3.2. Riemannian Geometry of Lie Groups
A CRASH COURSE.

<u>Def</u>. A group G is called a Lie group if the following holds:

(1) G is a differentiable manifold

(2) Both the multiplication $(g,h) \mapsto g \cdot h$ and the inversion $g \mapsto g^{-1}$ are C^∞.

On a Lie group G, one can define

$$L_g(h) = g \cdot h, \quad \forall h \in G$$

$$R_g(h) = h \cdot g, \quad \forall h \in G.$$

Both are diffeomorphisms.

<u>Def</u> (Left invariant vector field)

Let G be a Lie group. A vector field X is called left invariant if $\forall g, h \in G$

$$\underline{DL_g}(\underline{X_h}) = X_{g \cdot h} = X_{L_g(h)}.$$

Remark

$$X_g = dL_g(X_e), \quad \text{left invariant}$$

$$X_g = dR_g(X_e), \quad \text{right invariant}$$

Def (Lie algebra) A vector space V is called a Lie algebra if a bilinear operation $[\cdot,\cdot] : V \times V \to V$ s.t.

$\forall\, u, v, w \in V$

(1) $[v,w] = -[w,v]$

(2) $[[u,v],w] + [[v,w],u] + [[w,u],v] = 0$.

Ex Let M^n be a manifold. Show that $\forall\, X, Y, Z \in \mathfrak{X}(M^n)$, $[X,Y] = XY - YX$.

(1) $[Y,X] = -[X,Y]$

(2) $[[X,Y],Z] + [[Y,Z],X] + [[Z,X],Y] = 0$

$\mathfrak{X}(M^n)$ is a Lie algebra of dim ∞.

Lemma. If X, Y are left invariant, then $[X,Y] = XY - YX$ is also left invariant.

The subspace of left invariant vector fields is a Lie algebra of dimension equal to $\underline{\dim(G)}$

$$\begin{array}{c} \dim(G) \\ \shortparallel \\ \dim(T_e G). \end{array}$$

Ex Rotations of $\mathbb{R}^2$

$$SO(2) \equiv \left\{ \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix} \middle| \; t \in [0, 2\pi] \right\}$$

homeomorphic to $\underline{S^1}$.

Taking any curve

$$\gamma(t) = \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix}$$

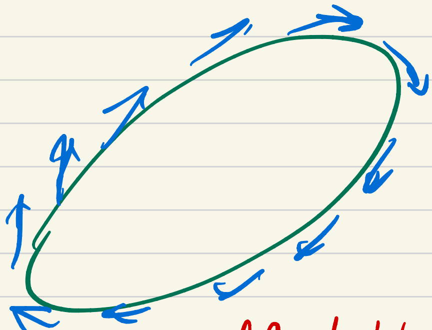
$$\gamma'(0) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

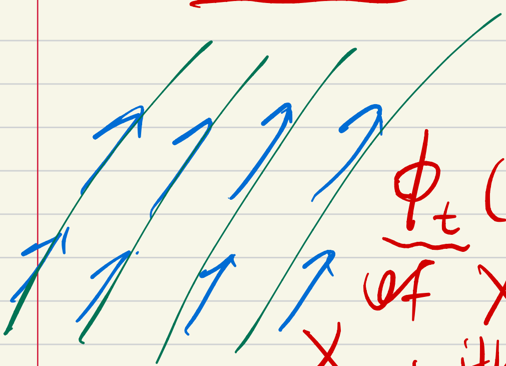
$$\mathfrak{so}(2) \equiv \left\{ \underbrace{\begin{pmatrix} 0 & -x \\ x & 0 \end{pmatrix}}_{\text{"}A\text{"}} \;\middle|\; x \in \mathbb{R} \right\}$$

$$\boxed{A + A^T = 0}$$

Definition. Let $X \in \mathfrak{X}(M^n)$. A curve $\gamma : [0,1] \to X$ is called the integral curve of X if $\forall t \in [0,1]$

$$\frac{d\gamma}{dt}(t) = X(t).$$





$\phi_t(p) \equiv \gamma(t)$ is called the flow of X, where γ is the integral curve X with $\gamma(0) = p$,

Theorem. Let G be a Lie group. $\forall v \in T_eG$. $\exists$ a unique 1-parameter subgroup of G s.t. $\gamma'(0) = v$.

