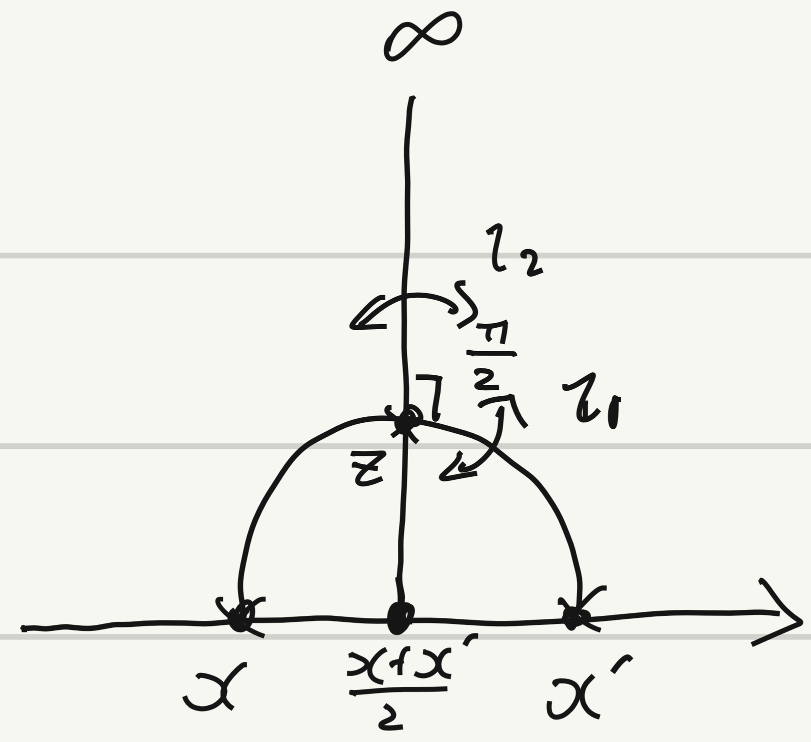
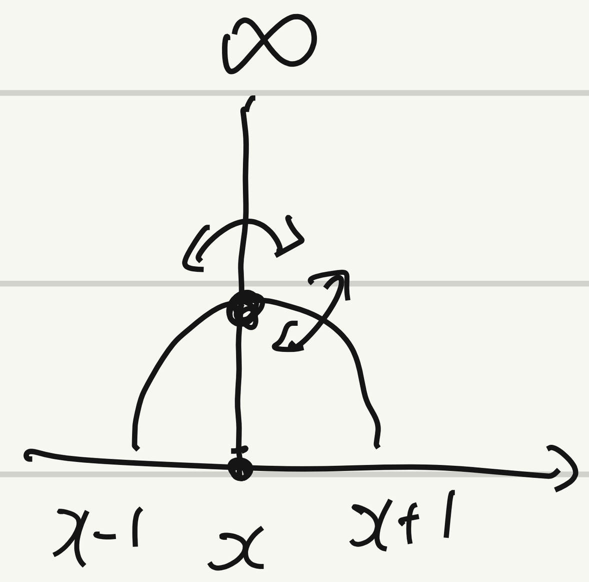
Ex 4.

2. a)





