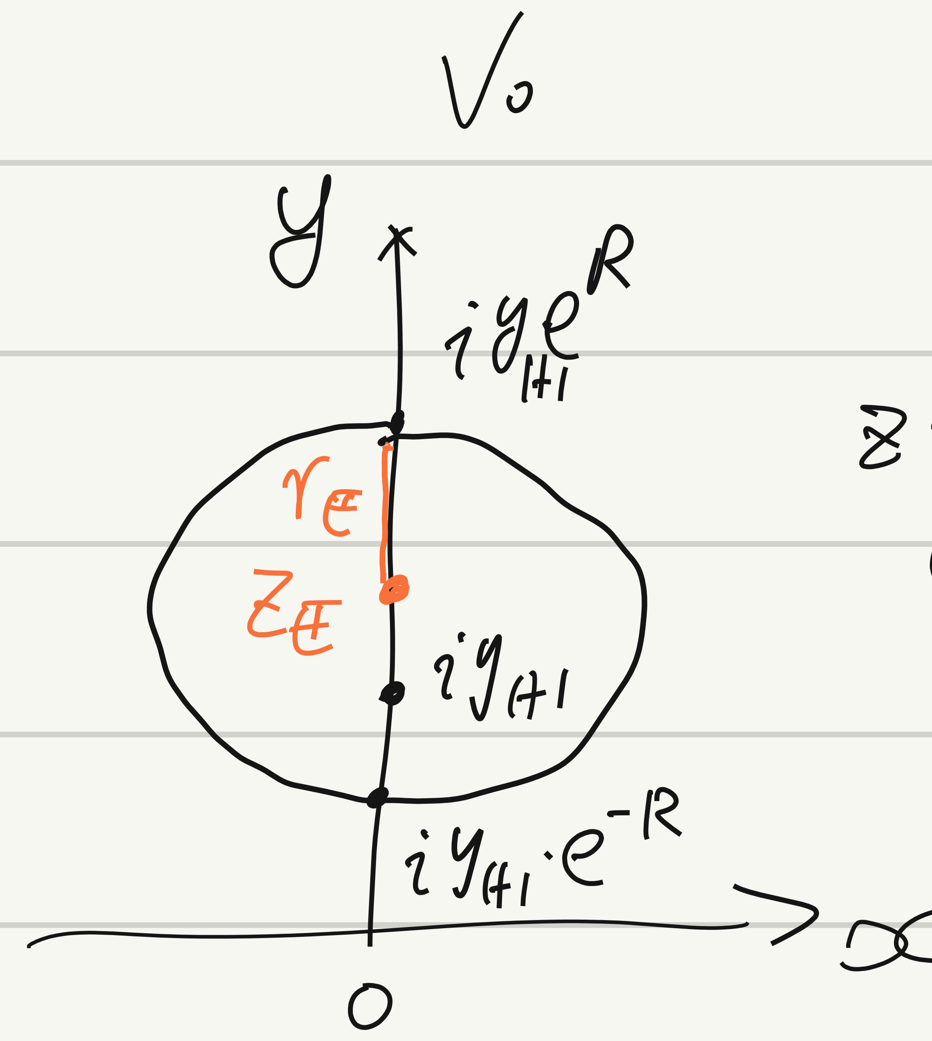
8 Horocycles and hypercycles

- Let $z_{\mathbb{H}} = iy_{\mathbb{H}}$, consider $C(z_{\mathbb{H}}, R)$

Then the Euclidean data:

$$z_{\mathbb{E}} = iy_{\mathbb{E}} \quad \begin{cases} y_{\mathbb{E}} = y_{\mathbb{H}} \cdot \mathrm{ch} R \\ r = y_{\mathbb{H}} \cdot \mathrm{sh} R \end{cases}$$

$\mathbb{E}$ $\mathbb{H}$



$z = x+iy$
(x, y)

$$\Rightarrow \bigstar \begin{cases} y_{\mathbb{H}} = \sqrt{y_{\mathbb{E}}^2 - r^2} \\ e^R = \sqrt{\dfrac{y_{\mathbb{E}} + r}{y_{\mathbb{E}} - r}} \end{cases} \qquad (y_{\mathbb{E}} > r)$$

- drop the ball in $\mathbb{E}$ and see how $y_{\mathbb{H}}$ and R change.

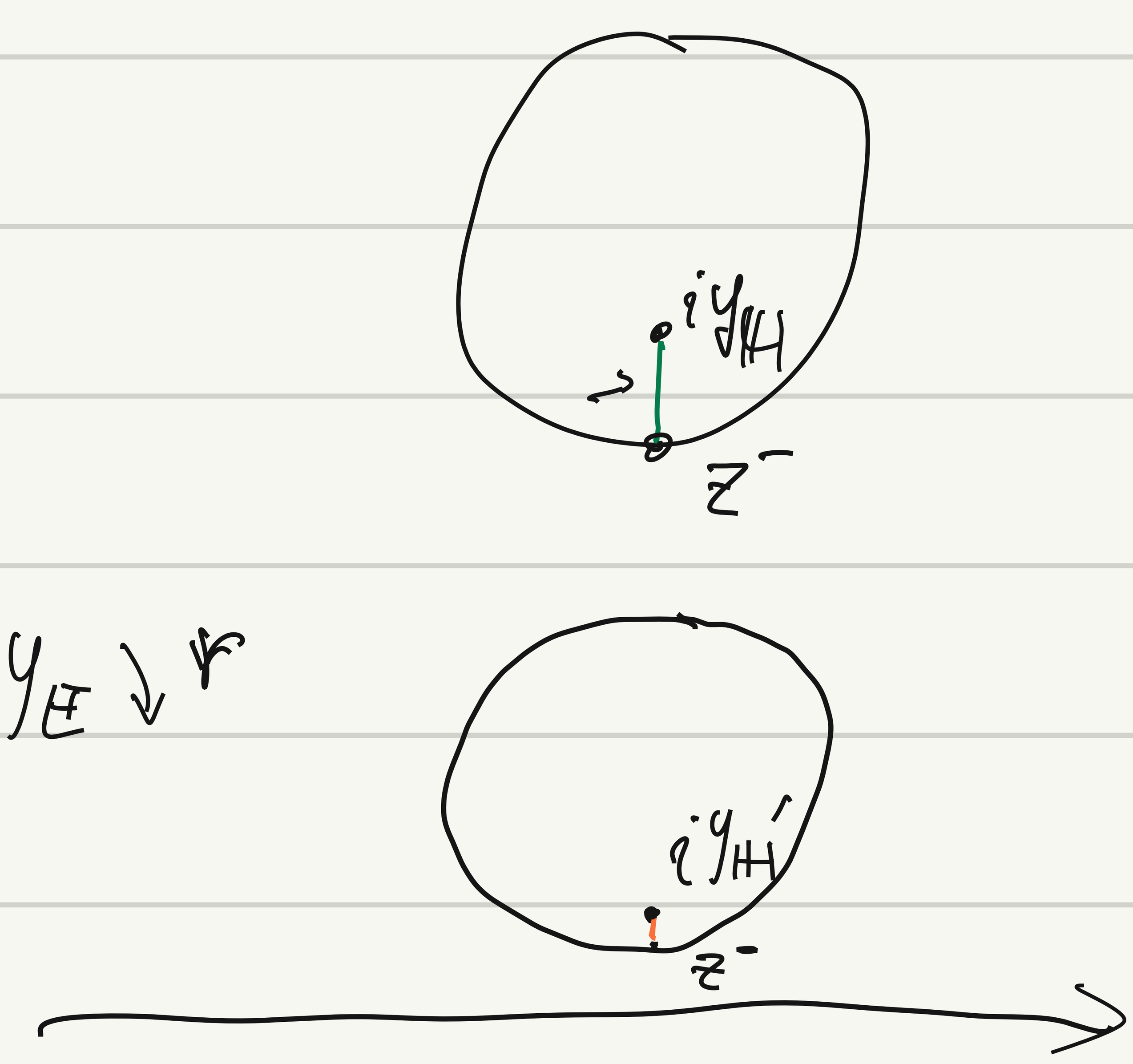
$r \equiv cst \qquad y_{\mathbb{E}} \downarrow r \quad (y_{\mathbb{E}} \geq r)$.

