Introduction to hyperbolic surfaces

Exercises V

1. We consider the map $f_{\mathbb{D}}(z) = (z-i)/(z+i)$ from $\mathbb{H}$ to $\mathbb{D}$, and the matrix

$$A_{\mathbb{D}} = \begin{bmatrix} 1 & -i \\ 1 & i \end{bmatrix}.$$

   a) (Easy) Show that f can be extended to $\widehat{\mathbb{R}}$, and sends $\widehat{\mathbb{R}}$ to the unit circle.

   (Hint: To compute $f(\infty)$, consider a sequence z_n with $|z_n| \to +\infty$, as $n \to +\infty$, then take the limit of $f(z_n)$ as the image $f(\infty)$.)

   b) (Easy) Show that $A_{\mathbb{D}}\mathrm{SL}(2,\mathbb{R})A_{\mathbb{D}}^{-1} = \mathrm{U}(1,1)$, i.e.

      i. For any matrix $A \in \mathrm{SL}(2,\mathbb{R})$, we have $A_{\mathbb{D}}AA_{\mathbb{D}}^{-1} \in \mathrm{U}(1,1)$;

      ii. For any matrix $B \in \mathrm{U}(1,1)$, there is a matrix $A \in \mathrm{SL}(2,\mathbb{R})$ such that $A_{\mathbb{D}}AA_{\mathbb{D}}^{-1} = B$.

2. Let γ and η be a pair of disjoint geodesics..

   a) (Normal) Using the extreme value theorem and the convexity of the distance function, show that for any $z \in \gamma$, the distance $\mathrm{d}_{\mathbb{H}}(z,\eta)$ can be realized by a unique point $w_z \in \eta$.

   b) (Normal) Using the extreme value theorem and the convexity of the distance function, show that $\inf\{\mathrm{d}_{\mathbb{H}}(z,\eta) \mid z \in \gamma\}$ can be realized by a unique point $z_0 \in \gamma$.

   c) (Easy) Conclude that the distance $\mathrm{d}_{\mathbb{H}}(\gamma,\eta)$ are realized by a unique pair of points $(z_0, w_{z_0}) \in \gamma \times \eta$.

3. (Hard) Let $z_1, \ldots, z_n$ be n distinct points in $\mathbb{H}$ with $n > 2$. We define a function d on $\mathbb{H}$ as follows:

$$d(z) = \sum_{j=1}^{n} \mathrm{d}_{\mathbb{H}}(z, z_j).$$

   Using the extreme value theorem and the convexity of distance function, show that the infimum of $d(\mathbb{H})$ can be realized by a unique point in $\mathbb{H}$.

   (Hint: To use the extreme value theorem, we can separate $\mathbb{H}$ into two parts. One is R neighborhood U of $\{z_1, \ldots, z_n\}$ for R large so that U is simply connected and $R > d(z_1)$. The other one is the complement of U.)

4. We would like to compute some trigonometry formulas:

   a) (Easy) Let $\alpha \in (0,\pi)$. Consider the triangle with vertices $z_1 = \infty$, $z_2 = i$ and $z_3 = e^{i\alpha}$. Let l denote the length of the side I_1. Use the distance formula to show:

$$\cosh l \sin\alpha = 1$$

b) (Normal) Let $\alpha, \beta \in (0, \pi)$. Consider the triangle with vertices $z_1 = \infty$, $z_2 = e^{i\alpha}$ and $z_3 = e^{i(\pi-\beta)}$ with $\alpha < \pi - \beta$. Let l denote the length of the side I_1. Use a) to show the following relations

$$\cosh l = \frac{1 + \cos\alpha\cos\beta}{\sin\alpha\sin\beta},$$
$$\sinh l = \frac{\cos\alpha + \cos\beta}{\sin\alpha\sin\beta}.$$

(Hint: Consider the triangle as the union (or the difference) of two triangles similar to the one considered in a).)