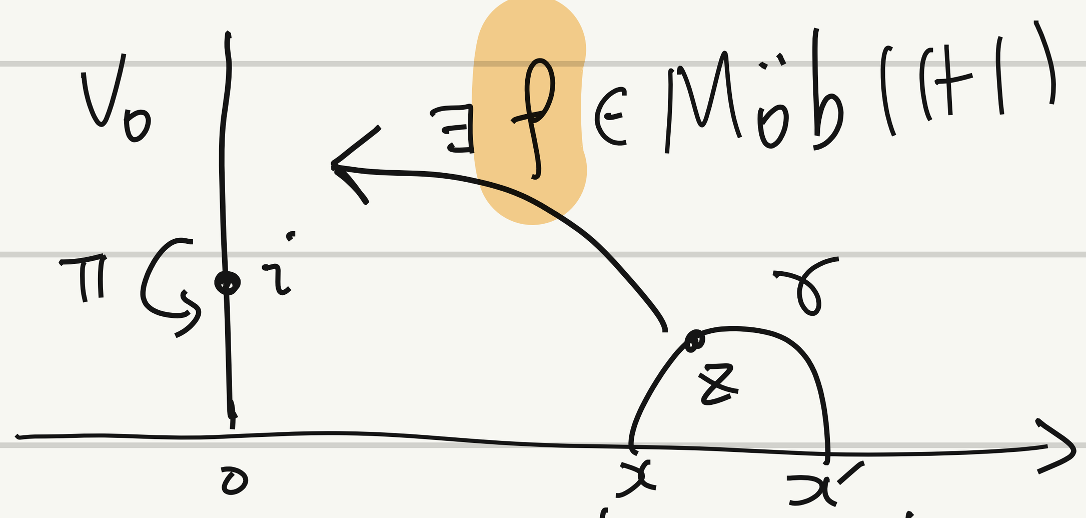
$C\left(\frac{x'+x}{2}, \frac{x'-x}{2}\right)$ $\quad l_1(z) = \dfrac{\frac{x'+x}{2}\bar{z} + \left(\frac{x'-x}{2}\right)^2 - \left(\frac{x'+x}{2}\right)^2}{\bar{z} - \frac{x'+x}{2}} = \dfrac{(x'+x)\bar{z} - 2xx'}{2\bar{z} - (x'+x)}$

$l_2(z) = -\bar{z} + (x+x')$ $\quad l_2 \circ l_1(z) = -\dfrac{(x'+x)z - 2xx'}{2z - (x+x')} + (x+x')$