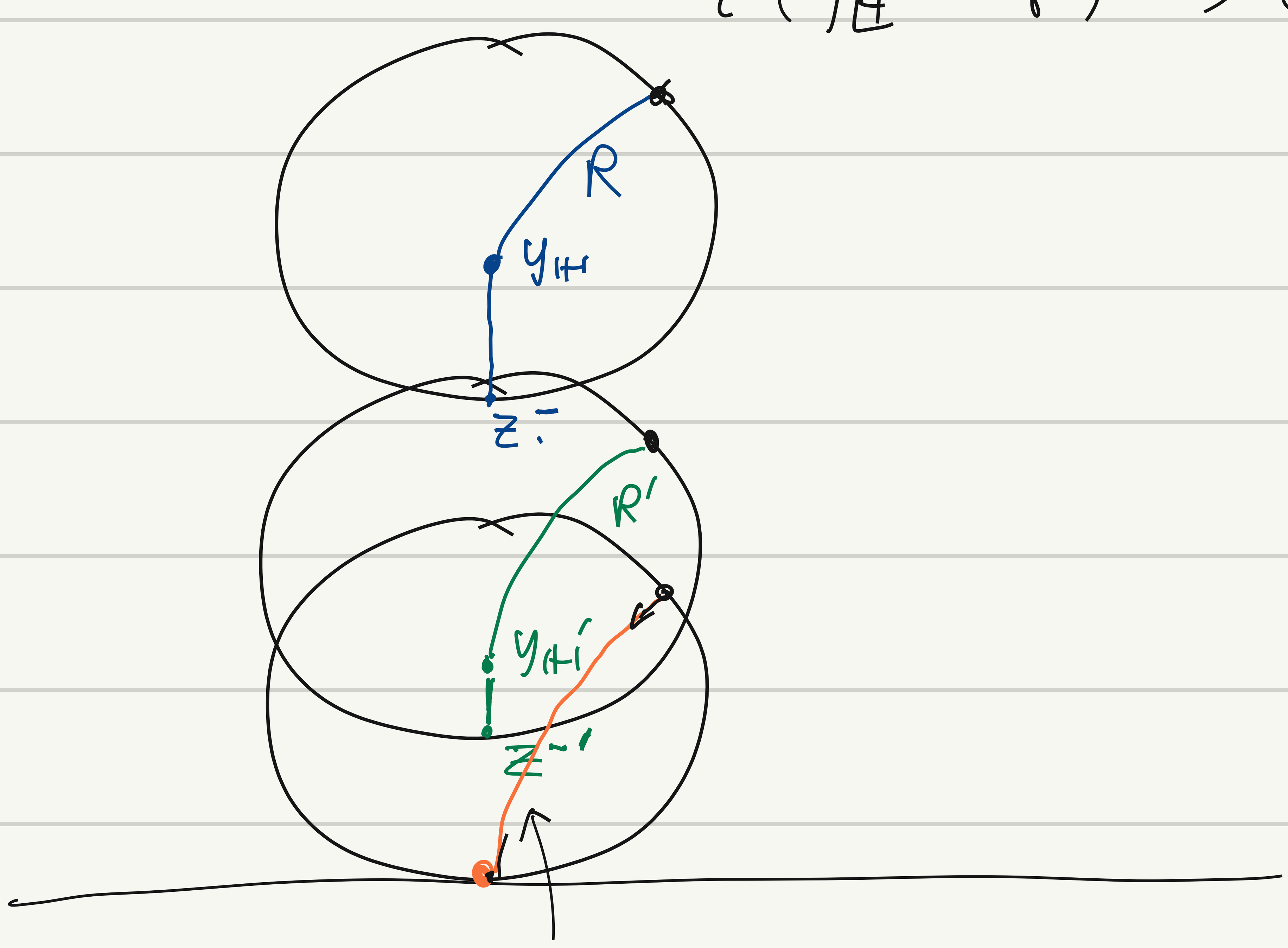
Ob1: $y_{\mathbb{H}} \downarrow 0$, as $y_{\mathbb{E}} \downarrow r$

Ob2: $R \nearrow \infty$, as $y_{\mathbb{E}} \downarrow r$

Ob3: $z_{\mathbb{H}} - z^- = iy_{\mathbb{H}} - iy_{\mathbb{H}}e^{-R}$

$= i(y_{\mathbb{E}} - r) \to 0$ as $y_{\mathbb{E}} \downarrow r$





Ob4:
$$e^R = \sqrt{\frac{y_{\mathbb{E}} + r}{y_{\mathbb{E}} - r}} \longrightarrow \infty, \quad \text{as } y_{\mathbb{E}} \to r$$

