2.3. Operators on a Riemannian manifold

Definition (Gradient) Given a function $f: M^n \to \mathbb{R}$, then ∇f is the vector field which is uniquely determined by

$$\langle \nabla f, X \rangle = X(f), \quad X \in \mathcal{X}(M^n).$$

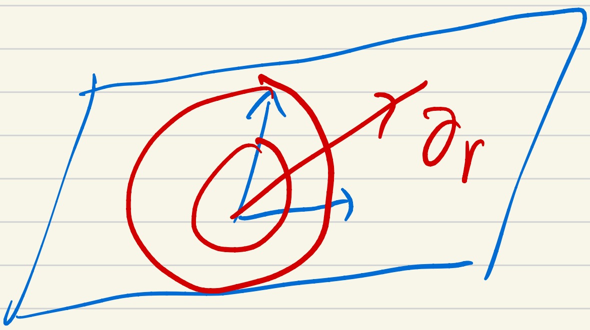
In coordinates, we can write

$$\boxed{\nabla f = g^{ij} \cdot \frac{\partial f}{\partial x_i} \frac{\partial}{\partial x_j}}, \qquad |\nabla f|^2 = g^{ij} \frac{\partial f}{\partial x_i} \frac{\partial f}{\partial x_j}.$$

Lemma. $\forall p \in M^n$, $r(x) \equiv d_g(x, p)$.

Then $|\nabla r(x)| = 1$ if r is smooth at x.

Pf.



$$g_{rr} = \langle \partial_r, \partial_r \rangle = 1$$

$$\begin{bmatrix} 1 & 0 \cdots 0 \\ 0 & \\ \vdots & \\ 0 & \end{bmatrix}$$

$$\nabla r = g^{rr} \frac{\partial r}{\partial r} \frac{\partial}{\partial r}$$

$$= 1 \cdot 1 \cdot \frac{\partial}{\partial r}$$

$$|\nabla r| = 1$$

Definition (Hessian and Laplacian)

- $\nabla^2 f(X,Y) = \langle \nabla_X \nabla f, Y\rangle$, $X,Y \in \mathfrak{X}(M)$ 1

  Hess(f)

- $\Delta f = \mathrm{Tr}(\nabla^2 f)$.

∇ : Levi-Civita connection.

$$\nabla^2 f(X,Y) = X\langle \nabla f, Y\rangle - \langle \nabla f, \nabla_X Y\rangle$$

3 2

$$= XY(f) - (\nabla_X Y)(f)$$

$\nabla^2 f$ is symmetric:

$$\nabla^2 f(Y,X) = YX(f) - (\nabla_Y X)(f)$$

$$\nabla^2 f(X,Y) - \nabla^2 f(Y,X) = [X,Y]f - (\nabla_X Y - \nabla_Y X)(f)$$

torsion-free $= 0$

$$\Delta f = \sum_{i=1}^{n} E_i E_i f - \nabla_{\nabla_{E_i} E_i} f.$$

In Geodesic Normal Coordinates at p

$$\Delta f = \sum_{i=1}^{n} \frac{\partial^2 f}{\partial x_i^2} \quad \text{at } p$$

$$\nabla f = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \frac{\partial}{\partial x_i}.$$

Ex. $\gamma : [a,b] \to M^n$ geodesic, then

$$\nabla^2 f(\gamma'(t), \gamma'(t)) = \frac{d^2 f}{dt^2}.$$

2.4. Constructing Riemannian metrics

- <u>Example</u> (Product manifold)

$(M^n \times N^k, g \oplus h)$ Riemannian product, where

$$g \oplus h : TM^n \oplus TN^k \to \mathbb{R}$$

$$g \oplus h\left(X_1 + Y_1, X_2 + Y_2\right) = g(X_1, X_2) + h(Y_1, Y_2)$$

$$X_i + Y_i \in TM^n \oplus TN^k, \quad i = 1, 2.$$

<u>Ex</u> $$\nabla^{g\oplus h}_{Y_1+Y_2}(X_1 + X_2) = \nabla^g_{Y_1} X_1 + \nabla^h_{Y_2} X_2 .$$