Def. A 1-parameter subgroup of G is a Lie group homomorphism $\gamma : \mathbb{R} \to G$ from the additive group of $\mathbb{R}$ into G.

Def (Exponential map)

The exponential map $\exp : \mathfrak{g} \to G$ is the map defined by

$$\boxed{\exp(X) \equiv \exp_X(1)},$$

where $\exp_X(\cdot)$ denotes the integral curve of X through e with

$$\exp_X(0) = e, \quad \exp_X'(0) = X_e.$$

Proposition Let G be a Lie group, $X \in \mathfrak{g}$, $s, t \in \mathbb{R}$, $g \in G$.

(1) $\exp(tX) = \exp_X(t) \quad \forall t \in \mathbb{R}$.

(2) $\exp((t+s)X) = \exp(sX) \cdot \exp(tX)$

(3) $L_g \circ \exp_X(\cdot)$ is the unique integral curve of X through g

(4) Let φ_t be flow of X. Then

$$\boxed{\varphi_t = R_{\exp_X(t)}.}$$

Pf. $\forall g \in G$, $\varphi_t(g) = \gamma_g(t)$,

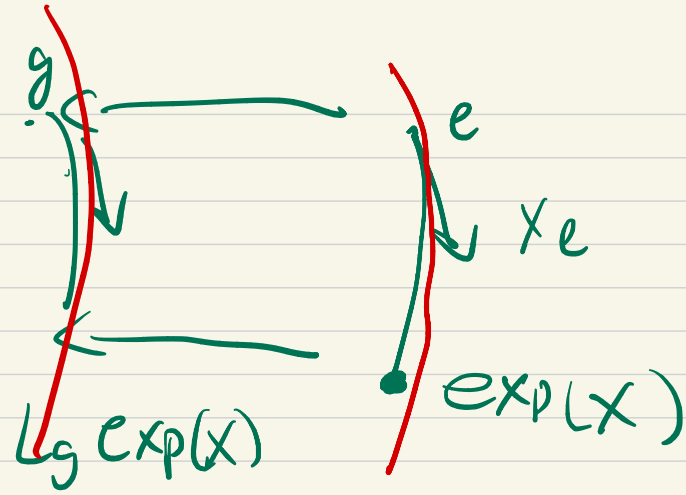
where γ_g is the integral curve of X through g.

$$\gamma_g(t) = L_g\left(\exp_X(t)\right)$$

$$= g \cdot \underline{\exp_X(t)}$$

$$= R_{\exp_X(t)}(g)$$

$$\underline{\varphi_t}(g) = \underline{R_{\exp_X(t)}}(g). \qquad \square$$



Example

$$SL(n) = \{ A \in GL(n) \mid \det(A) = 1 \}$$

$$\mathfrak{sl}(n) = \{ tr(A) = 0 \}.$$

$\gamma(t) = \exp(tA)$ is a 1-parameter subgroup of $\underline{SL(n)}$.

$$\underline{\det(\gamma(t)) = 1}$$

$$\det(\exp(tA)) = \exp(\underline{tr(tA)}) \quad \forall t \in \mathbb{R}$$

$$\boxed{tr(A) = 0}.$$

Ex The Lie algebra of $O(n)$ is $\underline{\mathfrak{o}(n)}$.

Ex $SU(n)$, $U(n)$

$SU(n) \subset U(n)$ $\quad \boxed{A \cdot A^* = Id}$

Definition

A Riemannian metric $\langle \cdot, \cdot \rangle$ is called left invariant if any left translation is isometric, i.e.,

$$\langle u, v \rangle_g = \langle (DL_h)_g u, (DL_h)_g v \rangle_h$$

$$\forall g, h \in G, \quad \forall u, v \in T_g G.$$

Any inner product $\langle \cdot, \cdot \rangle_e$ on $T_e G$ can be extended to be a left invariant metric on G by defining

$$\langle u, v \rangle_g = \langle (DL_{g^{-1}})_g u, (DL_{g^{-1}})_g v \rangle_e .$$

Definition

Bi-invariant: both left invariant and right invariant.

