Overview

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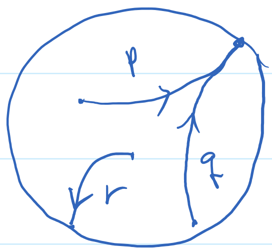
δ-Hyperbolic space $\rightsquigarrow$ Gromov boundary

$X \longmapsto \partial X$ as a set

geodesic metric space $\triangleq$ { geodesic rays up to finite hausdorff distance }

- $\underline{X \text{ proper}}$: $\underline{(\partial X \sqcup X) \text{ is compactification}}$ (open & dense)



$\mathbb{H}^2$ $\quad \exists R > 0$ s.t. $\begin{cases} p \subseteq N_R(q) \\ q \subseteq N_R(p) \end{cases}$

- $\varphi: X \xrightarrow{\text{Q.I.E.}} Y \rightsquigarrow \partial\varphi: \partial X \longrightarrow \partial Y$

  is top. embedding

So quasi-isometries induce homeo. between bdries.

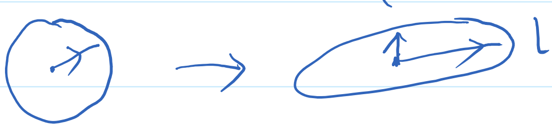
Gromov bdry is Q.I. invariant.

- $\exists$ a family of $\underline{\text{visual metrics}}$ $\{\rho_\varepsilon\}$ on $\underline{\partial X}$

s.t.

$\underline{\partial\phi}: (\partial X, \rho_\varepsilon) \longrightarrow (\partial Y, \rho_\varepsilon)$

is $\underline{\text{quasi-conformal}}$.



$\left|\frac{L}{\ell}\right| \leq M < \infty$

典型例子

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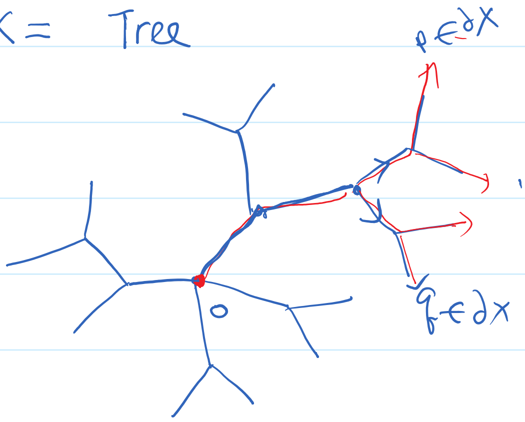
$\overset{x}{\text{———}} \longrightarrow \partial X = \{+, -\}$

$[0, +\infty) \xrightarrow{\text{isom.}} X$

Ex 1: $X =$ Tree



$\partial X = \{$ geod rays from the same pt o $\}$

$d(p, q) \triangleq 2^{-\mathrm{Len}(p \cap q)}$

is a metric on ∂X.

$\forall\, p_n \in \partial X$ $\underline{\exists\, p_{n_i}}$ s.t. $\underline{\forall R > 0}$ $\underline{p_{n_i}|_{B(o,R)} \equiv \text{const}}$.

$\underline{\text{for all but finite } n_i > 0}$

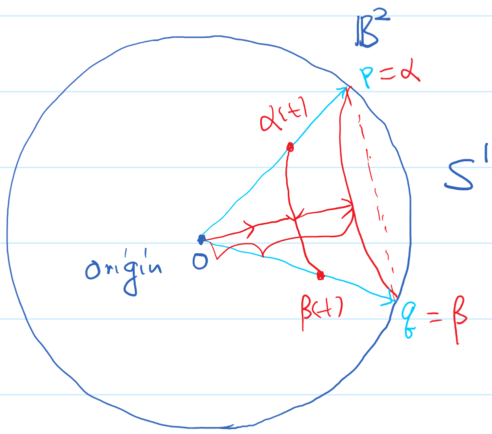
$\rightsquigarrow \underline{\exists\, p_\infty}$ s.t. $p_{n_i}|_{B(o,R)} = p_\infty|_{B(o,R)}$

for $\cdots\cdots\cdots$ $n_i > o$.

- If X is $\underline{\text{proper}}$ ($\Leftrightarrow$ loc. finite) then $\underline{\partial X \text{ is compact}}$.
- If $3 \leq \deg(\text{any vertex}) \leq \underline{M}$

  then $\underline{\partial X \cong \text{Cantor set}}$.

