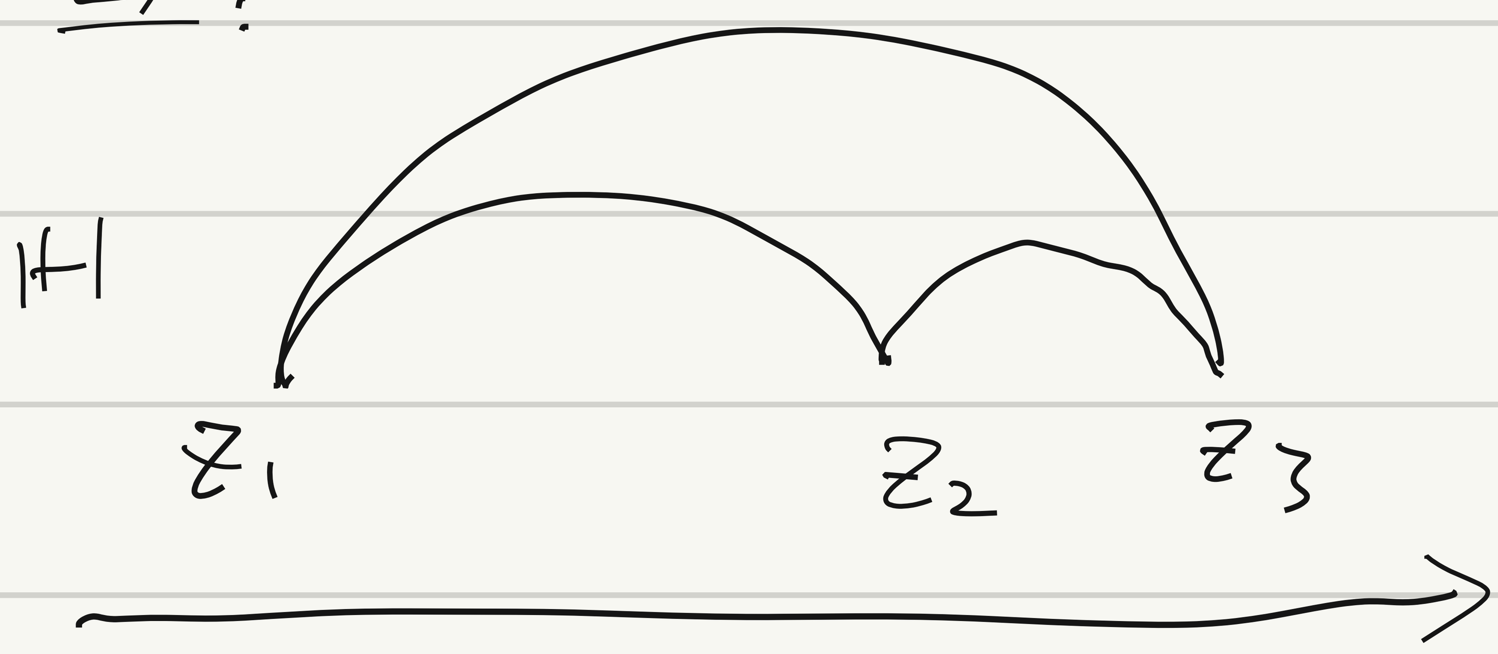
IV Polygone:

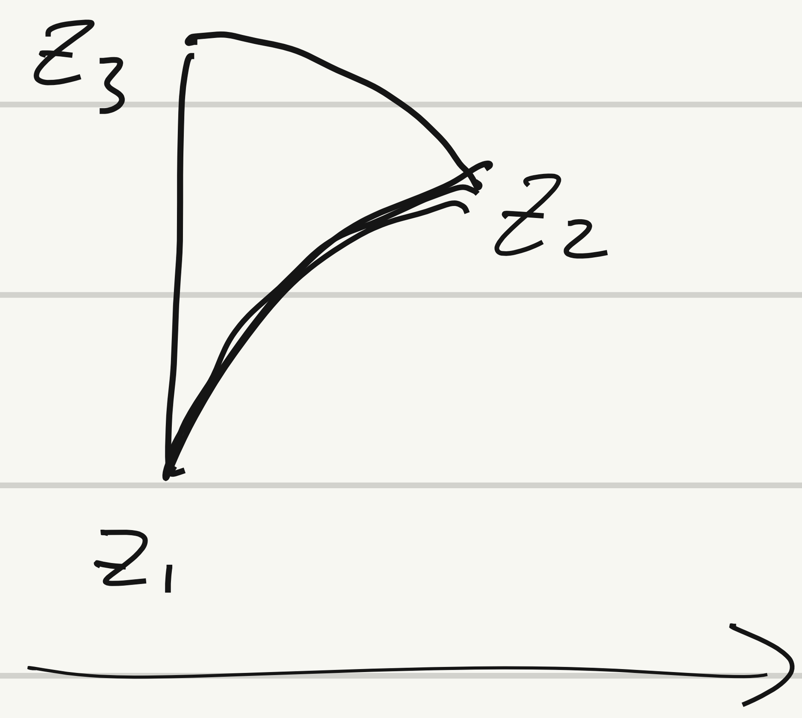
1. Triangle:

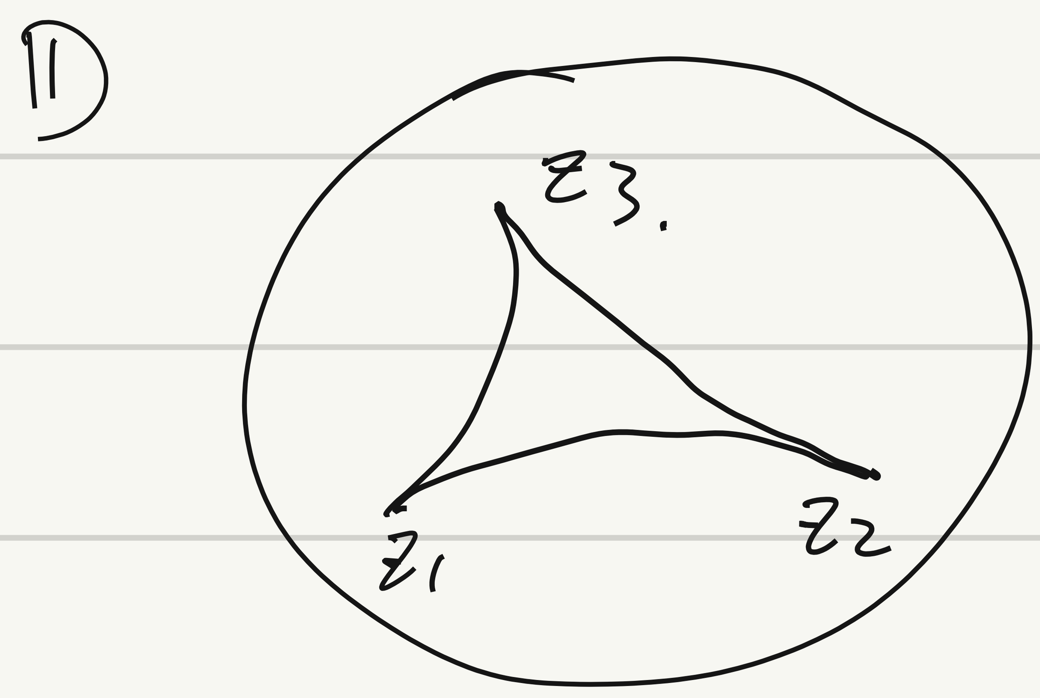
- Def: $\forall z_1, z_2, z_3 \in \overline{\mathbb{H}}$, $\Delta(z_1 z_2 z_3) := [z_1, z_2] \cup [z_2, z_3] \cup [z_1, z_3]$ ← geodesic segment.