Theorem (Milnor) G Lie group

(1) If G compact, then G admits a bi-invariant metric.

(2) A Lie group admits a bi-invariant metric iff $G \cong \underset{\text{compact}}{K} \times \underset{\text{Abelian}}{H}$

<u>Proposition</u> $\langle \cdot, \cdot \rangle$ bi-invariant metric on G. $\forall\, X, Y, Z \in \mathfrak{g}$

(1) $\langle [X,Y], Z \rangle + \langle Y, [X,Z] \rangle = 0$

(2) If a left invariant metric G satisfies (1), then it is a bi-invariant.

(3) $\nabla_X X \equiv 0$. That is, any 1-parameter subgroup of G is a geodesic.

$$\underset{\text{Lie theoretic}}{\exp} \equiv \underset{\text{Riemannian}}{\mathrm{Exp}}$$

(4) $\nabla_X Y = \frac{1}{2}[X, Y]$

(5) $R(X, Y, Z, W) = \frac{1}{4}\langle [X, Y], [W, Z]\rangle$.

Consequently, $\underline{\sec_G \geq 0}$

An important example

Hopf fibration (Berger sphere)

$$SU(2) = \left\{ \begin{bmatrix} z & -\bar{w} \\ w & \bar{z} \end{bmatrix} \middle| z, w \in \mathbb{C},\ |z|^2 + |w|^2 = 1 \right\}$$

- $SU(2) \approx S^3$.
- $\pi(z, w) = z/w = \mathbb{C} \cup \{\infty\} = S^2$

$$S^1 \longrightarrow S^3 \xrightarrow{\pi} S^2$$

$$\bullet \quad (z,w) \mapsto \left(\mathrm{Re}(zw), \mathrm{Im}(zw), \frac{|z|^2-|w|^2}{2} \right) \in \mathbb{R}^3$$

$X, Y, Z \in \mathfrak{g}$, s.t.

$[X,Y]=Z, \quad [Y,Z]=X, \quad [Z,X]=Y.$

duals η_1, η_2, η_3

$$g_t = \underbrace{t^2 \eta_1 \otimes \eta_1}_{} + \eta_2 \otimes \eta_2 + \eta_3 \otimes \eta_3$$

$\boxed{\nabla_X X = 0}$, $\quad \nabla_X Y = (2-t^2)Z, \quad \nabla_X Z = (-2+t^2)Y$

$\nabla_Y X = -t^2 Z, \quad \nabla_Y Y = 0, \quad \nabla_Y Z = X$

$\nabla_Z X = t^2 Y \quad \nabla_Z Y = -X, \quad \nabla_Z Z = 0$

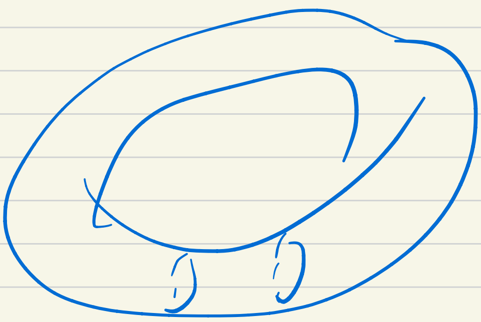
$\Downarrow$

$$\boxed{\begin{gathered} \mathrm{Sec}(X,Y) = t^2, \quad \mathrm{Sec}(X,Z) = t^2, \\ \underline{\mathrm{Sec}(Y,Z) = 4-3t^2} \end{gathered}}$$

Let $t \to 0$ the direction X is collapsing since its integral curve has length $2t\pi$.

The "limiting metric" is the round metric of curvature $+4$ on the 2-sphere of radius $\frac{1}{2}$.

$$S^1_\varepsilon \times S^1 \longrightarrow S^1$$



- $(\mathbb{R}^4, g_t) \xrightarrow{GH} (\mathbb{R}^3, g_0)$