$\delta$-Hyperbolic space $\rightarrow$ Gromov boundary

$X \rightarrow \partial X$ as a set

godesic metric space $\triangleq \{ \text{geodesic rays up to finite hausdorff distance} \}$

- $X$ proper: $(\partial X \cup \overline{X})$ is compactification

- $\psi: X \xrightarrow{\text{quasi-isometric}} Y$ and $\phi: \partial X \rightarrow \partial Y$

  is top. embedding

So quasi-isometries induce homeo. between bodies.

Gromov boundary is $\delta$-1 invariant.

- $\exists$ a family of visual metrics $\{ p_c \}$ on $\partial X$

  s.t.

  $\phi: (\partial X, p_c) \rightarrow (\partial Y, p_c)$

  is quasi-conformal.
Exercise 0.4 (Visual metric). Let $\alpha, \beta : [0, \infty) \to (\mathbb{B}^n, \rho)$ be two distinct geodesic rays (i.e.: isometric embedding) from $o$ ending at two points $p, q \in S^{n-1}$ respectively. Define

$$\delta(\alpha, \beta) := \lim_{t \to \infty} e^{-\frac{\langle \alpha(t), \beta(t) \rangle_\rho}{\rho}} \cdot \frac{d(o, (\alpha(t), \beta(t)))}{\text{hyp. metric}}.$$
Define

\[ \delta(\alpha, \beta) := \lim_{t \to \infty} e^{-\langle \alpha(t), \beta(t) \rangle_\omega} \]

Prove that \(2\delta(\alpha, \beta) = |p - q|\). The so-defined function \(\delta\) is called visual metric.

(Tips: use Hyperbolic Cosine Rule (in Beardon’s book))
Definition 8.1. Let \( p, q : [0, \infty) \rightarrow X \) be two geodesic rays. We say that \( p, q \) are \textit{asymptotic} if there exists \( D > 0 \) such that \( p \subset N_D(q) \).

Lemma 8.2 (Uniformity of asymptotic rays). Let \( p, q \) be two asymptotic geodesic rays. Then there exist \( t_0, s_0 > 0 \) such that

\[
p([t_0, \infty)) \subset N_{4\delta}(q([s_0, \infty)))
\]

and

\[
q([s_0, \infty)) \subset N_{4\delta}(p([t_0, \infty)))
\]

\[
\max \{d(q, x), d(p, y)\} \gg 2D + 4\delta
\]

Fix a basepoint \( o \in X \)

\[
\mathcal{D}_o X \triangleq \{ \text{Asymptotic geodesic rays from } o \}
\]

\[
X \triangleq \bigcup \left\{ [o, x] : \forall x \in X, [o x] \text{ all geodesics between } o \text{ and } x \right\}
\]

\[
\overline{X} \triangleq \mathcal{D}_o X \cup X
\]

Def. • A bi-infinite geodesic \( d \) connects \( [c] \in [c] \in \mathcal{D}_o X \) if the two half rays of \( d \) end at \( [c] \) and \( \bar{c'}.\)

• A geodesic ray \( \gamma \) \text{ ends at } \[c] \in \mathcal{D}_o X \) if \( \gamma \sim c \).

\[
\gamma \sim c \quad \gamma \in \mathcal{D}_o X
\]
Lemma 8.8 (Visibility of boundary). In a hyperbolic space $X$, there exists a geodesic between any two distinct $x, y \in \overline{X}$.

Proof:

Let $d_n \triangleq d(\circ, \cdot n)$, $n \in \mathbb{N}$.

Observe: $d_n \to \infty$ loc. unif. (bi-infinite)

Remark: In $\mathbb{E}^2$, $\infty$ may not exist.

Proof of obs. $\text{d}(\circ, d_n) \leq D$ (independently of $n$) (by definition)

Use Ascoli–Arzelà Lemma to

$\overline{d_n \cap B(\circ, R)} = \overline{R = D}$

Then, let $R \to +\infty$ to get $d_{\infty}$.