$-\dfrac{(x'+x)x - 2xx'}{2x - (x+x')} + (x+x') \overset{?}{=} x'$

$= -\dfrac{x^2 - xx'}{x - x'} + (x+x')$

$= -x + (x+x') = x'$

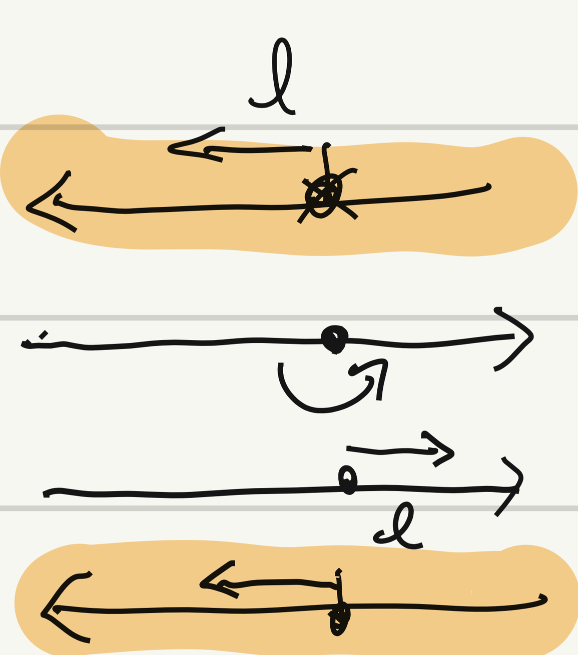


$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

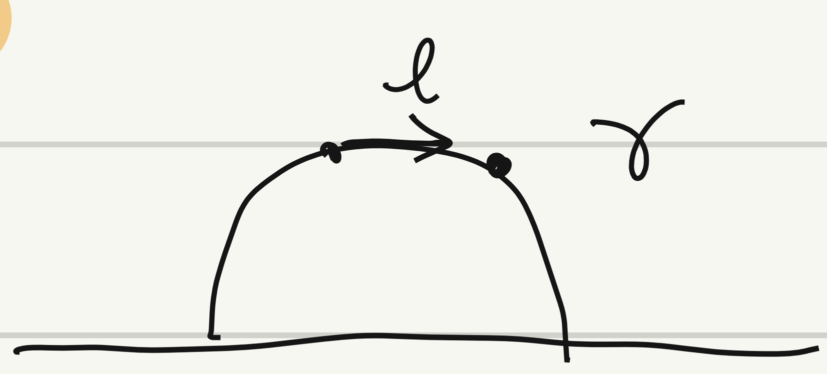
$\rho_{\frac{\pi}{2}}(z) = -\dfrac{1}{z}$ $\quad f^{-1} \circ \rho_{\frac{\pi}{2}} \circ f$ exchanges x and x'

b). $A \sim \begin{bmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{bmatrix}$ ✓ or $\begin{bmatrix} -\lambda & 0 \\ 0 & -\lambda^{-1} \end{bmatrix}$ $\quad f_A$.

$f \in \text{Möb}(\mathbb{H})$ $\exists P \in SL(2, \mathbb{R})$ s.t. $f = f_P$.



$f^{-1} \circ \rho_{\frac{\pi}{2}} \circ f = f_B$ $\quad B = P^{-1}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} P$ $\quad (B^2 = -Id \Rightarrow B^{-1} = -B)$



$f_{B^{-1}AB} = f_{A^{-1}}$ $\quad f = f_A \leftarrow \begin{matrix} A \\ -A \end{matrix}$ $\quad \text{tr} A = \text{tr} A^{-1}$

$\text{tr}\, B^{-1}AB = \text{tr}\, A.$ $\quad \left.\begin{matrix} M \in \{A^{-1}, -A^{-1}\} \\ \text{tr} M = \text{tr} A \neq 0 \end{matrix}\right\} \Rightarrow M = A^{-1}$

$\Rightarrow B^{-1}AB = A^{-1}$

证.

$A^{-1} = B^{-1}AB = BAB^{-1}$

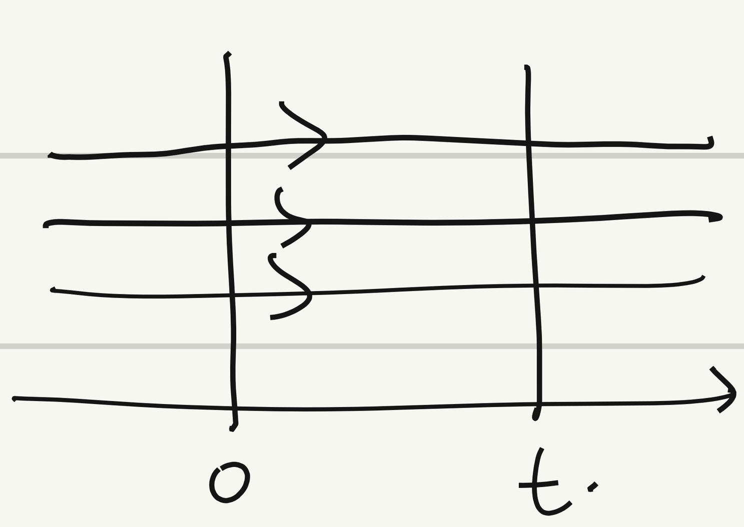
$\nexists\, f_0 \in \text{Möb}(\mathbb{H})$ $\quad f_{-A^{-1}}$