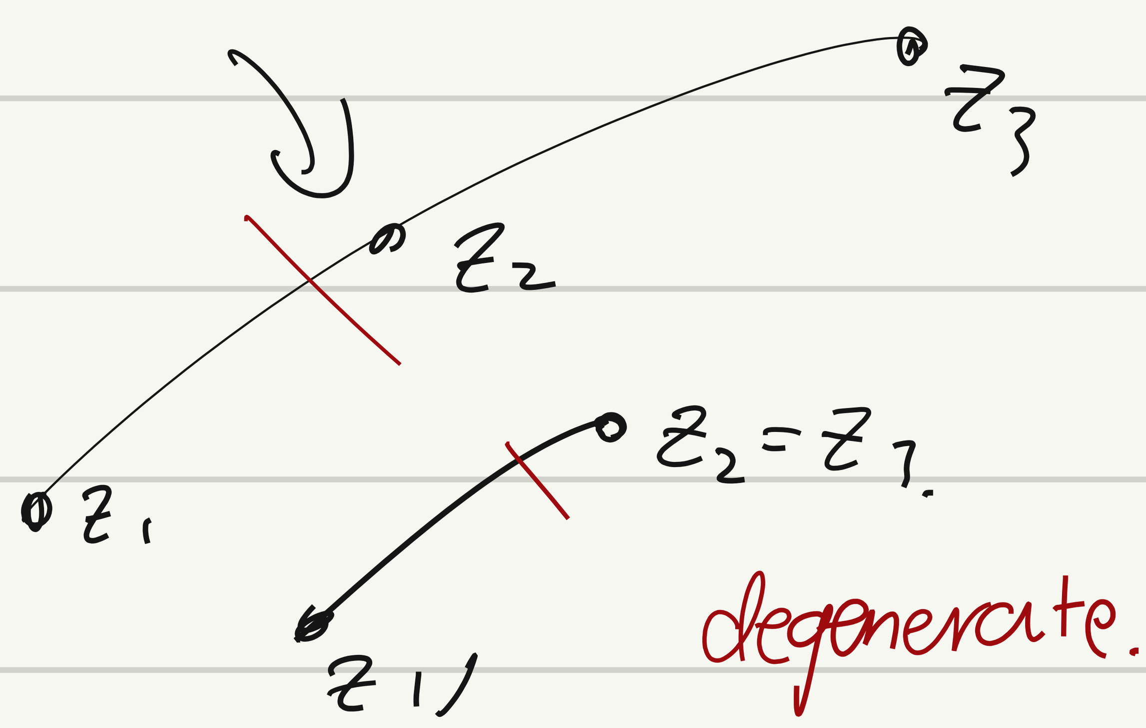
  $\Delta(z_1 z_2 z_3)$ degenerate if z_j's colinear.

- Ex:

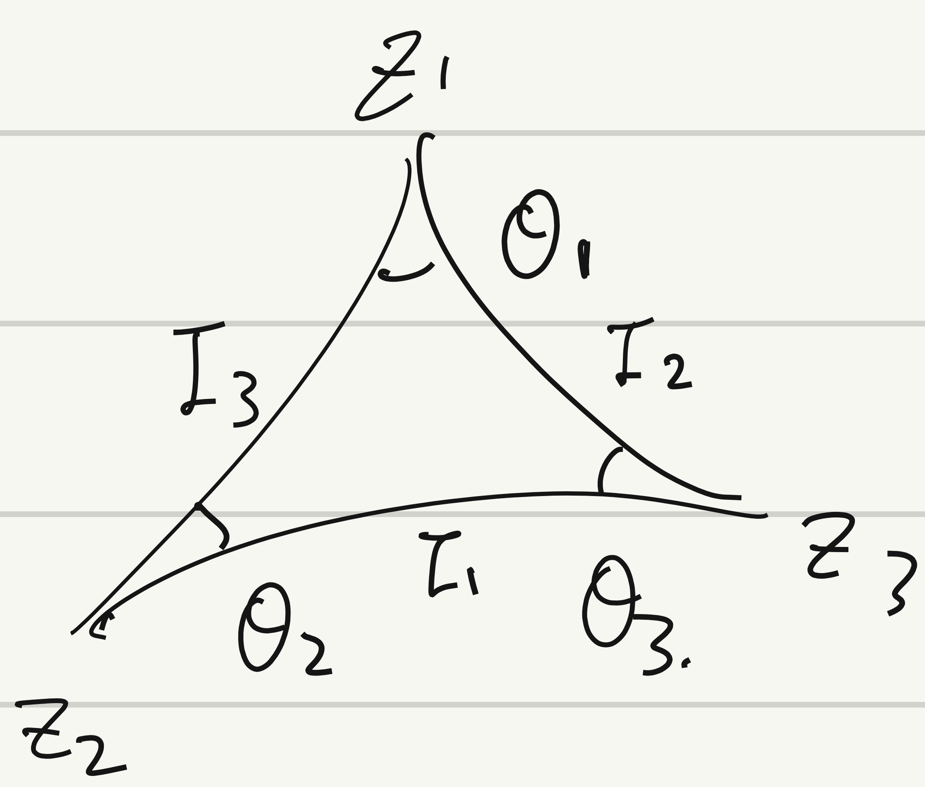








Δ in $\mathbb{D}$



Vertices $z_1 \; z_2 \; z_3$

sides $I_1 \; I_2 \; I_3 \; (l_1 \; l_2 \; l_3)$

internal angles $\theta_1 \; \theta_2 \; \theta_3 \in [0, \pi)$

- Vertex on $\partial\mathbb{H}$

Def: If $\exists z_j \in \partial\mathbb{H}$, Δ is a triangle with ideal vertices

Def: If $z_1, z_2, z_3 \in \partial\mathbb{H}$, Δ is an ideal triangle.

