3.2. Riemannian Geometry of Lie Groups

A CRASH COURSE.

Def. A group $G$ is called a Lie group if the following holds:

1. $G$ is a differentiable manifold
2. Both the multiplication $(g, h) \mapsto gh$ and the inversion $g \mapsto g^{-1}$ are $C^\infty$.

On a Lie group $G$, one can define

$L_g(h) = gh, \quad \forall h \in G$

$R_g(h) = hg, \quad \forall h \in G$

Both are diffeomorphisms.

Def (Left invariant vector field)

Let $G$ be a Lie group. A vector field $X$ is called left invariant if

$\forall g, h \in G$

$DL_g(X_h) = X_{g \cdot h} = X_{L_g(h)}$. 
Remark
\[ X_g = dL_g(X_e), \text{ left invariant} \]
\[ X_g = dR_g(X_e), \text{ right invariant} \]

Def (Lie algebra) A vector space \( V \) is called a Lie algebra if a bilinear operation \( [\cdot, \cdot] : V \times V \to V \) s.t.

\[ \forall \ u, v, w \in V \]

(1) \[ [v, w] = - [w, v] \]

(2) \[ [u, [v, w]] + [v, [w, u]] + [w, [u, v]] = 0 \]

Ex. Let \( M^n \) be a manifold. Show that \( \forall X, Y, Z \in \mathfrak{X}(M^n), \ [X, Y] = X Y - Y X \)

(1) \[ [Y, X] = - [X, Y] \]

(2) \[ [[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0 \]

\( \mathfrak{X}(M^n) \) is a Lie algebra of \( \dim = \infty \).
Lemma. If $X, Y$ are left invariant, then $[X, Y] = XY - YX$ is also left invariant.

The subspace of left invariant vector fields is a Lie algebra of dimension equal to $\dim(G) - \dim(T_e G)$.

Example. Rotations of $\mathbb{R}^2$

$SO(2) = \left\{ \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix} \mid t \in [0, 2\pi] \right\}$

homeomorphic to $\mathbb{S}^1$.

Taking any curve

$\sigma(t) = \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix}$
\[ \vec{x}'(0) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \]

\[ \mathfrak{so}(2) = \left\{ \begin{pmatrix} 0 & -x \\ x & 0 \end{pmatrix} \mid x \in \mathbb{R} \right\} \]

A curve \( \vec{x} : [0, 1] \to X \) is called the integral curve of \( \vec{X} \) if \( X(t) \) is the integral curve of \( \vec{X} \) with \( \vec{x}(0) = \vec{X} \).

Definition: Let \( \vec{X} \in \mathfrak{X}(M^n) \). A curve \( \vec{x} : [0, 1] \to X \) is called the integral curve of \( \vec{X} \) if \( \forall t \in [0, 1] \)

\[ \frac{d \vec{x}}{dt}(t) = \vec{X}(t) . \]

Theorem: Let \( G \) be a Lie group.

\[ \forall \vec{X} \in T_e G \exists \ a \ unique \ 1- \ parameter \]
Subgroup of $G$ s.t. $\gamma'(0) = \mathbf{1}$.

**Def.** A 1-parameter subgroup of $G$ is a Lie group homomorphism $\gamma : \mathbb{R} \to G$ from the additive group of $\mathbb{R}$ into $G$.

**Def.** (Exponential map) The exponential map $\exp : B \to G$ is the map defined by

$$\exp (X) = \exp X (1),$$

where $\exp X (\cdot)$ denotes the integral curve of $X$ through $e$ with $\exp X (0) = e$, $\exp X' (0) = X e$. 
Proposition Let $G$ be a Lie group, $X \in \mathfrak{g}$, $s,t \in \mathbb{R}$, $g \in G$.

(1) $\exp(tX) = \exp_s(t)$ $\forall t \in \mathbb{R}$.

(2) $\exp((t+s)X) = \exp(sX) \cdot \exp(tX)$

(3) $L_g \cdot \exp_x(\cdot)$ is the unique integral curve of $X$ through $g$.

(4) Let $P_t$ be flow of $X$. Then $P_t = R_{\exp_x(t)}$.

Proof. $\forall g \in G$, $P_t(g) = \gamma_g(t)$, where $\gamma_g$ is the integral curve of $X$ through $g$.