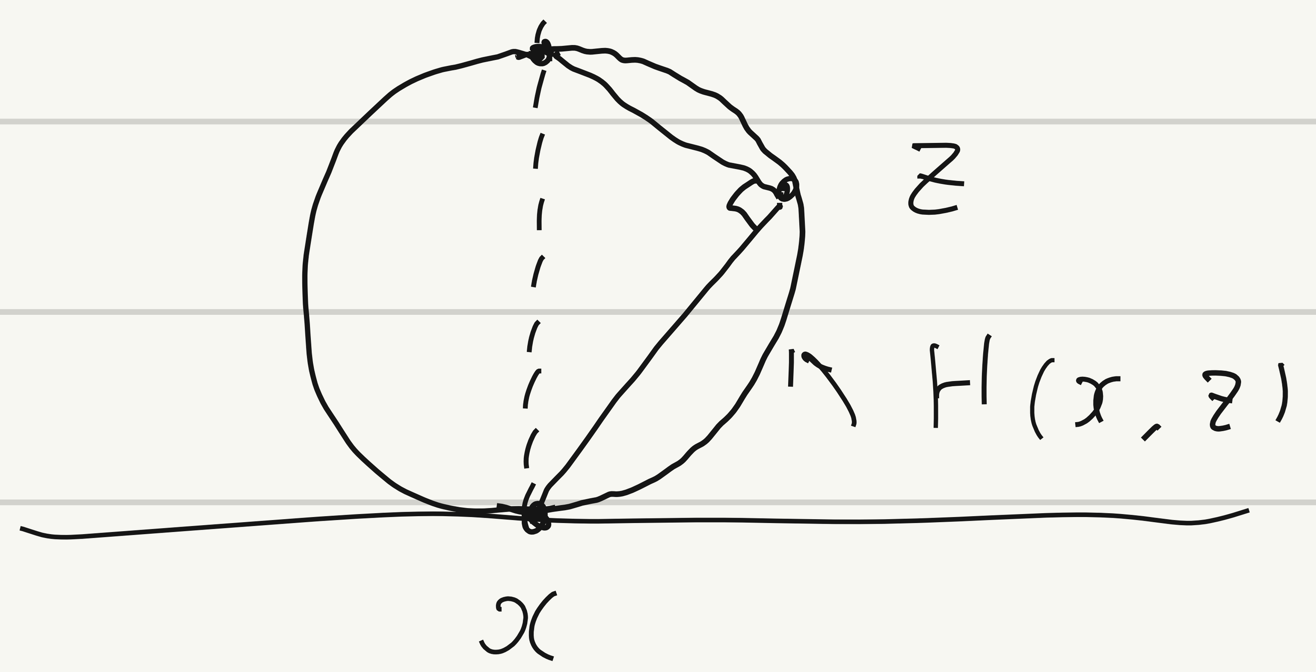
$\Leftrightarrow R \longrightarrow \infty$

When C and real axis tangent.

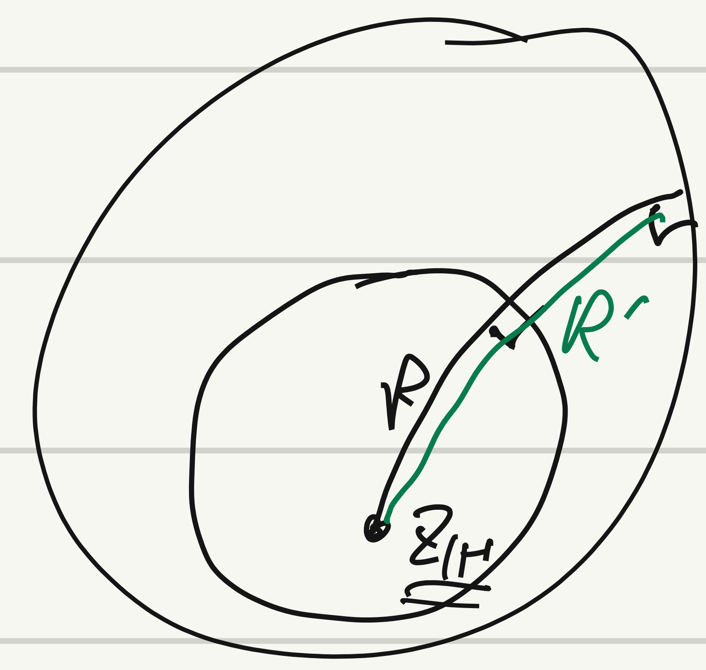
"center" at ∞, "radius" is ∞.

<u>Def</u>: We call such an Euclidean circle tangent to $\mathbb{R}$ at x passing $z \in \mathbb{H}$, a <u>horocycle</u> centered at x passing z.

<u>Ex</u>:



<u>Horocycles with the same center</u>



- $C(z_{\mathbb{H}}, R)$, $C(z_{\mathbb{H}}, R')$.

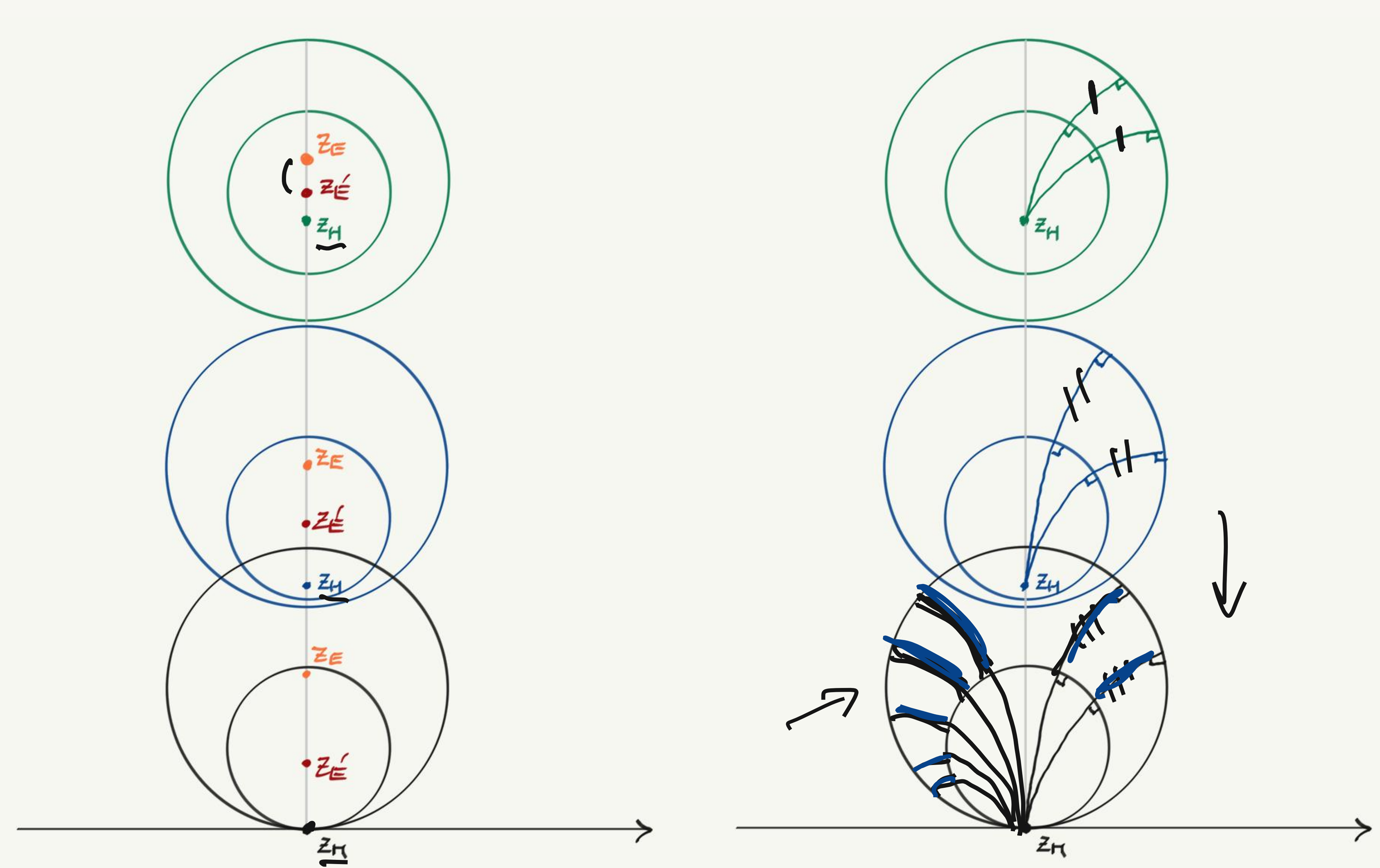
drop them keep hyperbolic cent incide.