$$l_1 = l_2 = l_3 = \infty$$

$$\theta_1 = \theta_2 = \theta_3 = 0$$

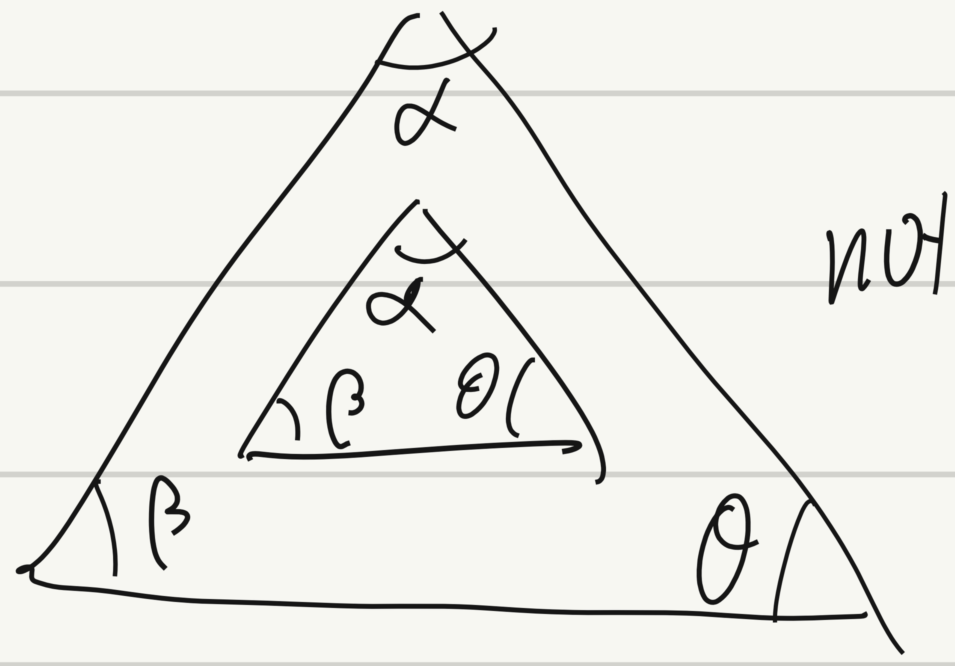
2. Determine Δ by θ_j's.

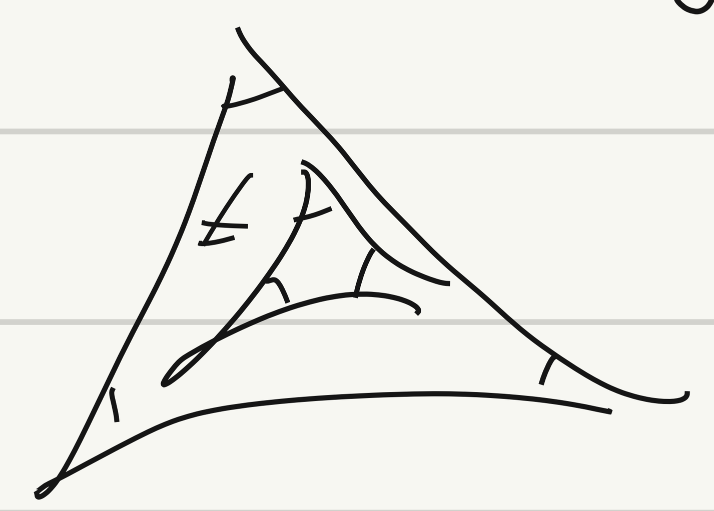
Prop: Two triangle Δ and Δ' are isometric iff $\theta_j = \theta_j'$ for $j = 1, 2, 3$.

i.e. $\exists f \in \mathrm{Isom}(\mathbb{H})$ s.t. $f(\Delta) = \Delta'$

Rmk:

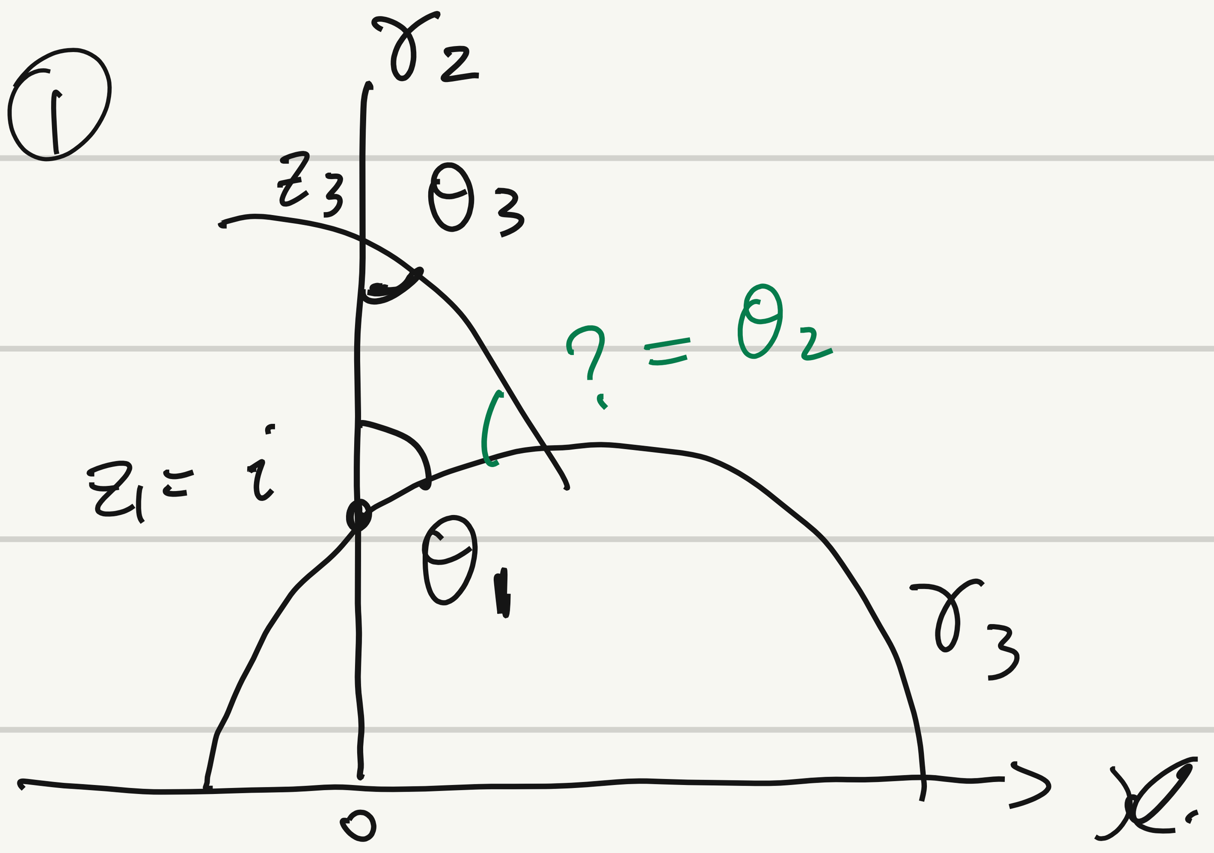
	Euclidean	Hyperbolic
$\mathbb{E}$	not isometric.	no rescaling (hyperbolic)

