# Introduction to hyperbolic surfaces

## Exercises II

Let $l_{\mathbb{E}}$, $l_{\mathbb{H}}$ and $A_{\mathbb{H}}$ be the notations for the Euclidean length, the hyperbolic length and the hyperbolic area respectively. Let $H_y$ be the horizontal line passing $iy$ and $V_x$ be the vertical geodesic ending at $x$ and $\infty$.

1. (Easy) Let $C$ denote a circle in $\mathbb{H}$ with Euclidean center $z_{\mathbb{E}} = x + iy_{\mathbb{E}} \in \mathbb{H}$, of Euclidean radius $r$.
   a) Compute the hyperbolic radius $R$ of $C$ in term of $x$, $y_{\mathbb{E}}$ and $r$.
   b) For each $y_{\mathbb{E}}$, find $r$ such that $l_{\mathbb{H}}(C) = l_{\mathbb{E}}(C)$.

2. Recall the definitions and some properties of the hyperbolic cosine function and the hyperbolic sine functions: for $x$ and $y$ in $\mathbb{R}$, we have

$$\begin{aligned}\cosh x &= \frac{e^x + e^{-x}}{2}, \\ \sinh x &= \frac{e^x - e^{-x}}{2}, \\ 1 &= \cosh^2 x - \sinh^2 x, \\ \cosh(x+y) &= \cosh x \cosh y + \sinh x \sinh y, \\ \sinh(x+y) &= \cosh x \sinh y + \sinh x \cosh y.\end{aligned}$$

   These functions can be extended to $\mathbb{C}$. Using $e^{i\theta} = \cos\theta + i\sin\theta$ to verify the following equalities:

   a) (Easy) For any $\theta \in [0, 2\pi]$, we have

$$\begin{aligned}\sinh(i\theta) &= i\sin\theta, \\ \cosh(i\theta) &= \cos\theta.\end{aligned}$$

   b) (Easy) For any $x \in \mathbb{R}$, we have

$$\begin{aligned}\sin(ix) &= i\sinh x, \\ \cos(ix) &= \cosh x\end{aligned}$$

3. (Easy) Let $C_R$ denote a circle in $\mathbb{H}$ of hyperbolic radius $R$. Let $D_R$ denote the closed disk bounded by $C_R$. Let $l(R) = l_{\mathbb{H}}(C_R)$ and $A(R) = A_{\mathbb{H}}(D_R)$.

   a) Compute the following limits:

$$\lim_{R \to 0} l(R) - A(R).$$
$$\lim_{R \to +\infty} l(R) - A(R).$$

   b) Verify the following equality.

$$(l(R))^2 = 4\pi A(R) + (A(R))^2.$$

*Remark 0.0.1.*
We denote by $l$ the hyperbolic length of a closed curve bounding a simply connected region in $\mathbb{H}$, and by $A$ the area of this region. The *isoperimetric inequality* for hyperbolic plane is as follows:

$$l^2 \geq 4\pi A + A^2.$$

Moreover, the equality holds if and only if the region is a disk.

4. We would like to get the formula for the length of an arc in a circle, a horocycle or a hypercycle, and compare them.
   a) (Easy) Let $C$ be a circle in $\mathbb{H}$ of hyperbolic radius $R$. Let $c$ be an arc on $C$ with central angle $\theta$. Compute the length of $c$ in term of $R$ and $\theta$.
   b) (Normal) We consider horocycles $H_y$'s with center $\infty$. Let $c$ be an arc on $H_1$ between $V_0$ and $V_x$.
      i. Compute the length of $c$ in term of $x$.
      ii. Compute the distance $R$ between $H_y$ and $H_1$ in term of $y$.
      iii. Let $c_y$ denote the horocycle arc on $H_y$ between $V_0$ and $V_x$. Compute the length of $c_y$ in terms of $R$ and $x$.
   c) (Hard) Consider hypercycles of center $V_0$. We denote by $L_\theta$ the hypercycle having angle $\theta$ with $V_0$. We consider the radius geodesics $\gamma_r$ which is a geodesic with Euclidean center 0 and Euclidean radius $r$.
      i. Compute the distance $R$ between $L_\theta$ and $V_0$ in term of $\theta$.
      ii. Compute the distance $d$ between $\gamma_1$ ($r = 1$) and $\gamma_r$ in term of $r$.
      iii. Compute the length of the arc $c$ in $L_\theta$ between radius $\gamma_1$ and $\gamma_r$, in terms of $\theta$ and $r$. (Hint: use polar coordinates.)
      iv. Rewrite the length $c$ in term of $R$ and $d$.

(Hint: Draw pictures.)