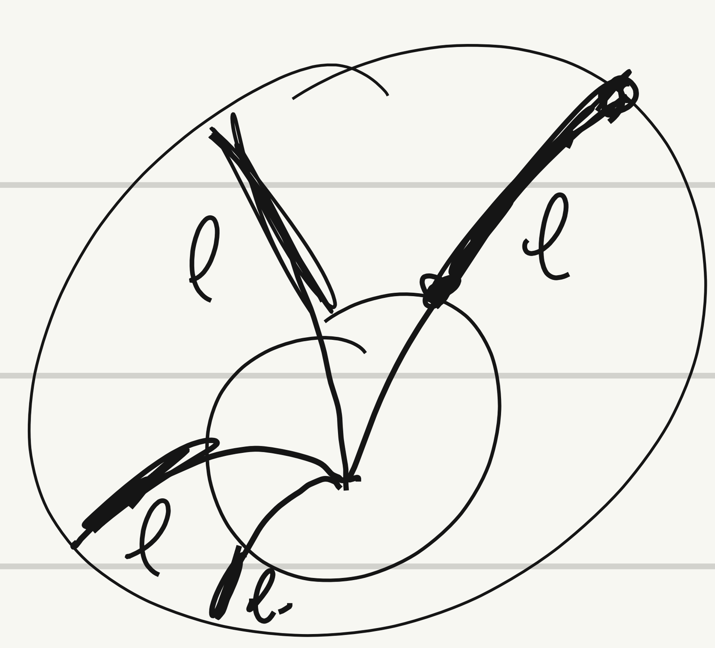
$$\begin{cases} y_{\mathbb{H}} = \sqrt{y_{\mathbb{E}}^2 - r^2} \\ e^R = \sqrt{\dfrac{y_{\mathbb{E}} + r}{y_{\mathbb{E}} - r}} \end{cases}$$

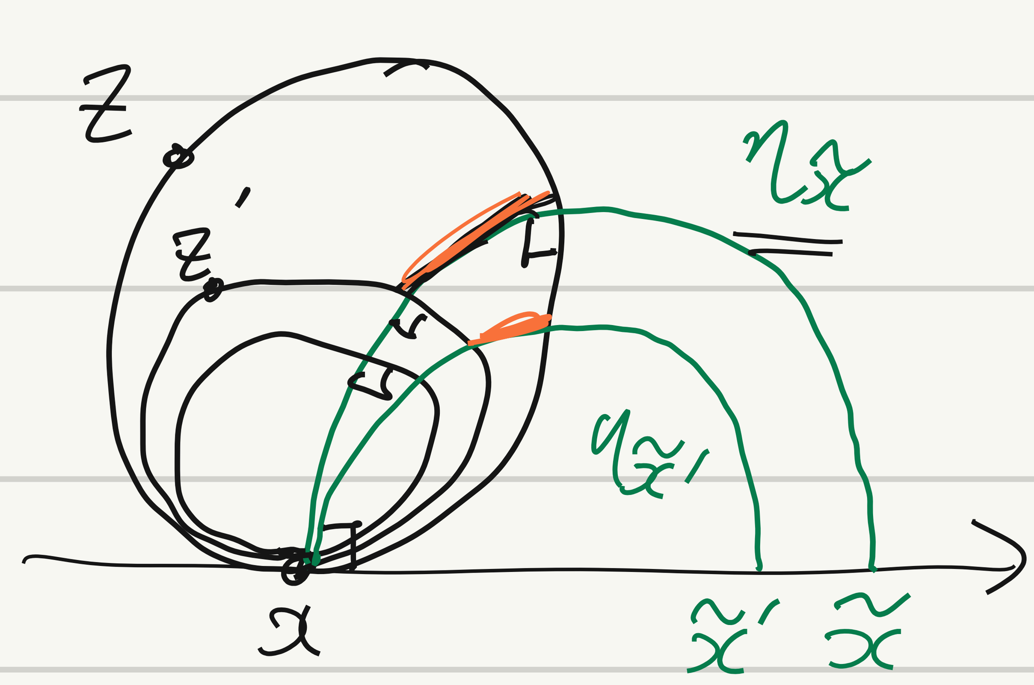
$C(z_{\mathbb{H}}, R)$ Euc. data $y_{\mathbb{E}}, r$

$C(z_{\mathbb{H}}, R')$ Euc data $y_{\mathbb{E}}', r'$

$$0 = y_{\mathbb{H}} = \sqrt{\underline{y_{\mathbb{E}}}^2 - r^2} = \sqrt{\underline{y_{\mathbb{E}}'}^2 - r'^2}$$







Let $x \in \mathbb{R}$, $H(x, z)$ be a horocycle.

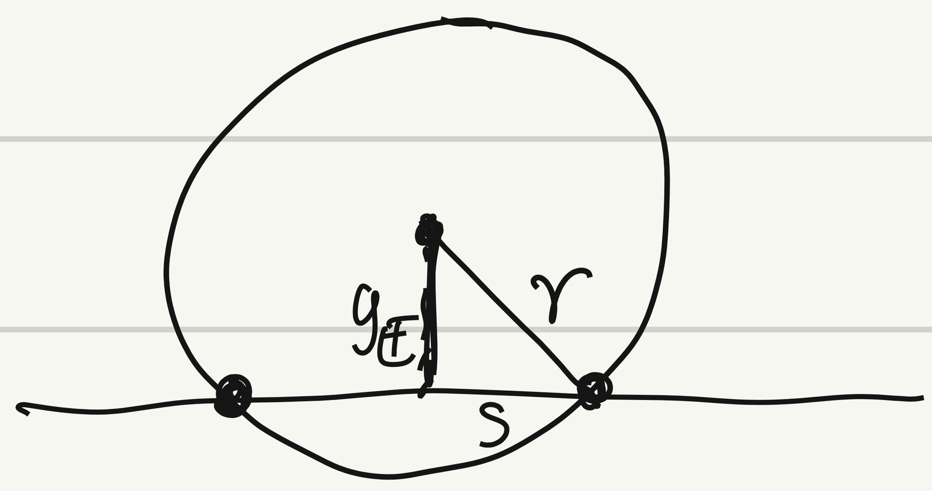
Let $\eta_{\tilde{x}}$ be a complete geod with endpts x and $\tilde{x}$