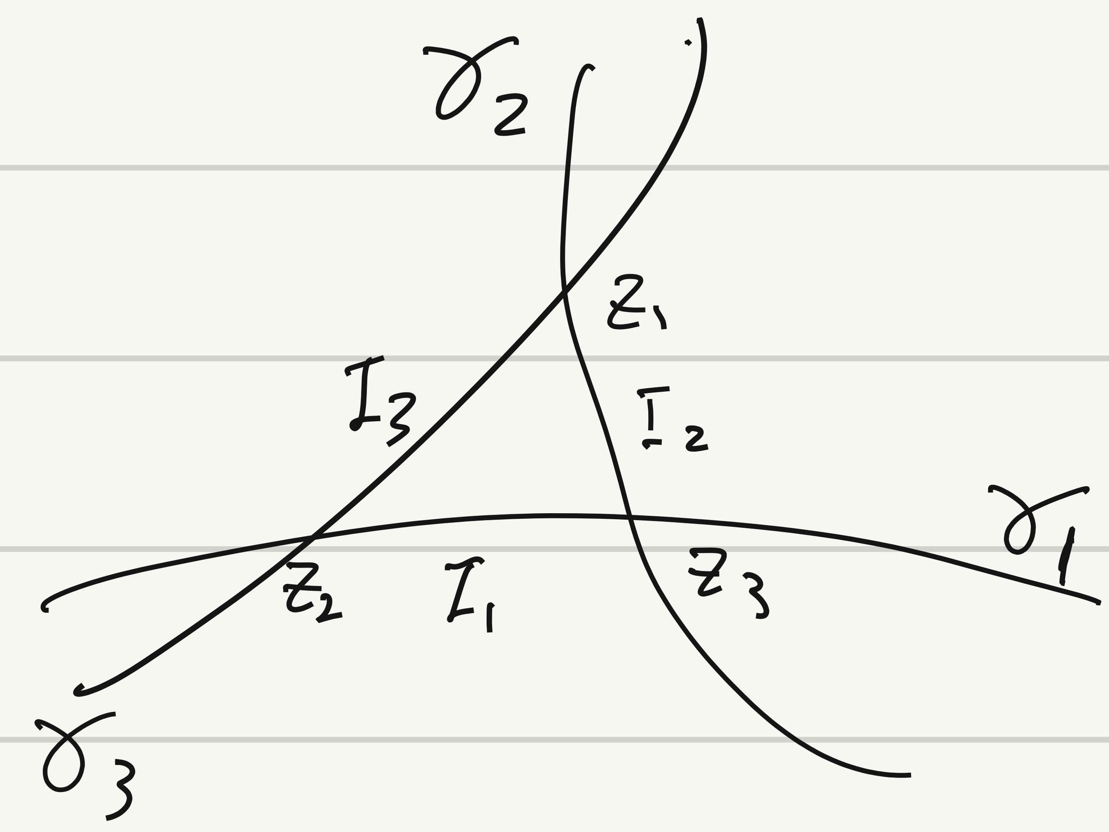




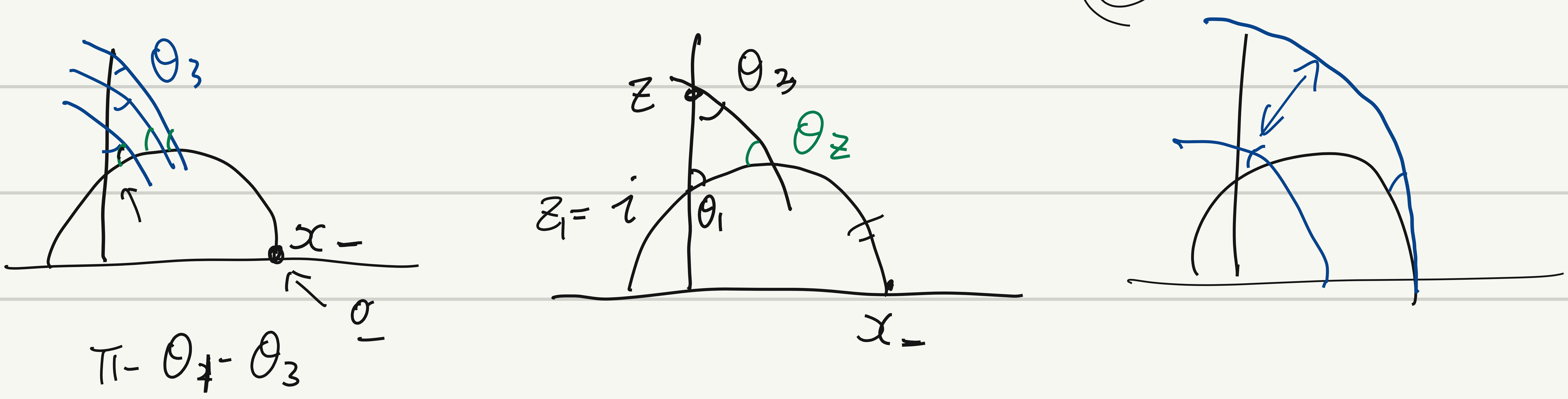
No ideal vertex.