$\gamma_g(t) = L_g \left( \exp_x(t) \right)$

$= g \cdot \exp_x(t)$

$= R_{\exp_x(t)}(g)$

$P_t(g) = R_{\exp_x(t)}(g)$ \qed
Example

\[ \text{SL}(n) = \left\{ A \in \text{GL}(n) \mid \det(A) = 1 \right\} \]
\[ \mathfrak{sl}(n) = \left\{ \text{tr}(A) = 0 \right\}. \]

\( x(t) = \exp(tA) \) is a 1-parameter subgroup of \( \text{SL}(n) \).

\[ \det(x(t)) = 1 \]
\[ \det(\exp(tA)) = \exp(\text{tr}(tA)) \forall t \in \mathbb{R} \]
\[ \text{tr}(A) = 0 \]

Ex: The Lie algebra of \( \text{O}(n) \) is \( \mathfrak{so}(n) \).
**Ex** $SU(n), \quad U(n)$
$\begin{array}{c}
\text{SU(n)} \\
\text{U(n)}
\end{array}$

(A.A* = Id)

**Definition**

A Riemannian metric $\langle \cdot, \cdot \rangle$ is called left invariant if any left translation is isometric, i.e.,

$$\langle u, v \rangle_g = \langle (DL_h)_g u, (DL_h)_g v \rangle_h$$

for all $g, h \in G$, $u, v \in T_g G$.

Any inner product $\langle \cdot, \cdot \rangle_e$ on $T_e G$ can be extended to be a left invariant metric on $G$ by defining

$$\langle u, v \rangle_g = \langle (DL_{s^{-1}})_g u, (DL_{s^{-1}})_g v \rangle_e$$

**Definition** Bi-invariant:

both left invariant and right invariant.
Theorem (Milnor) \( G \) Lie group

1. If \( G \) compact, then \( G \) admits a bi-invariant metric.
2. A Lie group admits a bi-invariant metric if \( G \cong K \times H \) where \( K \) is compact and \( H \) is Abelian.

Proposition \( \langle \cdot , \cdot \rangle \) bi-invariant metric on \( G \). \( \forall X,Y,Z \in G \)

1. \( \langle [X,Y], Z \rangle + \langle Y, [X,Z] \rangle = 0 \)
2. If a left invariant metric \( G \) satisfies (1), then it is a bi-invariant.
3. \( \Box_X X \equiv 0 \). That is, any 1-parameter subgroup of \( G \) is a geodesic.

\( \exp \equiv \text{Exp} \) Lie theoretic Riemannian
(4) $\nabla_{x} y = \frac{1}{2} [x, y]$

(5) $R(x_1, x_2, x) = \frac{1}{4} \langle [x, y], [w, w] \rangle$

Consequently, $\sec G \geq 0$

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An important example

Hopf fibration (Berger sphere)

$SU(2) = \left\{ \begin{bmatrix} 0 & -w \\ w & \bar{z} \end{bmatrix} \right\} \in \mathbb{C} \setminus \{0\}$

- $SU(2) \cong S^3$

- $\pi(\mathbb{C}) = \mathbb{C} \setminus \{0\} \cong S^2$

$S^1 \rightarrow S^3 \overset{\pi}{\rightarrow} S^2$
\((z, w) \mapsto (\Re(zw), \Im(zw), \frac{z^3 - (zw)^2}{2}) \in \mathbb{R}^3\)

\(X, Y, Z \in g\), s.t.


duals \(\theta_1, \theta_2, \theta_3\)

\(g = t^2 \theta_1 \otimes \theta_1 + t_2 \otimes \theta_2 + t_3 \otimes \theta_3\)

\(\nabla_X X = 0\)

\(\nabla_X Y = (2-t^2)Z, \nabla_X Z = (2-t^2)Y\)

\(\nabla_Y X = -t^2Z, \nabla_Y Y = 0, \nabla_Y Z = X\)

\(\nabla_Z X = t^2 Y, \nabla_Z Y = -X, \nabla_Z Z = 0\)

\(\text{Sec}(X, Y) = t^2, \text{Sec}(X, Z) = t^2, \text{Sec}(Y, Z) = 4 - 3t^2\)
Let \( t \to 0 \) the direction \( X \) is collapsing since its integral curve has length \( 2\pi a \).

The "limiting metric" is the round metric of curvature \( +4 \) on the 2-sphere of radius \( \frac{1}{2} \).

\[
S^1 \times S^1 \rightarrow S^1
\]
\((\mathbb{R}^4, g_t) \xrightarrow{GH} (\mathbb{R}^3, g_0)\)