<u>Prop</u>: $\forall z$, $\forall \tilde{x} \neq x$, $\eta_{\tilde{x}} \perp H(x, z)$

<u>Prop</u>: $\forall z, z' \in \mathbb{H}$, $d_{\mathbb{H}}\left(\eta_{\tilde{x}} \cap H(x,z), \eta_{\tilde{x}} \cap H(x,z')\right) = c^{st}$ indep of $\tilde{x}$

Hyperaycle:

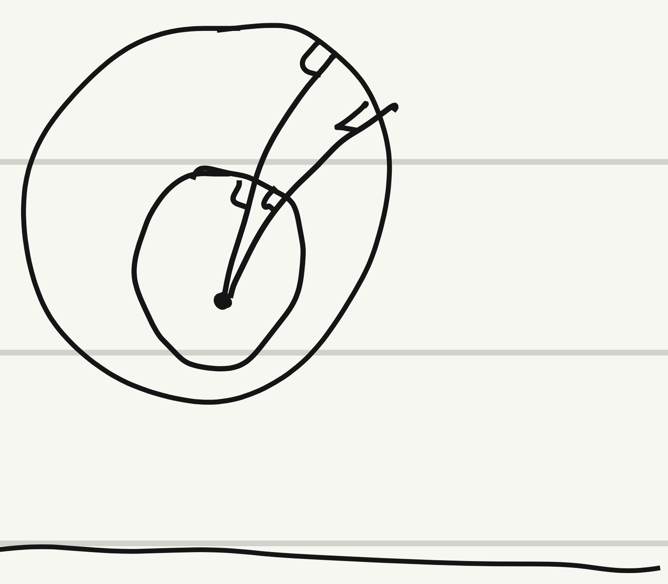
Let the Euclidean circle fall beyond $\mathbb{R}$ $\quad (0 < y_E < r)$