①

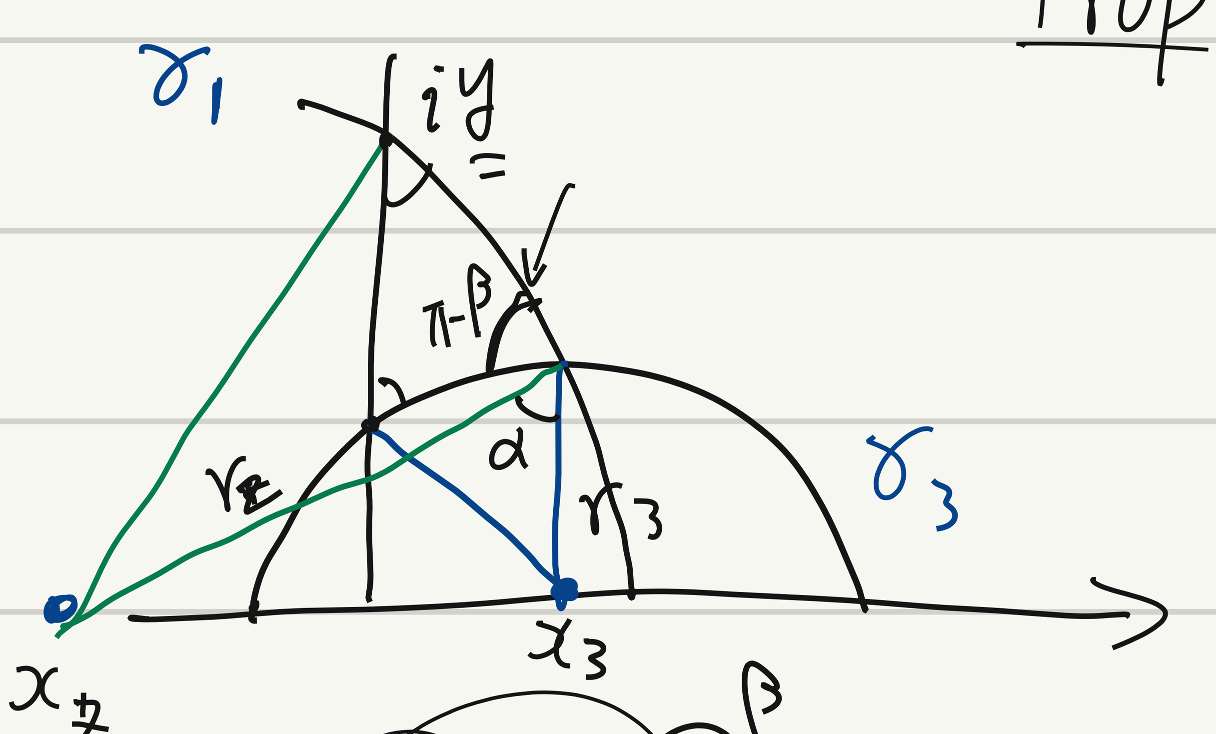




② $z_3 = iy$ move y from 1 to ∞

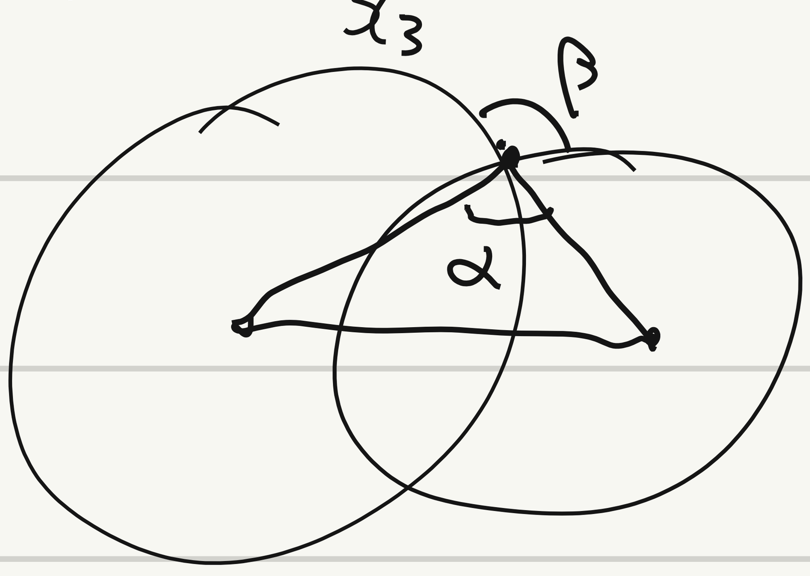


Prop: θ_z is strictly decreasing as $y \to x_-$



$$\cos\theta_z = -\cos\alpha$$

$$= -\frac{r_z^2 + r_3^2 - |x_z - x_3|^2}{2 r_z r_3} \qquad z = iy.$$



$\alpha + \beta = \pi$

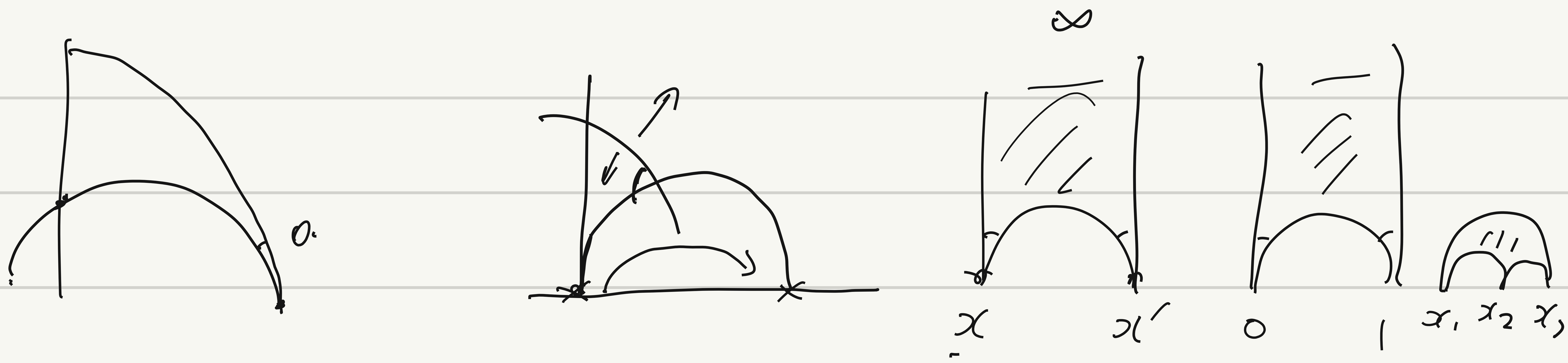
$$= \underbrace{\underbrace{\frac{\sin\theta_3 \sin\theta_1}{2}}_{=cst}\left(y + \frac{1}{y}\right) - \underbrace{\cos\theta_3 \cos\theta_1}_{=cst}}$$

$\Rightarrow \forall\ 0 < \theta < \pi - \theta_1 - \theta_3$, $\exists\ z = iy$ s.t. $\theta_z = \theta$. ∎

Cor: ① $\forall$ non-deg Δ, $\theta_1 + \theta_2 + \theta_3 < \pi$

② $\theta_1\ \theta_2\ \theta_3$ determine Δ up to isometry.

③ $\forall\ \theta_1, \theta_2, \theta_3 \geq 0$ s.t. $\theta_1 + \theta_2 + \theta_3 < \pi$, $\exists !\ \Delta$ (up to isometry)

