Introduction to hyperbolic surfaces

Exercises IV

For $\theta \in [0, 2\pi)$, $\lambda > 0$ and $t \in \mathbb{R}$, we consider

$$K_\theta = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}, \quad A_\lambda = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{bmatrix} \text{ and } N_t = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}.$$

Their corresponding Möbius transformations are ρ_θ, ϕ_λ and T_t respectively. Recall that Möbius transformations on $\mathbb{H}$ are orientation preserving isometries of $\mathbb{H}$.

Let B and C be matrices in $\mathrm{SL}(2,\mathbb{R})$. Recall that B and C are *similar* to each other in $\mathrm{SL}(2,\mathbb{R})$ if there is a matrix $P \in \mathrm{SL}(2,\mathbb{R})$, such that $B = PCP^{-1}$, (i.e. B can be obtained by taking the conjugation of C by P). In the following, by being similar, we always mean being similar in $\mathrm{SL}(2,\mathbb{R})$.

1. Show that for any matrices B and C in $\mathrm{SL}(2,\mathbb{R})$, we have

$$\text{(Easy)} \quad \operatorname{tr} B = \operatorname{tr} B^{-1}$$
$$\text{(Normal)} \quad \operatorname{tr} B \operatorname{tr} C = \operatorname{tr} BC + \operatorname{tr} BC^{-1}$$

   (Hint: Traces of matrices are invariant under conjugation.)

2. Let $M \in \mathrm{SL}(2,\mathbb{R})$. We would like to check if M and M^{-1} are similar to each other in a geometric way.

   a) (Easy) Show that for any geodesic, there is a Möbius transformation exchanging its two end points.

      (Hint: Ex III 3)

   b) (Normal) Use a) to show that any matrix M associated to a hyperbolic Möbius transformation is similar to its inverse M^{-1}.

      (Hint: Use a) on the axis of M to find a Möbius transformation, and consider its associated matrix.)

   c) (Hard) The orientation on $\mathbb{H}$ induces an orientation on each cycle and each horocycle (described by giving a positive rotation direction). Moreover this orientation on a cycle or a horocycle is preserved by orientation preserving isometries. Use this fact to show that

      i. If $M = N_t$, then M is similar to M^{-1}, if and only if $t = 0$.

      ii. If $M = K_\theta$, then M is similar to M^{-1}, if and only if $\theta \in \{0, \pi/2, \pi, 3\pi/2\}$.

   (Hint: Understand (i) the difference between the isometries of $\mathbb{H}$ induced by M and M^{-1}, and (ii) the geometric meaning of taking conjugation)

3. We consider the matrix

$$M = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}.$$

   Let f be the Möbius transformation associated to M. We would like to use hyperbolic geometry to find $J(M)$ the Jordan normal form of M.

a) (Easy) Use the trace to show that M is hyperbolic.

b) (Normal) Compute its eigenvalues μ and μ^{-1} with $\mu > 1$.

c) (Normal) Find the fixed points x_1 and x_2 of f with $x_1 < x_2$.

d) (Easy) Find the parabolic isometry T_t, such that $T_t(x_1) = -T_t(x_2)$, and write down the matrix corresponding N_t.

e) (Hard) Find an elliptic isometry sending $T_t(x_1)$ to 0 and $T_t(x_2)$ to ∞, and write down its matrix B.

   (Hint: Try to get the rotation around iy for an angle θ from the rotation around i for an angle θ)

f) (Easy) Compute $P = BN_t$ and verify that P satisfies:

$$PMP^{-1} = J(M).$$

g) (Easy) Compare $J(M)$ with A_μ.

h) (Easy) Let $P_\lambda = A_\lambda P$. Show that for any $\lambda > 0$, we have

$$P_\lambda M P_\lambda^{-1} = J(M).$$

4. We consider the following 3 subgroups of $\mathrm{SL}(2, \mathbb{R})$:

$$K = \{K_\theta \mid \theta \in [0, 2\pi)\},$$
$$A = \{A_\lambda \mid \lambda > 0\},$$
$$N = \{N_t \mid t \in \mathbb{R}\}.$$

The KAN decomposition (also called Iwasawa decomposition) of $\mathrm{SL}(2, \mathbb{R})$ states that: every $M \in \mathrm{SL}(2, \mathbb{R})$ can be written as a product $K_\theta A_\lambda N_t$ in a unique way (i.e. θ, λ and t are unique).

We would like to show this in a geometric way.

a) (Normal) By considering the algorithm that used for determining an isometry, show that for any matrix $M \in \mathrm{SL}(2, \mathbb{R})$ with $\operatorname{tr} M \geq 0$, we can find matrices K_θ, A_λ and N_t, such that $M = K_\theta A_\lambda N_t$, for some $\theta \in [0, \pi)$, $\lambda > 0$ and $t \in \mathbb{R}$.

b) (Easy) Show that $K_\pi = -Id$. (Hence the associated Möbius transformation is the identity map.)

c) (Normal) Show that for any $\theta \in [0, 2\pi)$ and $t \in \mathbb{R}$. If $K_\theta N_t$ preserves the vertical geodesic V_0, then we have $\theta = 0$, $\pi/2$ or π and $t = 0$.

   (Hint: Check if it preserves the end points up to exchange them.)

d) (Normal) Use c) to conclude that if $K_\theta A_\lambda N_t = A_\mu$, where $\theta \in [0, 2\pi)$, $\lambda > 0$, $\mu > 0$ and $t \in \mathbb{R}$, then we have $\theta = 0$, $\lambda = \mu$ and $t = 0$.

   (Hint: A necessary condition for being equal is that they preserve the same axis without exchanging the end points (i.e. the orientation on the axis).)

e) (Easy) Conclude that the KAN decomposition for any $M \in \mathrm{SL}(2, \mathbb{R})$ is unique.

   (Hint: K and N are subgroups of $\mathrm{SL}(2, \mathbb{R})$, hence are closed under multiplication.)