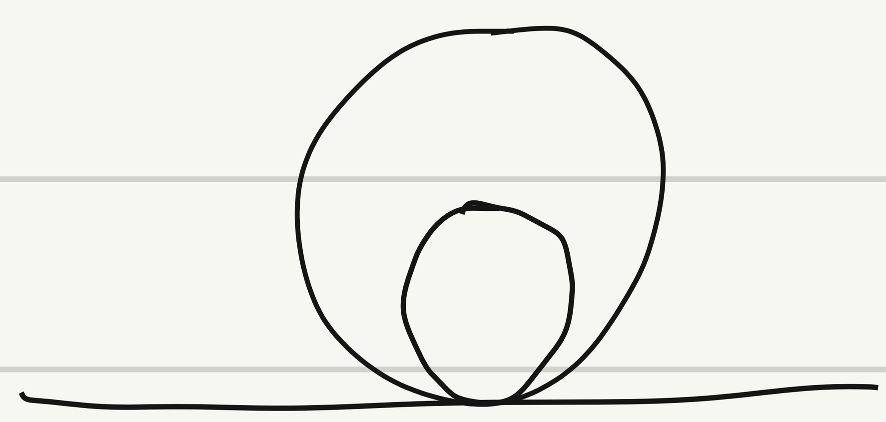


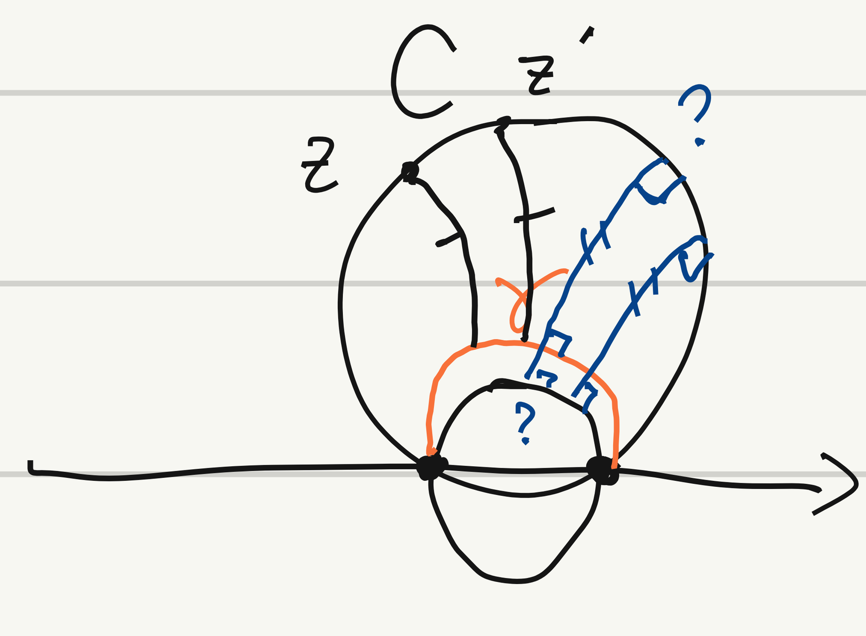
$$s = \sqrt{r^2 - y_E^2}$$

$$y_{\mathbb{H}} = \sqrt{y_E^2 - r^2} = \sqrt{y_E'^2 - r'^2}$$

$$r^2 - y_E^2 = r'^2 - y_E'^2 = s^2$$

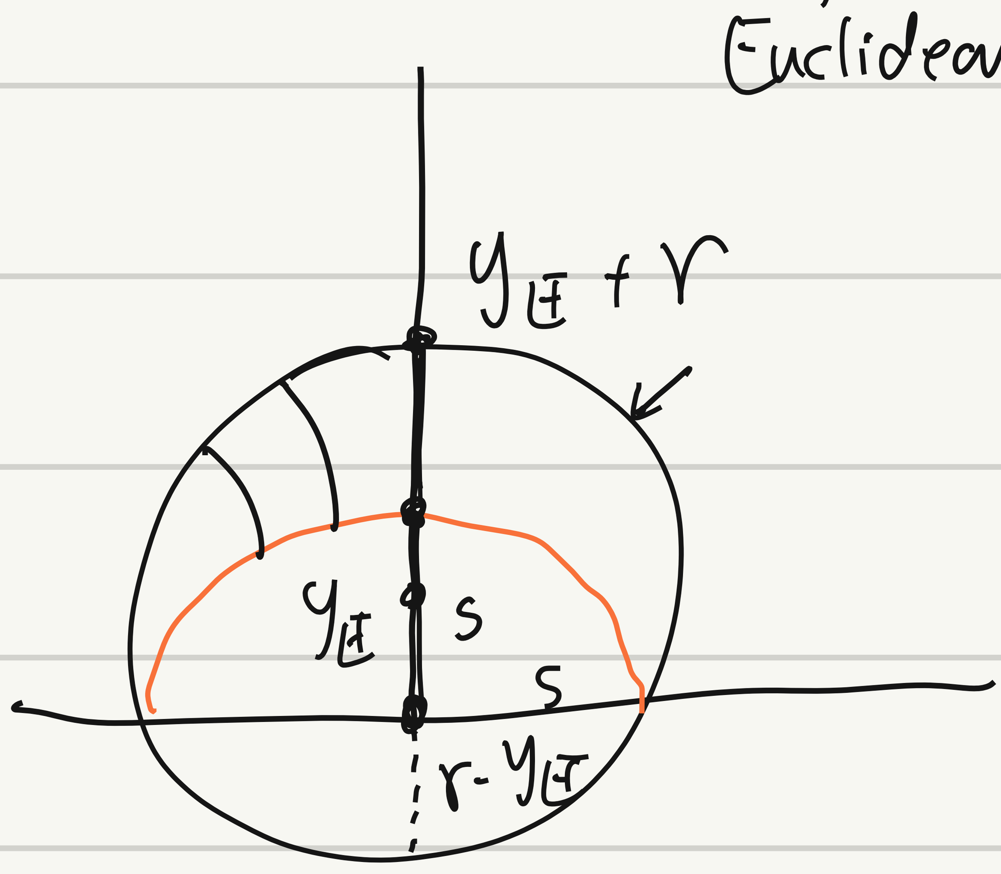






Prop: $\forall\, z, z' \in C \cap \mathbb{H}$, $\quad d_{\mathbb{H}}(z, \gamma) = d_{\mathbb{H}}(z', \gamma) = \log \dfrac{y_E + r}{s}$

$\uparrow$ Euclidean circle



$$s = \sqrt{r^2 - y_E^2}$$

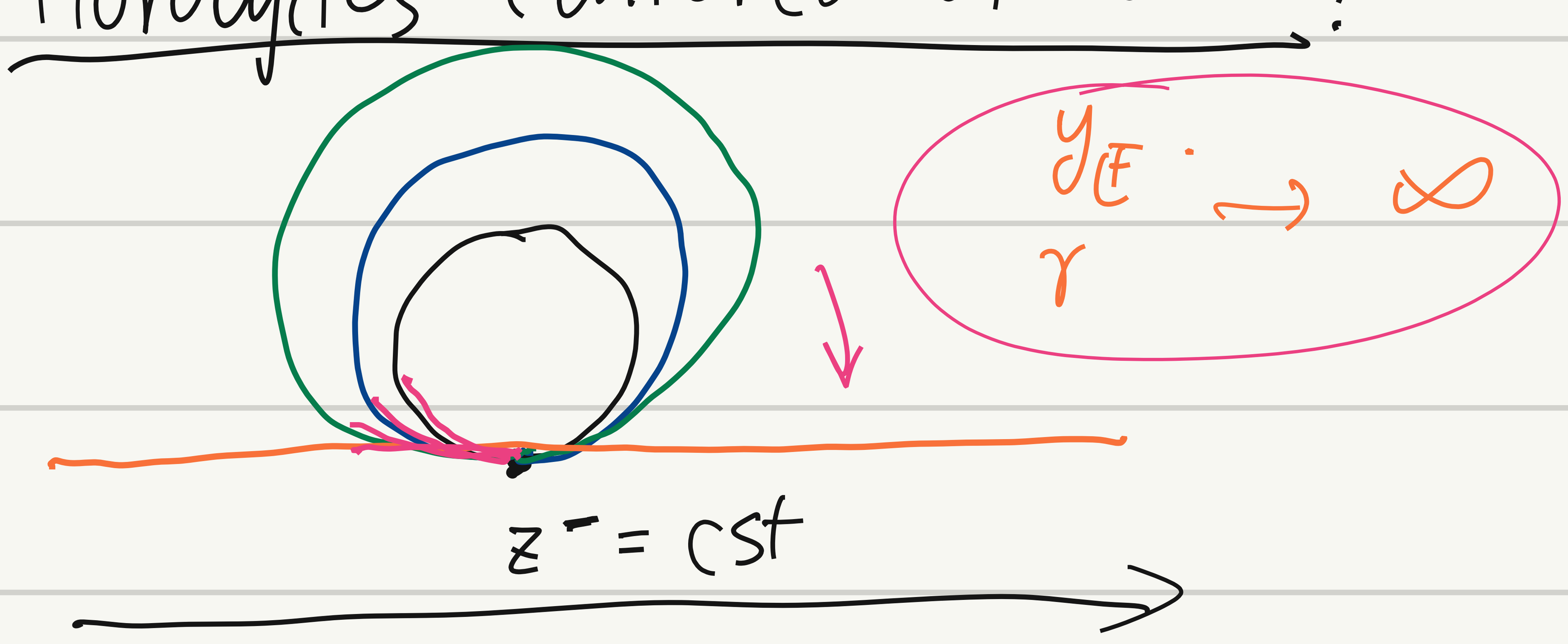
$$= \log \sqrt{\frac{(y_E + r)^2}{r^2 - y_E^2}}$$

$$= \frac{1}{2} \log \frac{r + y_E}{r - y_E}.$$

Def: $C \cap \mathbb{H}$ is called a hypercycle with "center" γ.

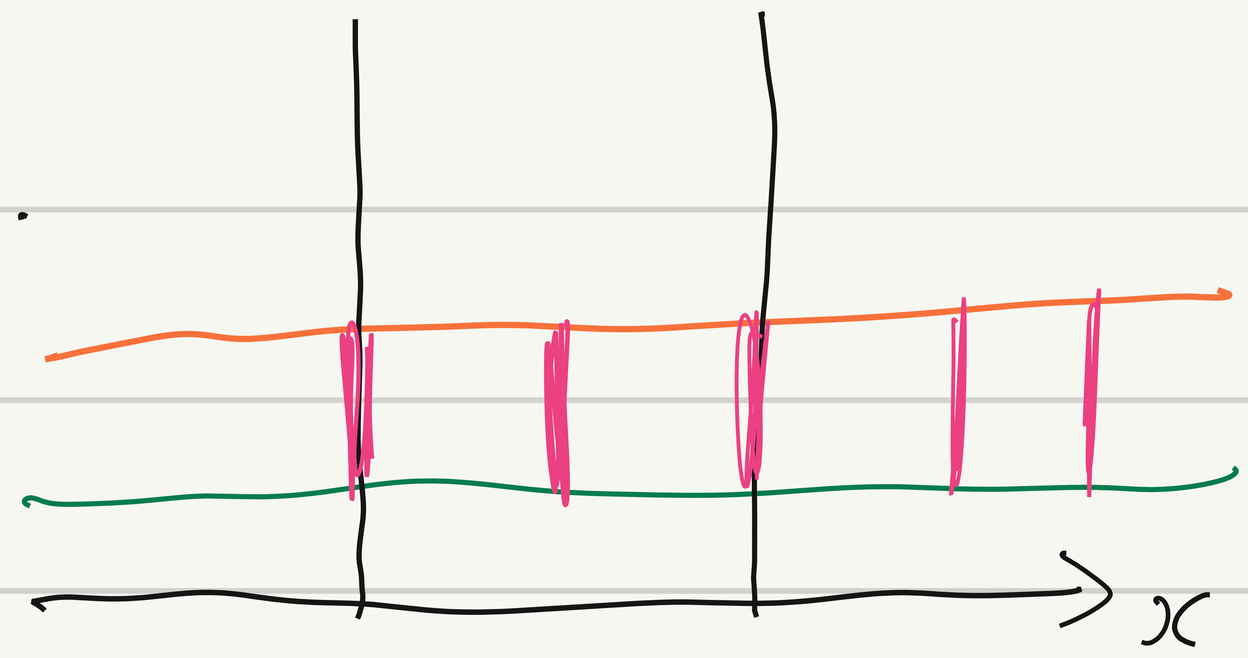
Rmk "equidistance curve."