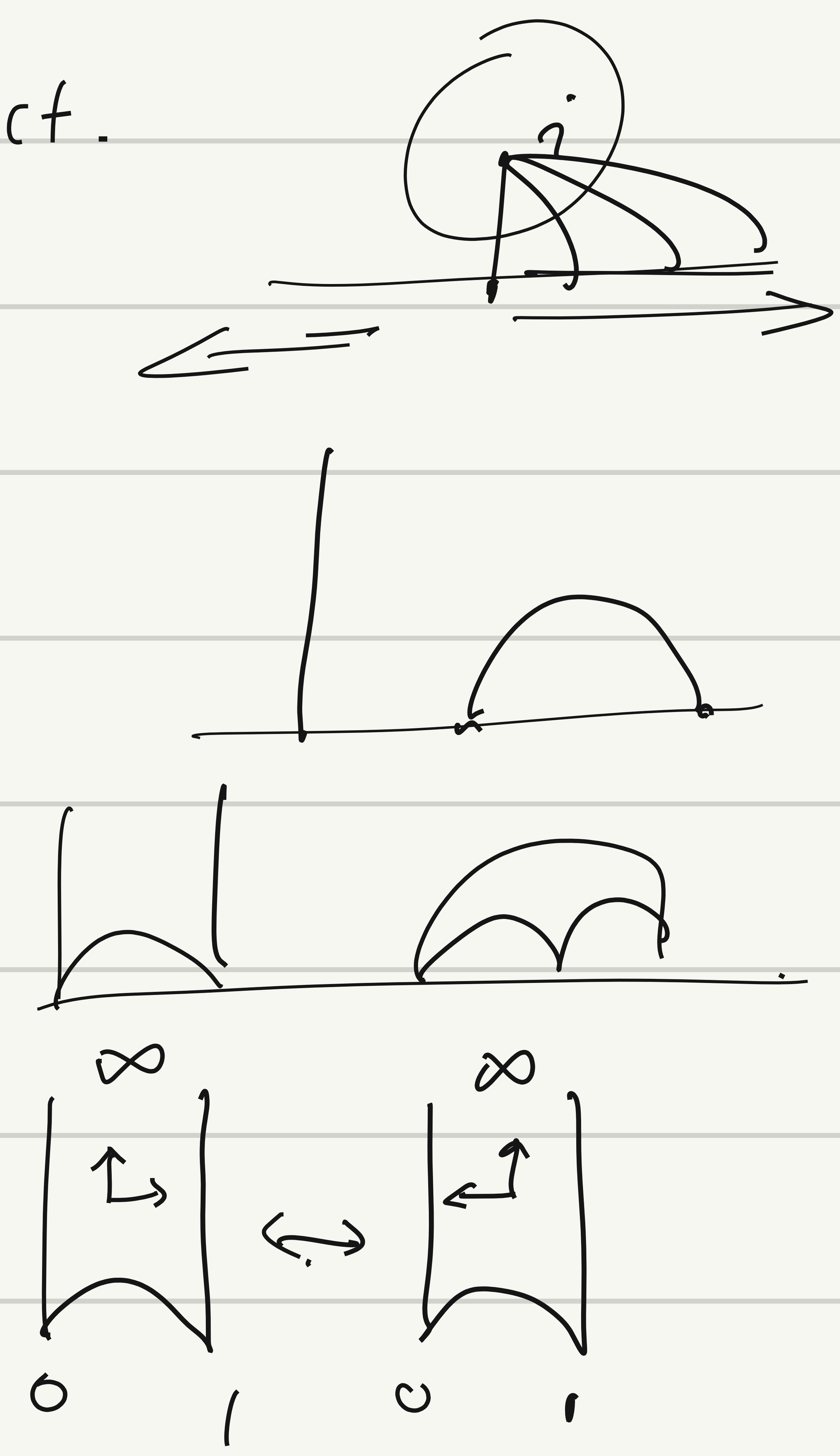


Prop. $\forall\ (x_1\ x_2\ x_3)\quad (x_1'\ x_2'\ x_3') \in (\partial\mathbb{H})^{(3)}$ ← distinct.