Ex 2: $\left(\mathbb{H}^2, \dfrac{\sqrt{dx^2 + dy^2}}{y}\right) \overset{\text{Isom}}{\cong} \left(\mathbb{B}^2, \rho = \dfrac{2|dz|}{1-|z|^2}\right)$



Exercise 0.4 (Visual metric). *Let $\alpha, \beta : [0, \infty) \to (\mathbb{B}^n, \rho)$ be two distinct geodesic rays (i.e.: isometric embedding) from o ending at two points $p, q \in \mathbb{S}^{n-1}$ respectively. Define*

$$\delta(\alpha, \beta) := \lim_{t \to \infty} e^{-\underbrace{\langle \alpha(t), \beta(t) \rangle_o}_{\text{hyp. metric}}}. \quad \doteq d(o, [\alpha(t), \beta(t)])$$

Define

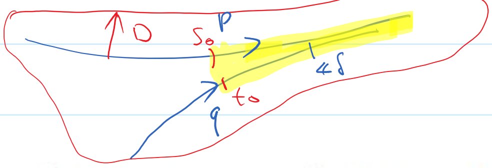
$$\delta(\alpha, \beta) := \lim_{t\to\infty} e^{-\underline{\langle \alpha(t), \beta(t) \rangle_o}}. \quad \asymp \; d(o, [\alpha(t), \beta(t)])$$

hyp. metric

Prove that $\underline{2\delta(\alpha, \beta) = |p - q|}$. *The so-defined function* δ *is called visual metric.*

(Tips: use Hyperbolic Cosine Rule (in Beardon's book))

Definition 8.1. Let $p, q : [0, \infty) \to X$ be two geodesic rays. We say that p, q are *asymptotic* if there exists $D > 0$ such that $p \subset N_D(q)$.

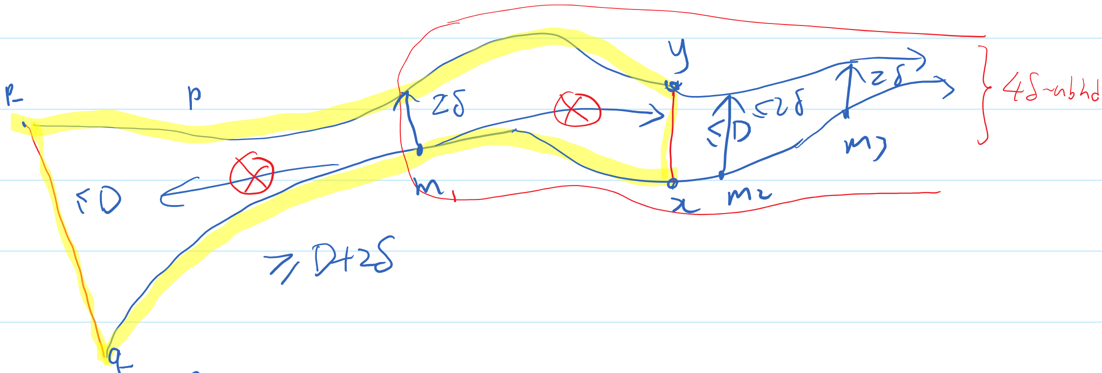


Lemma 8.2 (Uniformity of asymptotic rays). *Let p, q be two asymptotic geodesic rays. Then there exist $t_0, s_0 > 0$ such that* X is δ-hyp.

$$p([t_0, \infty)) \subset N_{4\delta}(q([s_0, \infty)))$$

and

$$q([s_0, \infty)) \subset N_{4\delta}(p([t_0, \infty)))$$



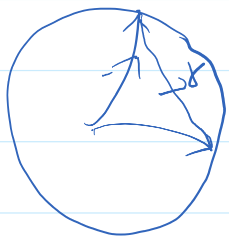
$$\max\{d(q_-, x), d(p_-, y)\} \gg 2D + 4\delta$$

Fix a basepoint $o \in X$

$$\partial_o X \triangleq \{\text{Asymptotic geodesic rays from } o\}$$

$$X \triangleq \{[[o, x]] : \forall x \in X,\ [[o,x]] \text{ all geodesics between } o \text{ and } x\}$$

$$\overline{X} \triangleq \partial_o X \sqcup X$$

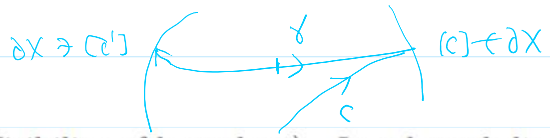


Def: • A bi-infinite geodesic γ connects $[c] \neq [c'] \in \partial X$ if the two half rays of γ end at $[c]$ and $[c']$.