Topology on $X \cup \circ X$:

$x_n \to x$ iff

$\exists [c_n] = x_n \cup [c] = x$

$s.t.

\forall \kappa \leq x : c_n \cap \kappa \Rightarrow c_n \kappa$

A subset $S \subseteq \overline{X_0}$ is closed if $S$ contains the limit points of any convergent sequences in $S$.

Fact: In a proper length space, any sequence of geodesic rays from a fixed point must sub-converge:

($\exists$ subsequence converges loc. unif. to a limiting geodesic ray)
\( x_n \to x \iff d(0, [x, x_n]) \to \infty \quad (\text{In } \mathbb{S} \text{-hyp space}) \)

**proof:** \( \Rightarrow: \) Thin-triangle

\[ \Delta \implies B(u, R) \]

\[ \implies x_n \text{ converges locally unif. to } c' \leftarrow c_n \text{ loc.unif.} \]

It is easy to see \( [c'] = x \).

Moreover, \( [c'] \) does not convergent subsequence of \( c_n \).

**Lemma:** Fix \( o, o' \in X \),

\[ X_0 \equiv X_{o'} \]

**proof:** \[ \text{Fact: } \]

Let \( [c] \) be asymptotic class of geod. ray from \( o \)

then for \( o' \in X \) \( \exists \) geod. ray \( c' \) from \( o' \)

\[ [c'] = [c] \]

so \( \pi: [c] \in \partial X \mapsto (c) \in \partial X \text{ is bijective.} \)

By the above fact \( \pi \text{ is continuous.} \)
Lemma 8.6. Let $\tilde{X}_o$ be endowed with the first or second topology. Then $\tilde{X}_o$ is a compactification of $X$. 
7.6. Approximation trees in hyperbolic spaces. In this section, we prove a very useful result, which gives a tree-like picture of any finite set in a hyperbolic space.

**Lemma 7.20.** Let $X$ be a hyperbolic space, and $F$ be a finite set. There exists a constant $c = c(|F|)$ and an embedded tree $T \subset X$ with $F \subset T^0$ such that the following holds

$$d_X(x, y) \leq d_T(x, y) \leq d_X(x, y) + c.$$

In other words, there exists an injective $(1, c)$-quasi-isometric map $i : T \to X$ with $F \subset i(T^0)$.

**Sketch:** Induction on $\#F$,

- **Obs:** draw a perpendicular to a $c$-taut path.
- Produces two $c'$-taut paths where $c' = c(c, 5)$.

**Fact:** projection pts are $D$-center!
Lemma 7.22. Suppose \((X, d)\) is a hyperbolic space. Then there exists \(\delta > 0\) such that the following holds
\[
(15) \quad d(x, y) + d(z, w) \leq \max\{d(x, z) + d(y, w), d(x, w) + d(y, z)\} + \delta
\]
for any four points \(x, y, z, w \in X\).
Equivalently, (15) is amount to saying that
\[
(16) \quad (x, y)_w \geq \min \{(x, z)_w, (y, z)_w\} - \delta/2.
\]
So the inequality (16) in Lemma 7.22 proves the following.

Lemma 8.13. There exists \(\delta > 0\) such that the following holds
\[
(17) \quad d(o, [x, y]) \geq \min\{d(o, [x, z]), d(o, [y, z])\} - \delta
\]
for any distinct \(x, y, z \in X\) and \(o \in X\).

Visual metric: \(X \cup \partial X \supseteq x, y, z\)

Fix \(a > 0\), define
\[
P_a(x, y) = e^{-d(o, [x, y])}
\]
Then set \(K = e^{a\delta}\), we have
\[
(18) \quad (x, y) \leq K \max\{P_a(x, z), P_a(z, y)\}
\]
\[
(19) \quad P_a(x, y) = P_a(y, x)
\]