$\exists f \in \mathrm{Isom}(\mathbb{H})\qquad f(x_1) = x_1'$

$f(x_2) = x_2'$

$f(x_3) = x_3'$

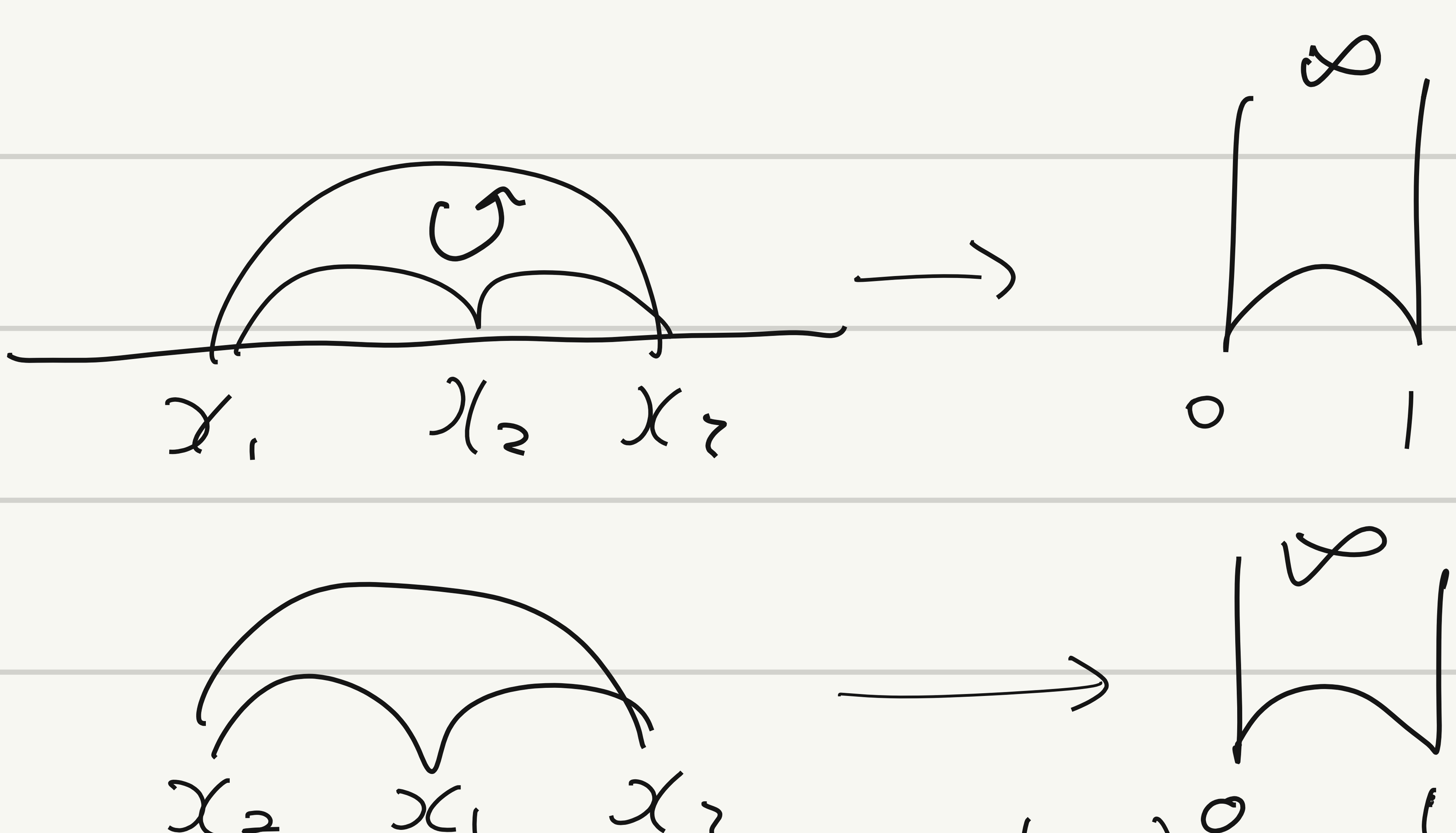


Rmk: $\mathrm{Isom}^+(\mathbb{H})$

$(x_1, x_2, x_3) \nearrow (0, 1, \infty)$

or

$\searrow (1, 0, \infty)$



$(x_1\ x_2\ x_3)$

$\downarrow$

$(0\ \ 1\ \ \infty)$

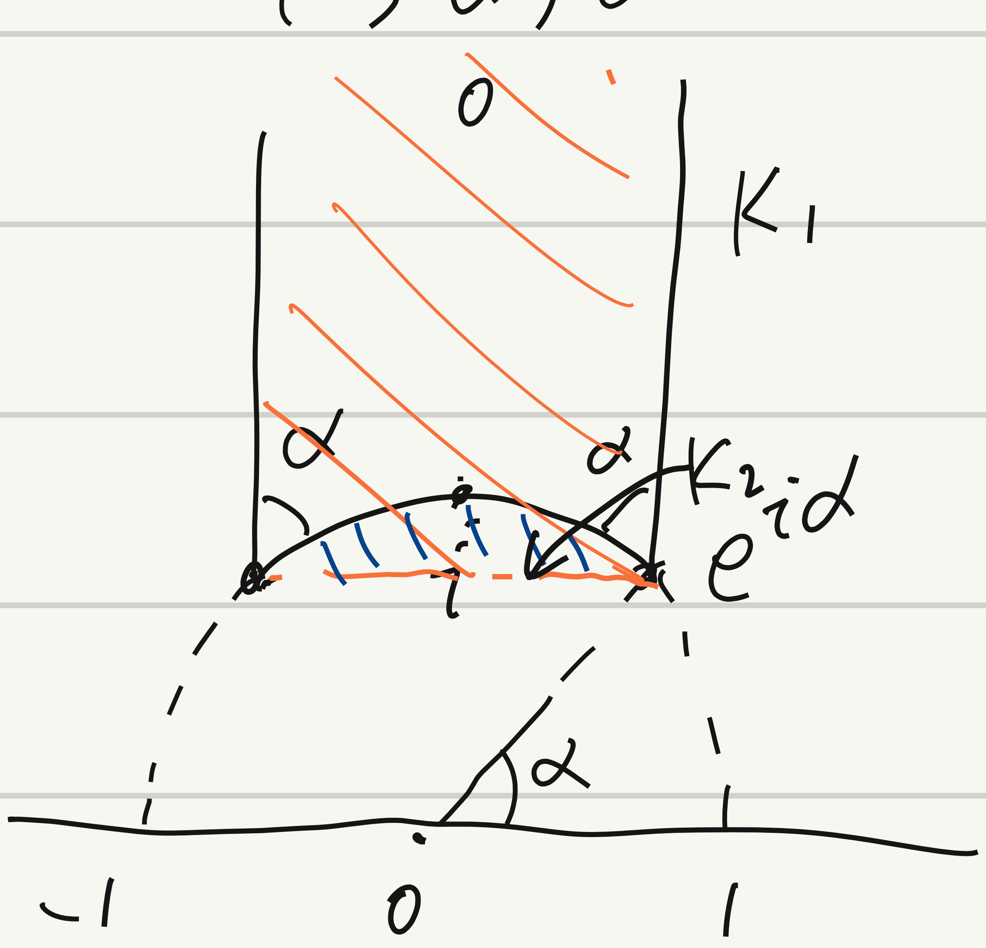
$(x_1\ x_2\ x_3)$

$(1\ \ 0\ \ \infty)$

$\{x_1\ x_2\ x_3\} \xrightarrow{\exists f \in \mathrm{Isom}^+(\mathbb{H})} \{0, 1, \infty\}$

3. Area of a triangle.

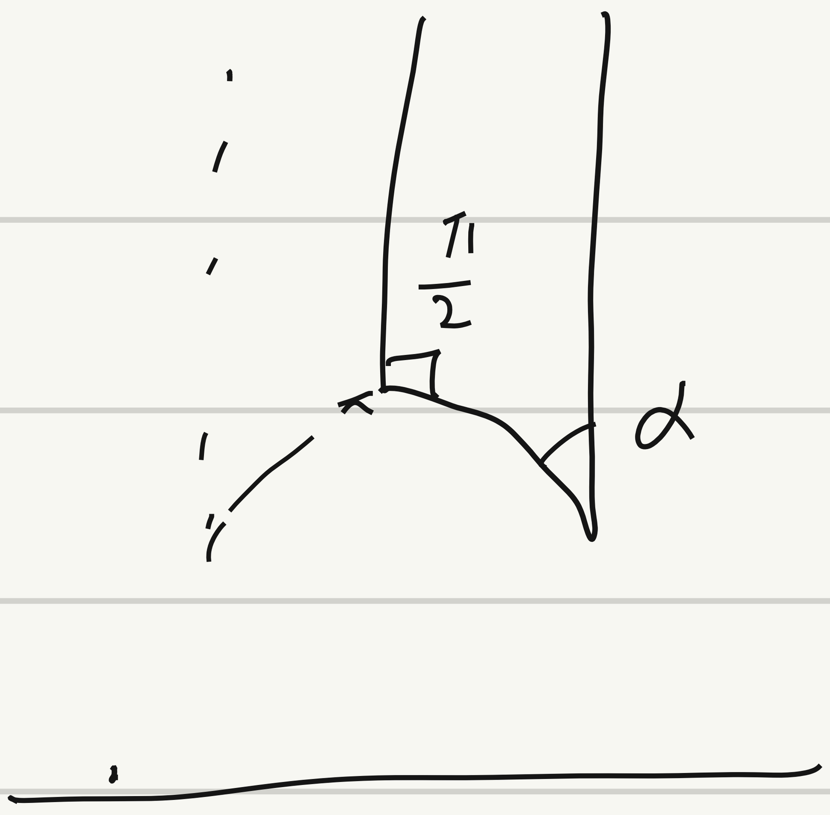
$(0, \alpha, \alpha)$



$$A_{\mathbb{H}}(\Delta) = A_{\mathbb{H}}(K_1) - A_{\mathbb{H}}(K_2)$$

$$= \int_{\sin\alpha}^{\infty}\int_{-\cos\alpha}^{\cos\alpha} \frac{dx\,dy}{y^2} - \int_{\sin\alpha}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \frac{dx\,dy}{y^2}$$

$$= 2\cot\alpha - \left(2\cot\alpha - (\pi - 2\alpha)\right) = \underline{\underline{\pi - 2\alpha}}$$