• A geodesic ray γ $\boxed{\text{ends at}}$ $[c] \in \partial_o X$ if $\gamma \sim c$.
asymptotic

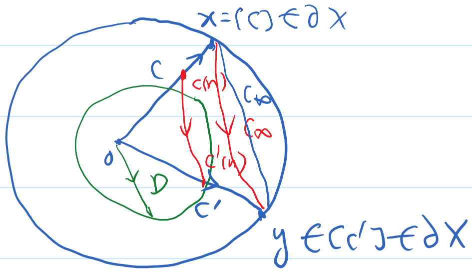
$\partial X \ni [c']$ γ $[c] \in \partial X$

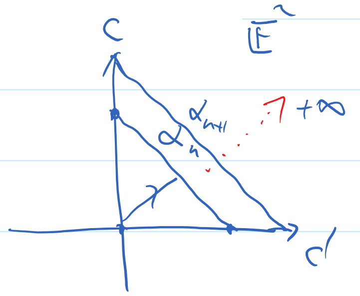
• A geodesic ray γ ends at $[c] \in \partial_0 X$ if $\gamma \sim c$.
asymptotic



Lemma 8.8 (Visibility of boundary). *In a hyperbolic space X, there exists a geodesic between any two distinct $x, y \in \bar{X}$.*

Proof:





Let $\alpha_n \triangleq [c(n), c'(n)]$ $n \in \mathbb{N}$.

Obs: $\alpha_n \longrightarrow \underline{c_\infty}$ loc. unif. (bi-infinite)

Rmk: In $\mathbb{E}^2$, c_∞ may not exist.

proof of Obs: $\boxed{d(o, \alpha_n) \le D \text{ (independ. of } n)}$ (反证法)

use Ascoli-Arzela Lemma to

$\alpha_n \cap \overline{B(o,R)}$ $R = D$

then let $R \to +\infty$ to get α_∞!

Topology on $X \sqcup \partial_0 X$:

$x_n \to x$ iff

$\exists [c_n] = x_n$ & $[c] = x$

s.t. $\boxed{\forall K \overset{\text{compact}}{\subset} X: \ c_n \cap K \Rightarrow c \cap K.}$

$c_n \to c$ locally uniformly

A subset $S \subseteq \bar{X}_0$ is closed if S contains the limit points of any convergent sequences in S.

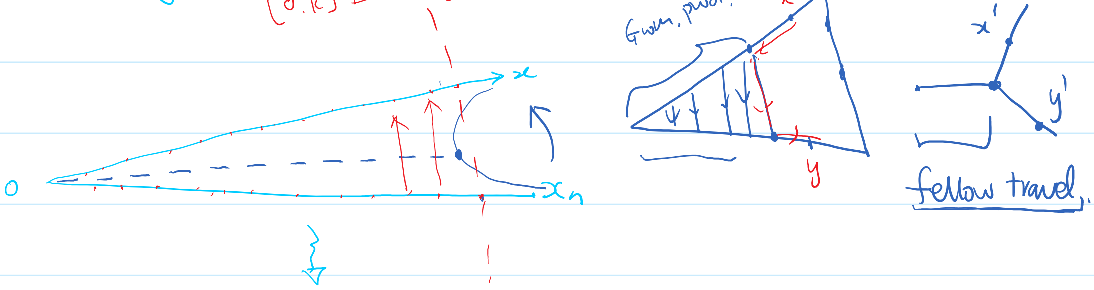
Fact: • In a proper length space (not necess. hyp.), any sequence of geod. rays from a fixed point must sub-converge:

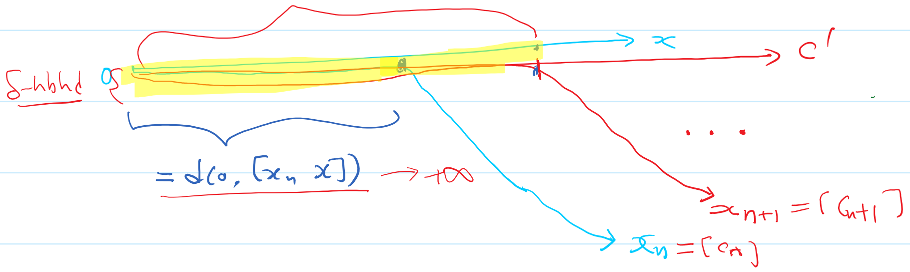
($\exists$ subsequence converges loc. unif. to a limiting geod. ray)

- $x_n \to x$ iff $d(o, [x\, x_n]) \to +\infty$ (In δ-hyp space)

proof: $\Rightarrow$: Thin-triangle ✓

$\Leftarrow$:





$\Rightarrow$ $\underset{\substack{\| \\ [c_n]}}{x_n}$ sub-converge loc. unif. to c' $\leftarrow$ c_{n_i} loc. unif.

It is easy to see $[c'] = x$.

Moreover, $[c']$ does not convergent subsequence of c_n!

Lemma: Fix $o, o' \in X$,

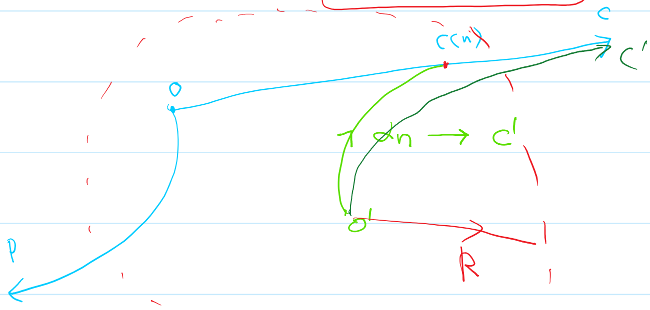
$$\overline{X_o} \cong \overline{X_{o'}}$$

proof: $\boxed{\text{Fact}}$ Let $[c]$ be asymptotic class of geod. ray from o

then for $\forall\, o' \in X$ $\exists$ geod. ray c' from o'

s.t.

$$\boxed{[c'] = [c]}$$



So $\pi: [c] \in \partial_o X \longmapsto [c'] \in \partial_{o'} X$ is bijective.

By the above fact π is continuous.

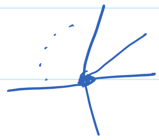
So $\pi : [c] \in \partial_o X \longmapsto [c] \in \partial_o' X$ is bijective.

By the above fact, π is continuous:

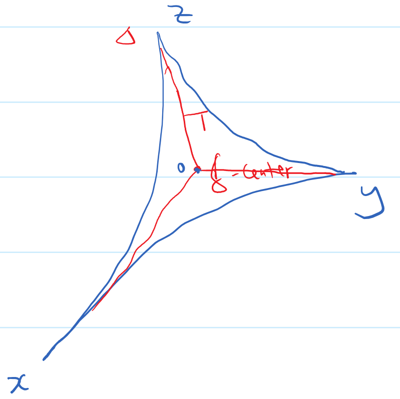
$$[c_n] \to c \Rightarrow [c_n'] \to [c'] \qquad \square$$

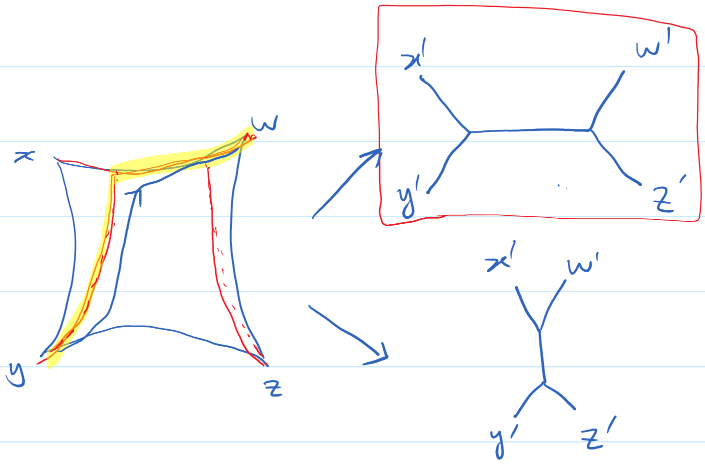
(X proper)

Lemma 8.6. *Let $\bar{X}_o$ be endowed with the first or second topology. Then $\bar{X}_o$ is a compactification of X.*



7.6. **Approximation trees in hyperbolic spaces.** In this section, we prove a very useful result, which gives a tree-like picture of any finite set in a hyerbolic space.