Lemma 8.14. Let \(\bar{\rho}: M \times M \to \mathbb{R}_{\geq 0}\) a function satisfying (18, 19) with \(1 \leq K \leq \sqrt{2}\). Then there exists a metric \(\rho\) on \(M\) such that
\[
\bar{\rho}(x, y) < \rho(o(x, u)) < \tilde{\rho}(x, u)
\]

\[ \sqrt{2}. \quad \text{Then there exists a metric } \rho \text{ on } M \text{ such that } \]
\[ \frac{\bar{\rho}(x, y)}{K} \leq \rho(x, y) \leq \bar{\rho}(x, y) \quad \text{ for any } x, y \in M. \]

Thus, we choose \( a \in (0, 1] \) small enough such that \( e^{a\delta} \leq \sqrt{2} \) (there is a critical value \( a_0 \) such that any \( a \in (0, a_0] \) works). Then we get a metric \( \rho_a \) on \( \bar{X} \) by Lemma 8.14 such that
\[ \frac{\bar{\rho}_a(x, y)}{2} \leq \rho_a(x, y) \leq \bar{\rho}_a(x, y) \quad (21) \]

**Lemma 8.15.** The induced topology on \( \partial X \) by \( \rho \) is the same as the topology defined in previous subsection.
**Theorem 8.19.** Let \( X, Y \) be two hyperbolic spaces. Assume that there exists a quasi-isometry between \( X \) and \( Y \). Then the Gromov boundary of \( X \) is homeomorphic to that of \( Y \).

**Proof.** \( \exists \phi : \partial X \rightarrow \partial Y \) by \( \partial \phi (c) \triangleq \overline{\phi (c)} \)

where \( \overline{\phi (c)} \) represents a geodesic ray in a finite word of \( \phi \).

\( \exists R \in \mathbb{R} \) such that \( d(\phi (x), \phi (y)) > R \) for \( x, y \in \partial X \).

\( \exists D \in \mathbb{R} \) such that \( d(\phi (x), \phi (y)) \leq D \).

**Fact:** The set of \( r \)-centers is of diameter \( D = D(r) \).

**Cor.** Set \( x' = \phi x, \ y' = \phi y, \ z' = \phi z, \ 0 = \phi 0 \). where \( \phi \) is \( (r, c) \)-p.s.

Set \( S(x, y, z) \triangleq |\langle x, z \rangle_0 - \langle x, y \rangle| \)

Then \( \exists c = c(x, y, z) \) s.t.
Definition 8.22. A homeomorphism $f : X \to Y$ is called quasi-conformal if there exists a constant $H$ so that

$$\limsup_{r \to 0} \frac{\sup_{d(x, y) = r} d(f(x), f(y))}{\inf_{d(x, y) = r} d(f(x), f(y))} \leq H < \infty$$

for all $x$ in $X$.

Theorem 8.23. Let $\phi : X \to X$ be a quasi-isometry between hyperbolic spaces. Then the induced map $\partial \phi : \partial X \to \partial X$ is a quasi-conformal map with respect to visual metric.

Sketch: WLOG $\phi(0) = 0$ (changing base points is quasi-conformal)

- Let $x, y, z \in \partial X$, $x' = \phi x$, $y' = \phi y$, $z' = \phi z \in \partial X$

Consider the small $r$-circle at $x$:

$$p_0(x, y) = r \propto e^{-c \cdot d(x, y)}$$

$$p_0(x, z) = r \propto e^{-c \cdot d(x, z)}$$

$$S(x, y, z) = |\langle x, z \rangle - \langle x, y \rangle| \leq D$$

...
\[ S(x', y', z') \leq AD + c. \]

\[ \Rightarrow H = \limsup_{r \to 0} \frac{\sup \{ \rho_0(x', y') \; : \; \rho_0(x, y) = r \}}{\inf \{ \rho_0(x', z') \; : \; \rho_0(x, z) = r \}} \]

\[ \leq \limsup_{r \to 0} \sup \{ e^{S(x', y', z')} \; : \; \rho_0(x, y) = \rho_0(x, y) = r \} \]

\[ \leq e^{2D + \epsilon} < \infty \]

Rank: An isometry on trees extend to conformal on Gromov, bdy.