$$X_i + Y_i \in TM^n \oplus TN^k, \quad i = 1, 2.$$

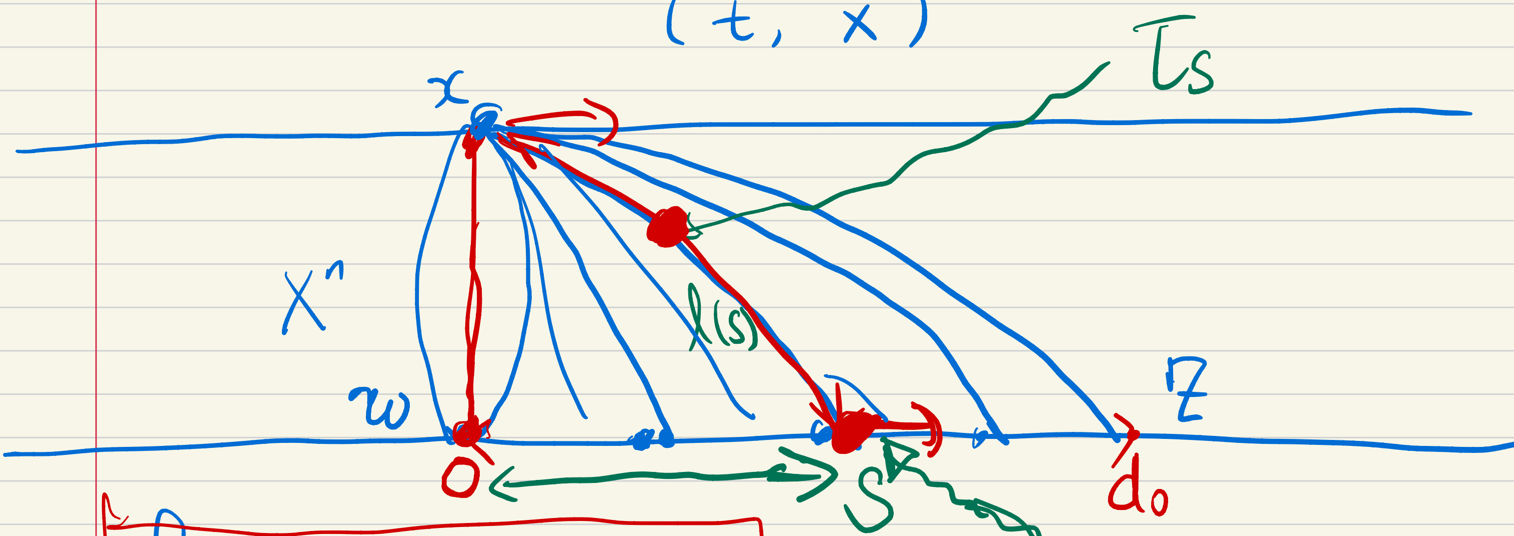
In particular, $\gamma(t) = (\gamma_1(t), \gamma_2(t))$ is a geodesic if γ_1, γ_2 are geodesics on M^n, N^k, respectively.

$$d^2_{g\oplus h}(p, q) = d^2_g(x_1, x_2) + d^2_h(y_1, y_2)$$

Theorem Let (M^n, g) be a complete Riemannian manifold. Then $\exists$ a full measure subset $\mathcal{R} \subseteq M^n \times M^n$ s.t. $\forall (p,q) \in \mathcal{R}$, $\exists$ unique minimal geodesic γ connecting p and q.

Example $Y^{n+1} \equiv \mathbb{R} \times X^n$

(t, x)



$$\boxed{h(t,x) = t}$$

Goal: $d^2(x,z) = d^2(w,z) + d^2(x,w)$

$|\nabla h| \equiv 1$, $\nabla^2 h \equiv 0$

$$d^2(w,z) = d_0^2 = \frac{1}{2}\int_0^{d_0} s\,ds$$

$\forall s \in [0, d_0]$, $\exists$ a geodesic

$\tau_s : [0, l(s)] \to Y^{n+1}$ with

$\tau_s(0) = x$, $\tau_s(l(s)) = \sigma(s)$.

$$\frac{1}{2}\int_0^{d_0} s\,ds = \frac{1}{2}\int_0^{d_0}\Big(h(\sigma(s)) - h(\sigma(0))\Big)ds$$

$$= \frac{1}{2}\int_0^{d_0}\Big(h(\tau_s(l_s)) - h(\tau_s(0))\Big)ds$$

$h(\sigma(0)) = h(\tau_s(0))$, i.e., $h(x) = h(w)$

$$= \frac{1}{2}\int_0^{d_0}\int_0^{l_s}\langle \nabla h(\tau_s(t)), \tau_s'(t)\rangle\,dt\,ds$$

$$\Big|\langle \nabla h(\tau_s(t)), \tau_s'(t)\rangle - \langle \nabla h, \tau_s'\rangle(\tau_s(l_s))\Big|$$

$$= \left| \int_t^s \left\langle \nabla_{\tau_s'(u)} \nabla h, \tau_s'(u) \right\rangle du \right|$$

$$= 0$$

$$= \frac{1}{2} \int_0^{d_0} \int_0^{l_s} \langle \nabla h, \tau_s' \rangle (\sigma(s))\, dt\, ds$$

$$= \frac{1}{2} \int_0^{d_0} l_s \cdot \langle \nabla h, \tau_s' \rangle (\sigma(s))\, ds$$

$$= \frac{1}{2} \int_0^{d_0} l_s \cdot \langle \sigma', \tau_s' \rangle (\sigma(s))\, ds$$

First variation formula

(τ_s is a geodesic)

$$= \frac{1}{2} \int_0^{d_0} l_s \cdot l_s'\, ds$$

$$= \ell^2(d_0) - \ell^2(0)$$

$$= d^2(x,z) - d^2(x,w)$$

- Warped product