Horocycles centered at ∞:

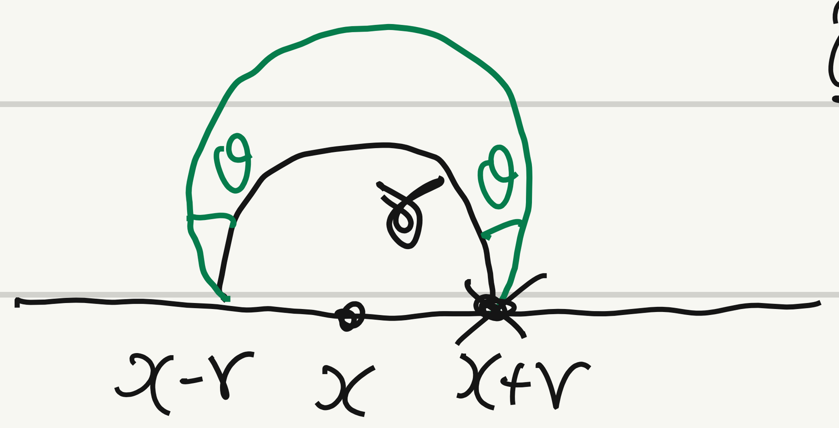


Def: A horocycle centered at ∞ is a horizontal line.

Ex.

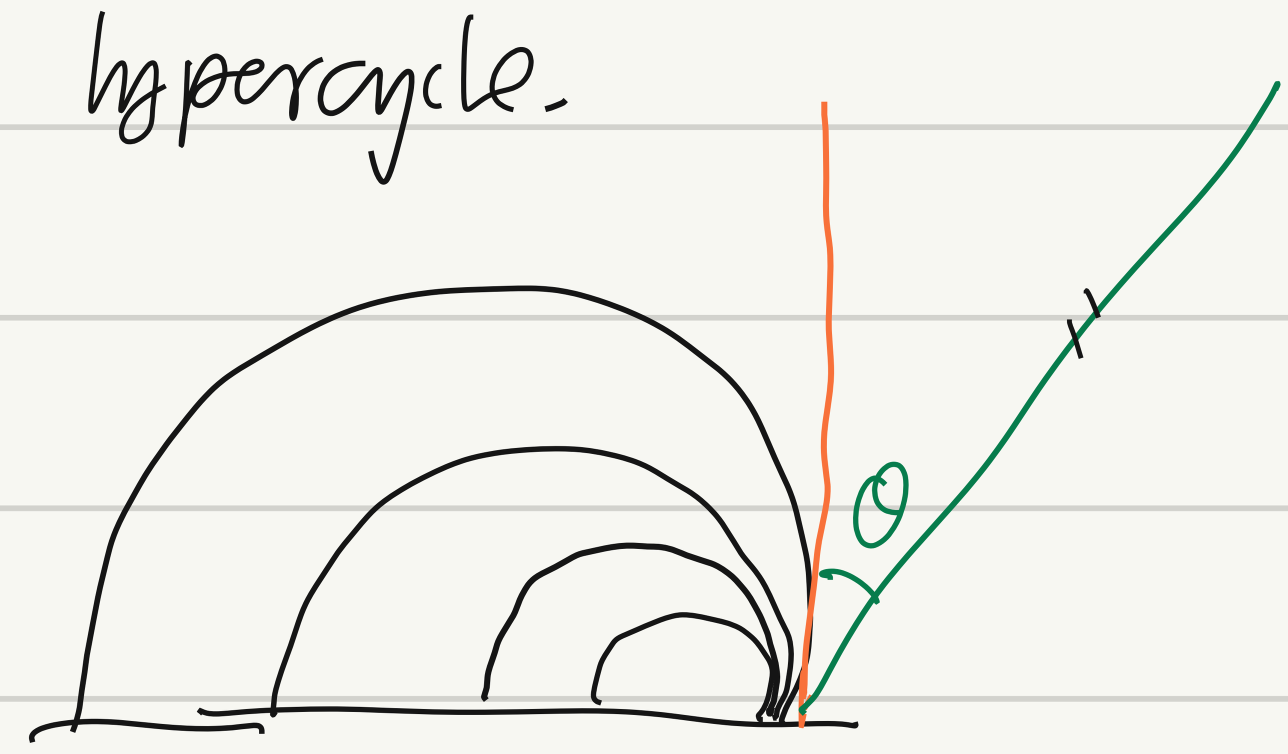


Hypercycles centered at V_x



Ob: θ determine the hypercycle.

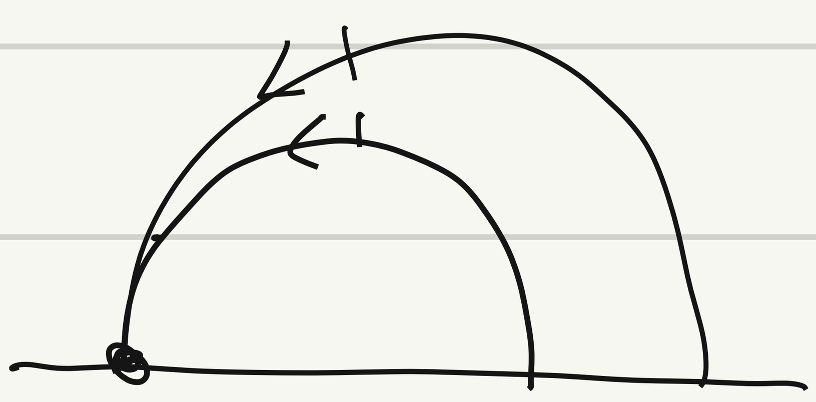
$x+r = cst$ $\quad$ $x \to -\infty$.



Def: A hypercycle centered at V_x is an Euclidean ray issued from x.

9. Boundary at infinity of $\mathbb{H}$ (ideal boundary)

γ η parallel.