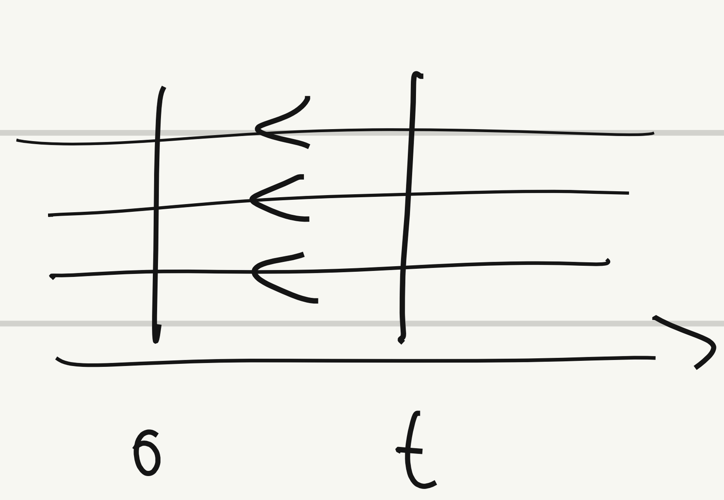
$f_0^{-1} \circ f_A \circ f_0 = f_A^{-1} = f_{A^{-1}}$

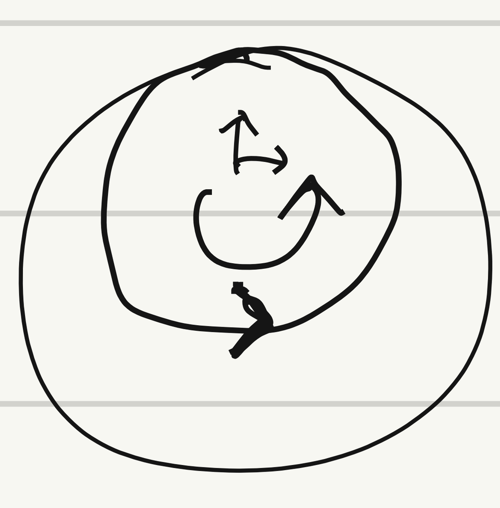
$\Rightarrow \nexists\, P$ s.t. $PAP^{-1} = \pm A^{-1}$

c) $A \sim \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$ or $A \sim \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$

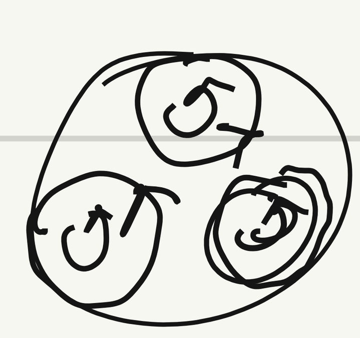
$A = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$ $\quad A^{-1} = \begin{bmatrix} 1 & -t \\ 0 & 1 \end{bmatrix}$ $\quad \forall f \in \text{Möb}(\mathbb{H})$







f presersenve orientation of $\mathbb{H}$



If $\exists P$ s.t. $PAP^{-1} = A^{-1}$ $\quad$ $t \neq 0$

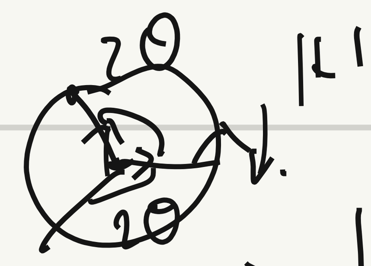
then $\underline{f_{PAP^{-1}}} = f_{A^{-1}}$

$$\begin{array}{c} \| \\ f_P \circ f_A \circ f_P^{-1} \end{array}$$

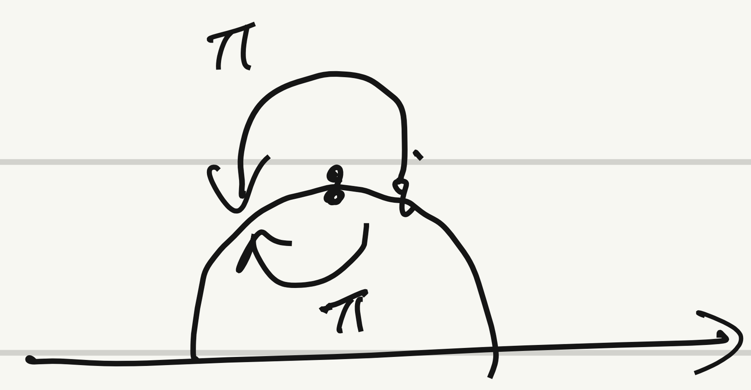
$\Rightarrow \exists f_P \in \text{Möb}(\mathbb{H})$ s.t. $f_P \circ f_A \circ f_P^{-1} = f_{A^{-1}}$ $\lightning$ $\Rightarrow \nexists P$ $PAP^{-1} = \pm A^{-1}$.