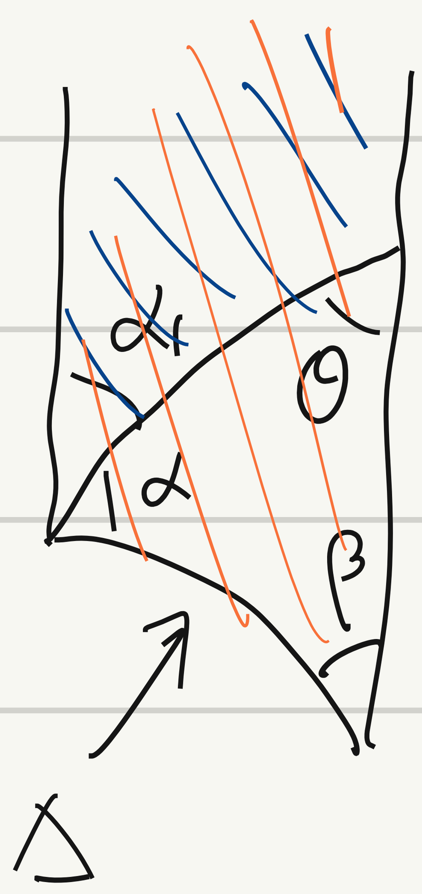


$\Delta(0, \frac{\pi}{2}, \alpha)$

$$A_{H1}(\Delta) = \frac{1}{2}(\pi - 2\alpha)$$

$\Delta(0, \alpha, \beta)$ $\quad A_{H1}(\Delta) = \frac{\pi - 2\alpha}{2} + \frac{\pi - 2\beta}{2}$

$$= \pi - \alpha - \beta.$$



$\Delta(\alpha, \beta, \theta)$ $\quad A_{H1}(\Delta) = (\pi - (\alpha_1 + \alpha) - \beta)$

$$- (\pi - \alpha_1 - (\pi - \theta))$$

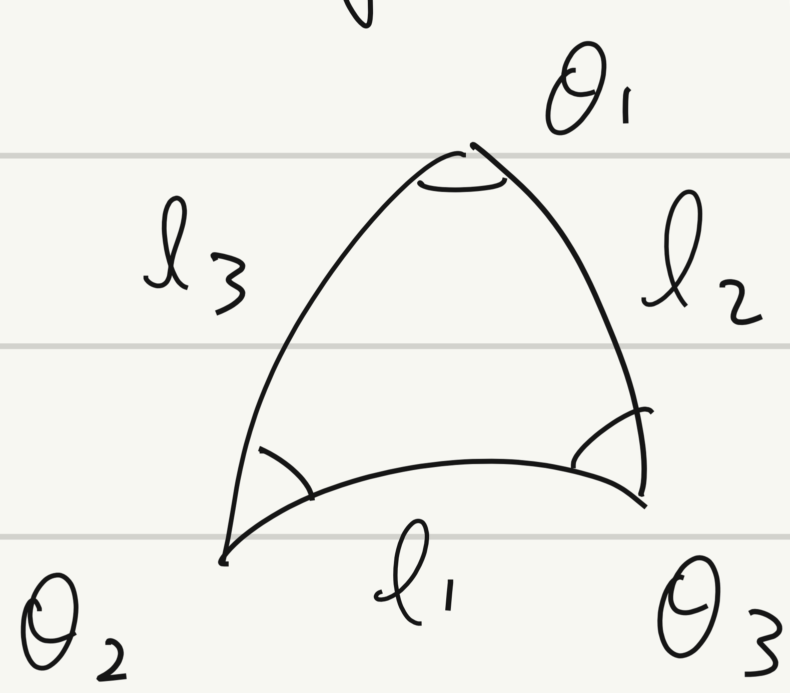
$$= \pi - \underbrace{(\alpha + \beta + \theta)}$$

$\Rightarrow \max A_{H1}(\Delta) = \pi.$

Elementary geometry in hyperbolic space — Fenchel.

Rigidity of polyhedral surfaces II — Ren Guo & Feng Luo

4. Trigonometry formula:

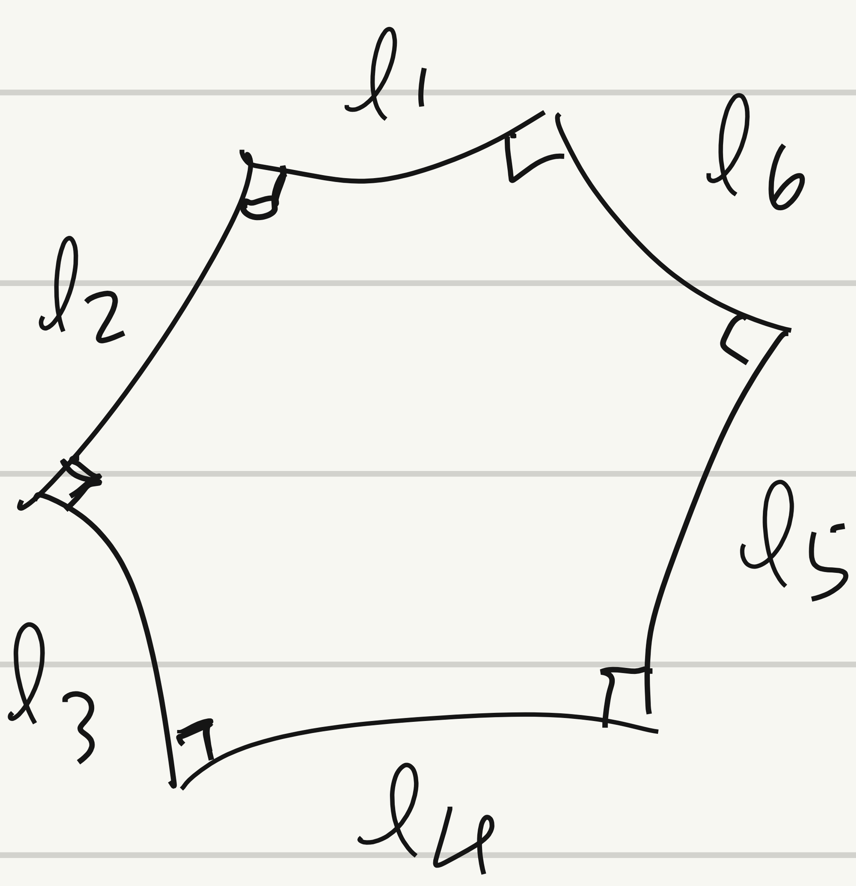


Sine rule: $\frac{\sinh l_1}{\sin \theta_1} = \frac{\sinh l_2}{\sin \theta_2} = \frac{\sinh l_3}{\sin \theta_3}$

Cosine rule I: $\cosh l_3 = \cosh l_1 \cosh l_2 - \sinh l_1 \sinh l_2 \cos \theta_3$

II: $\cos \theta_3 = \sin \theta_1 \sin \theta_2 \cosh l_3 - \cos \theta_1 \cos \theta_2$

5 Polygone:



sine rule $\quad \frac{\sinh l_1}{\sinh l_4} = \frac{\sinh l_2}{\sinh l_5} = \frac{\sinh l_3}{\sinh l_6}$

cosine rule $\quad \cosh l_3 = -\cosh l_1 \cosh l_2 + \sinh l_1 \sinh l_2 \cosh l_4$

$$n - (\theta_1 + \cdots + \theta_n)$$

$$A_D(P) = \underline{(n-2)\pi - (\theta_1 + \cdots + \theta_n)}$$

$$\underbrace{\int_P K \, dA}_{-A} + \underbrace{\int_{\partial P} k_\alpha \, dl} = \underline{\chi(P)} \cdot 2\pi = 2\pi$$