Lemma 7.20. *Let X be a hyperbolic space, and F be a finite set. There exists a constant $c = c(|F|)$ and an embedded tree $T \subset X$ with $F \subset T^0$ such that the following holds*

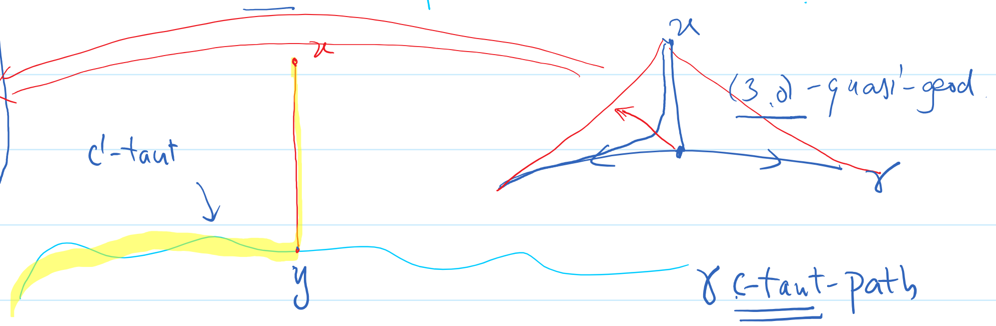
$$d_X(x,y) \le \boxed{d_T(x,y)} \le d_X(x,y) + c.$$

In other words, there exists an injective $(1,c)$-quasi-isometric map $\iota : T \to X$ with $F \subset \iota(T^0)$.

Sketch: Induction on #F.

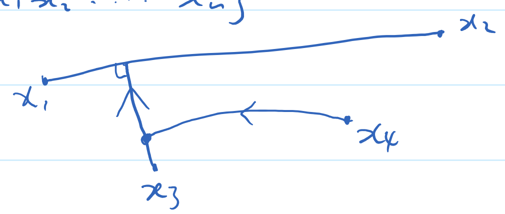
Obs: draw a perpendicular to a c-taut path produces two c'-taut paths where $c' = c(c,\delta)$.

Fact: projection pts are D-Center!



$F = \{x_1, x_2, \dots, x_n\}$

$T_3 \to T_4 \to \dots \to T_n$



Lemma 7.22. *Suppose (X, d) is a hyperbolic space. Then there exists $\delta > 0$ such that the following holds*

(15) $$d(x, y) + d(z, w) \le \max\{d(x, z) + d(y, w), d(x, w) + d(y, z)\} + \delta$$

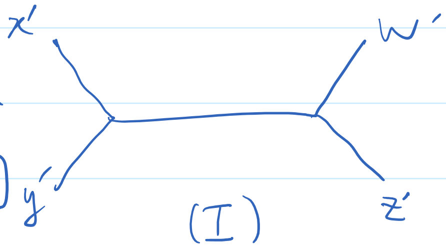
Gromov's def of hyp. space

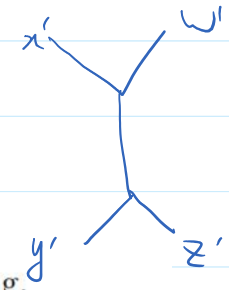
for any four points $x, y, z, w \in X$.

Equivalently, (15) is amount to saying that

(16) $$(x, y)_{\boxed{w}} \ge \min\{(x, z)_w, (y, z)_w\} - \delta/2.$$

(15) holds for without δ-error



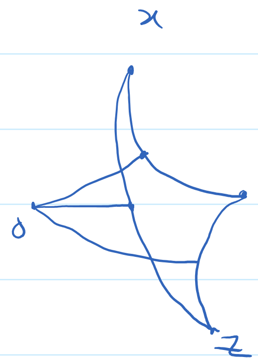


So the inequality (16) in Lemma 7.22 proves the following.