$$A \sim \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

Similarly $\Rightarrow$ $\forall \theta \notin \{0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}\}$, $\nexists P$ s.t. $PAP^{-1} = \pm A^{-1}$ $\quad \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$



$f_{\frac{\pi}{2}}$
$\|$
$f_{-\frac{\pi}{2}}$
$\in \text{Möb}(\mathbb{H})$



$$\begin{bmatrix} \cos\frac{\pi}{2} & \sin\frac{\pi}{2} \\ -\sin\frac{\pi}{2} & \cos\frac{\pi}{2} \end{bmatrix} = -\underbrace{\begin{bmatrix} \cos(-\frac{\pi}{2}) & \sin(-\frac{\pi}{2}) \\ -\sin(-\frac{\pi}{2}) & \cos(-\frac{\pi}{2}) \end{bmatrix}}_{\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}}$$

$\theta = 0$ $\quad A = Id$ $\checkmark$

$\underline{\theta = \frac{\pi}{2}}$ $\quad \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

$$= \begin{bmatrix} -b & a \\ -d & c \end{bmatrix}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} -bd-ac & b^2+a^2 \\ -d^2-c^2 & ac+bd \end{bmatrix}$$

$a^2+b^2 = -1$ $\lightning$. $\nexists P$ s.t. $PK_{\frac{\pi}{2}}P^{-1} = K_{-\frac{\pi}{2}}$.

$\theta = \frac{3\pi}{2}$ $\checkmark$

$\theta = \pi$. rotation of angle $2\pi \Rightarrow f_\pi$ action is trivial.

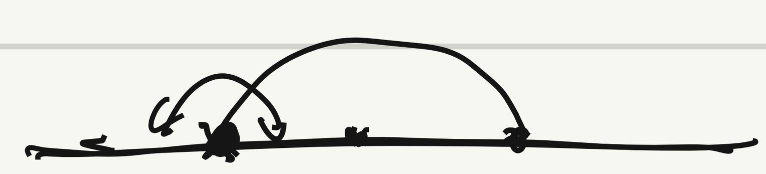
$K_\pi \in \{\lambda I_2 \mid \lambda \in \mathbb{R} \setminus \{0\}\} \cap SL_2\mathbb{R}$.

$\lambda = \pm 1$ $\quad K_\pi = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ $\quad \underline{K_\pi^{-1} = K_\pi}$

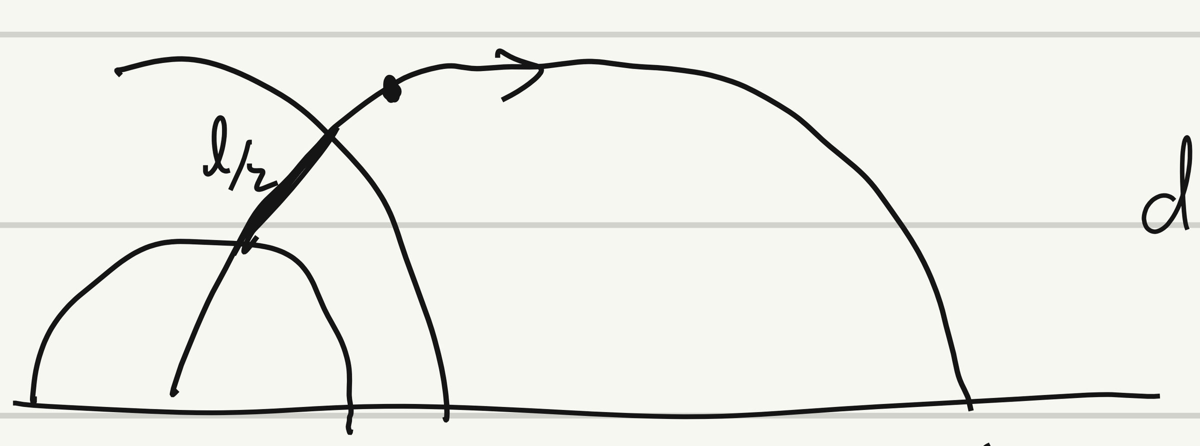
Conclusion:

① $A \sim \pm\begin{bmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{bmatrix}$ $\quad \exists P$ s.t. $PAP^{-1} = A^{-1}$

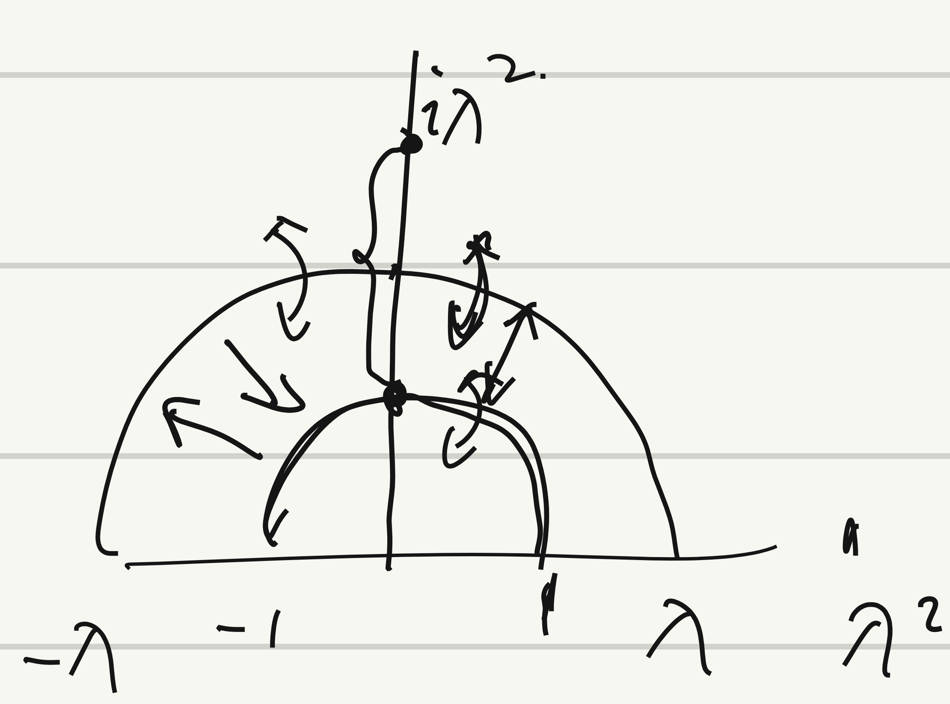
② $A \sim \pm\begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$ $t \neq 0$, $\nexists P$ s.t. $PAP^{-1} = A^{-1}$



③ $A \sim \pm\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ $\theta \neq 0, \pi$, $\nexists P$ s.t. $PAP^{-1} = A^{-1}$

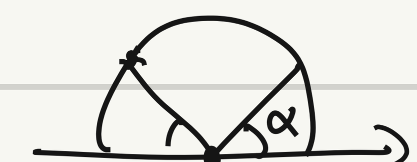


direction from x to x'.



$$f_3^{-1} \circ f_2 \circ f_1(z)$$
$$= f_1^{-1} \circ f_2 \circ (\lambda^{-1} z)$$
$$= f_1^{-1}\left(\frac{\lambda}{z}\right) = \frac{\lambda^2}{z}$$

$\frac{1}{z}$



$\boxed{0 < \alpha, \beta < \pi}$