"same direction"

↳ "bounded distance"

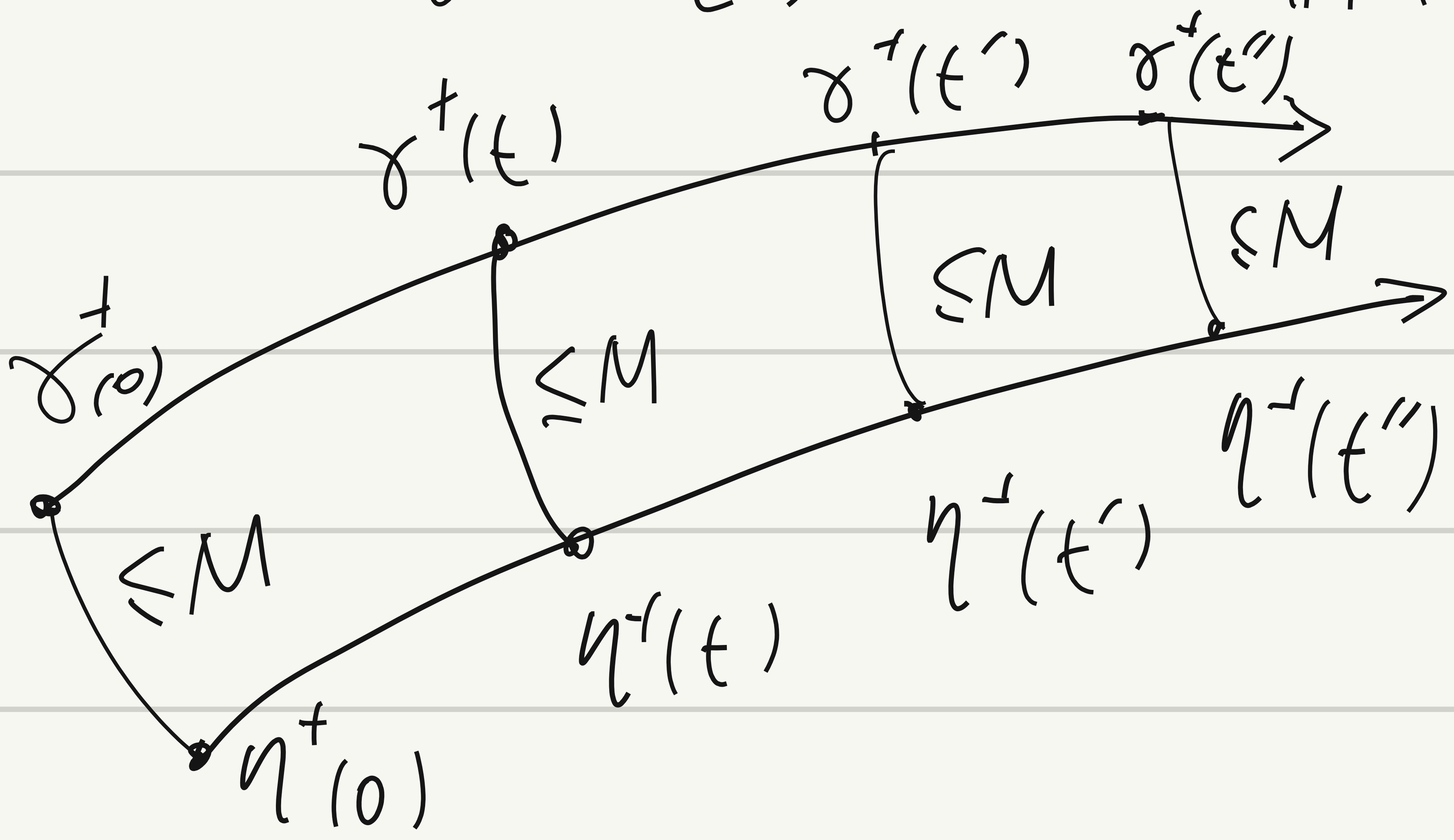
Let γ be a geod in $\mathbb{H}$, $z \in \gamma$.

Let $\underline{\gamma_z^+}: [0, \infty) \to \mathbb{H}$ be a half geod starting from z

- $\gamma_z^+(0) = z$
- $d_{\mathbb{H}}(\gamma_z^+(t), \gamma_z^+(t')) = |t - t'|$

Def: We say that $\gamma^+ \sim \eta^+$ if $\exists M > 0$ s.t. $\forall t \in [0, \infty)$, $d_{\mathbb{H}}(\gamma^+(t), \eta^+(t)) \le M$ (bounded distance)

Ex:



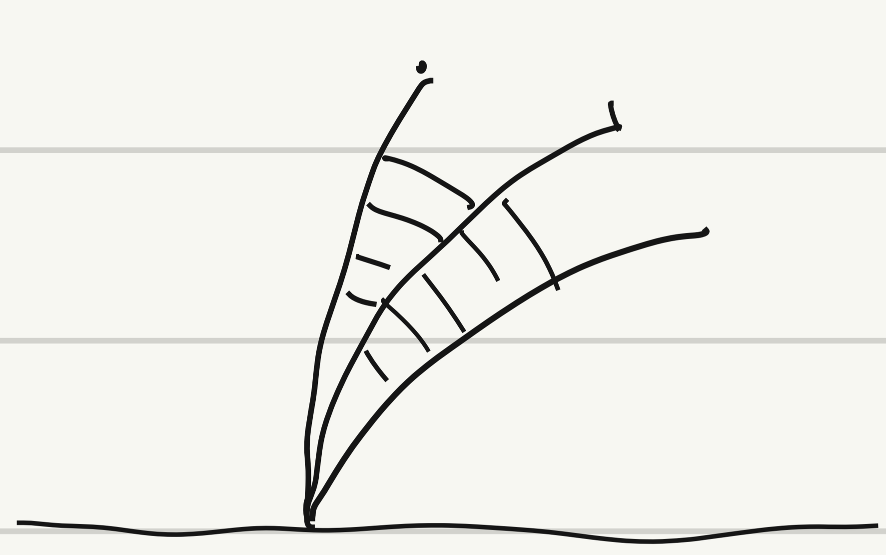
Prop: This defines an equi relation among all rays in $\mathbb{H}$!

$$\left| \begin{array}{l} \gamma^+ \sim \gamma^+ \\ \gamma^+ \sim \gamma'^+ \Rightarrow \gamma'^+ \sim \gamma^+ \\ \gamma^+ \sim \gamma'^+,\ \gamma'^+ \sim \gamma''^+ \Rightarrow \gamma^+ \sim \gamma''^+ \end{array} \right.$$

use triangular inequality.

Def: The boundary at infinity of $\mathbb{H}$ (ideal boundary) is defined to be

$$\{\gamma^+ \mid \gamma^+ \text{ ray in } \mathbb{H}\} / \sim$$



$\partial \mathbb{H}$ notation. $\forall x \in \partial \mathbb{H}$, we call x an ideal point.

Prop: $\partial \mathbb{H} \cong \hat{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$

next time.