Lemma 8.13. *There exists $\delta > 0$ such that the following holds*

(17) $$d(o, [x, y]) \ge \min\{d(o, [x, z]), d(o, [y, z])\} - \delta$$

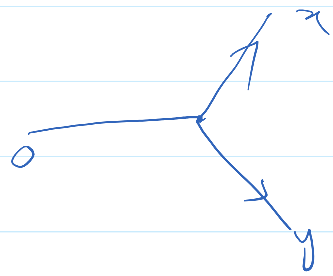
for any distinct $x, y, z \in \bar{X}$ and $o \in X$.



Visual metric: $X \sqcup \partial_o X \ni x, y$

Fix $a > 0$, define

$$\bar{\rho}_a(x, y) \triangleq e^{-d(o, [x, y])}$$



Then Set $K \triangleq e^{a\delta}$, we have

(18) $$(x, y) \le K \cdot \max\{\bar{\rho}_a(x, z), \bar{\rho}_a(z, y)\}$$

$K = 1$ ultra-metric

(19) $$\bar{\rho}_a(x, y) = \rho_a(y, x)$$

quasi-metric

Lemma 8.14. *Let $\bar{\rho} : M \times M \to \mathbb{R}_{\ge 0}$ a function satisfying (18, 19) with $1 \le K \le \sqrt{2}$. Then there exists a metric ρ on M such that*

$$\bar{\rho}(x, y) \le \boxed{\rho(x, y)} \le \bar{\rho}(x, y)$$

$M = \partial_o X$

$\sqrt{2}$. *Then there exists a metric ρ on M such that*

$$\frac{\bar{\rho}(x,y)}{K} \leq \boxed{\rho(x,y)} \leq \bar{\rho}(x,y)$$

$M = \partial_0 X$

for any $x, y \in M$.

Thus, we choose $a \in (0,1]$ small enough such that $e^{a\delta} \leq \sqrt{2}$ (there is a critical value a_0 such that any $a \in (0, a_0]$ works). Then we get a metric ρ_a on $\bar{X}$ by Lemma 8.14 such that

$$\bar{\rho}_a(x,y)/2 \leq \rho_a(x,y) \leq \bar{\rho}_a(x,y) \tag{21}$$

Lemma 8.15. *The induced topology on ∂X by ρ is the same as the topology defined in previous subsection.*

边界延拓

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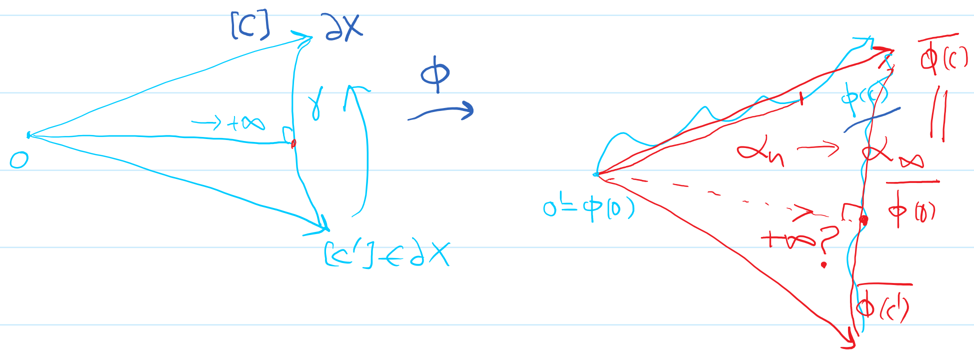
Theorem 8.19. *Let X, Y be two hyperbolic spaces. Assume that there exists a quasi-isometry between X and Y. Then the Gromov boundary of X is homeomorphic to that of Y.*

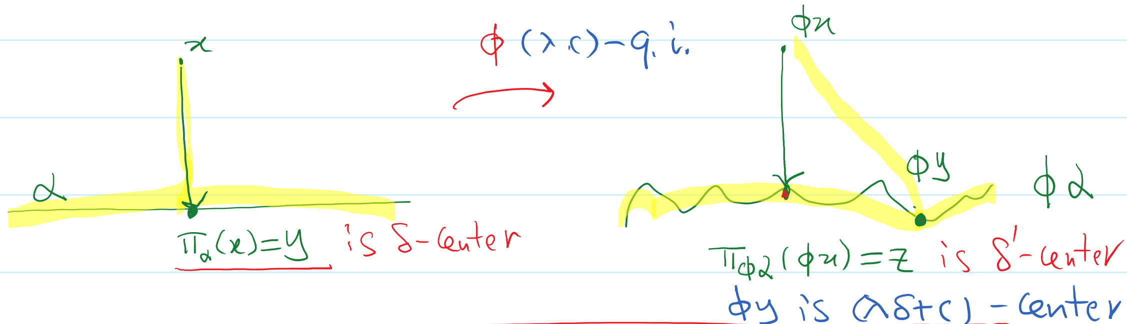
Proof: • $\partial\phi : \partial X \xrightarrow{1:1} \partial Y$ by $\partial\phi([c]) \triangleq [\overline{\phi(c)}]$

where $\overline{\phi(c)}$ represents a geodesic ray in a finite nbhd of $\phi(c)$.

[Morse Lemma & Paper]



• Quasi-isometries preserve projections up to finite error:



$\exists D = D(\lambda, c, \delta)$: $\boxed{d(\pi_{\phi\alpha}(\phi x), \phi(\pi_\alpha x)) \leq D}$

Fact: The set of r-centers is of diameter $D = D(r)$.

• Cor: Set $x' = \phi x$, $y' = \phi y$, $z' = \phi z$, $o' = \phi o$. where ϕ is (λ, c)-q.i.

Set $S(x, y, z) \triangleq |\langle x, z\rangle_o - \langle x, y\rangle_o|$

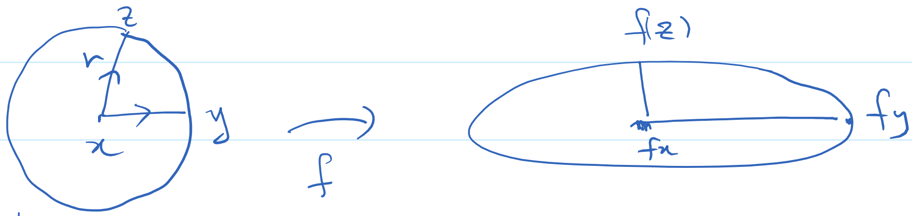
Then $\exists d = d(\lambda, c, \delta)$ s.t.

$$\boxed{\frac{1}{\lambda} S(x,y,z) - c \le S(x',y',z') \le \lambda S(x,y,z) + c}$$

Definition 8.22. A homeomorphism $f : X \to Y$ is called *quasi-conformal* if there exists a constant H so that

$$\limsup_{r\to 0} \frac{\sup_{d(x,y)=r} d(f(x),f(y))}{\inf_{d(x,y)=r} d(f(x),f(y))} \le H < \infty$$

forall x in X.



[Mostow $\mathbb{H}^n$] pseudo-isometries

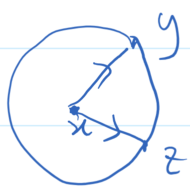
Theorem 8.23. *Let $\phi : X \to X$ be a quasi-isometry between hyperbolic spaces. Then the induced map $\partial\phi : \partial X \to \partial X$ is a quasi-conformal map with respect to visual metric.*

Sketch: • WLOG $\phi(o) = o$ (Changing base points is Quasi-conformal)

• Let $x, y, z \in \partial X$, $x' = \phi x$, $y' = \phi y$, $z' = \phi z \in \partial X$

Consider the small r-circle at x:

$$\boxed{\begin{aligned} \rho_o(x,y) &= r \asymp e^{-\langle x,y\rangle_o} \\ \rho_o(x,z) &= r \asymp e^{-\langle x,z\rangle_o} \end{aligned}}$$



$$\Updownarrow$$

$$\boxed{S(x,y,z) = |\langle x,z\rangle_o - \langle x,y\rangle_o| \le D}$$

$\Downarrow$ cor.

$$S(x',y',z') \le \lambda D + c.$$

$$\Rightarrow \quad H \triangleq \limsup_{r\to 0} \frac{\sup\{\rho_0(x',y') : \rho_0(x,y)=r\}}{\inf\{\rho_0(x',z') : \rho_0(x,z)=r\}}$$

$$\le \limsup_{r\to 0} \sup\{ e^{S(x',y',z')} : \rho_0(x,y)=\rho_0(x,z)=r\}$$

$$\le e^{\lambda D + c} < \infty$$

Rmk: An isometry on trees extend to conformal on Gromov. bdry.