I. Topology of $\mathbb{H}^1$

1. Gromov product on $\mathbb{H}^1$

Let $z, z'$ and $w \in \mathbb{H}^1$, the Gromov product of $z$ and $z'$ w.r.t. $w$ is defined as

$$\langle z, z' \rangle_w := d_H(z, w) + d_H(z', w) - d_H(z, z')$$

Remark: Drop the $\frac{1}{2}$ in the def to simplify the later discussion.

2. Sequences converging at infinity

Let $w$ be a point in $\mathbb{H}^1$. Let $(z_n)_{n \in \mathbb{N}}$ be a sequence of points in $\mathbb{H}^1$. We say that $(z_n)_{n \in \mathbb{N}}$ converges at infinity if

$$\lim_{n \to \infty} \inf \{ d(z_j, z_k)_w : j, k > N \} = \infty \quad \text{or} \quad (z_j, z_k)_w \to \infty, \text{ as } j, k \to \infty$$

$$d_H(z_j, w) + d_H(z_k, w) - d_H(z_j, z_k)$$

Prop.: $(z_n)_{n \in \mathbb{N}}$ converges at infinity iff there exist $x \in \mathbb{H}^1$ s.t.

- if $x \in \mathbb{R}$, $|z_n - x| \to 0$, as $n \to \infty$
- if $x = \infty$, $|z_n| \to \infty$, as $n \to \infty$

4. Proof of the proposition

"$\Leftarrow$" If $|z_n| \to \infty$, as $n \to \infty$.

\[ \frac{1}{z_n} \to 0, \text{ as } n \to \infty. \]

Hence it is enough to discuss the case when $x \in \mathbb{R}$.

Without loss of generality, we may assume that $x = 0$.

| $z_n$ | $\to$ 0 as $n \to \infty$ iff $\forall \varepsilon > 0, \exists N$ s.t. $\forall n > N$, $z_n \in D(0, \varepsilon)$. |
Lemma: \( \forall \ C(x, y) \) with \( x+y \in [-1, 1] \), \( \forall z=x+it \in C(x, y) \cap D(0,1) \)

We consider the angle \( \Theta \) between \( C(x, y) \) and \( V_x \).

\( \forall \delta > 0 \ \exists \ M > 0, \) s.t. \( \Theta < \delta \) if \( r > M \) independent of \( x \)

\( \forall C(x, y) \) with \( x-r \in [-1, 1] \), \( \cdots \) (the same)

Proof: \( \Theta < \alpha \). For \( x+y \in [-1, 1] \), we consider \( C(x_1, y) \) s.t. \( x_1+y=1 \)

We denote the intersection angle between \( C(x_1, y) \) and \( V_1 \) by \( \alpha_1 \).

- \( \alpha_1 \to 0, \) as \( r \to \infty \)
- \( \Theta < \alpha < \alpha_1 \) Hence the lemma.

For \( x-r \in [-1, 1] \), we consider \( C(x_2, y) \) with \( x_2-r=-1 \)

Let \( \alpha_2 \) be the intersection angle between \( C(x_2, y) \) and \( V_1 \)

- \( \alpha_2 \to 0, \) as \( r \to \infty \)
- \( \Theta < \alpha < \alpha_2 \) Hence the lemma

For each \( \varepsilon, \) we use \( \frac{\phi_{\varepsilon^{-1}}}{\varepsilon} \) and get

\[ W_\varepsilon = \phi_{\varepsilon^{-1}}(W) = \varepsilon^{-1} W \]

\[ Z_n^\varepsilon = \phi_{\varepsilon^{-1}}(Z_n) \]

Then we have \( |W_\varepsilon| \to \infty, \) as \( \varepsilon \to 0. \)

Then by lemma, \( \forall \varepsilon > 0, \exists \delta > 0 \)

s.t. \( \forall \varepsilon < \delta, \) we have \( \forall n, \Theta_n < \varepsilon. \)
There are several cases after rescaling

$r \to \infty$, as $|w_\varepsilon| \to \infty$

$w_\varepsilon = u_\varepsilon + i v_\varepsilon$

$v_\varepsilon \to \infty$, as $|w_\varepsilon| \to \infty$

The idea is to find some lower bound on $(z_j, z_k)_w$ going to 00, as $j,k \to \infty$.

This lower bound is obtained by replacing the length of pink geodesic by the length of green geodesic.

We will discuss only the second case. The first one and the last one can be treated similarly.
Let \( w_\epsilon = u_\epsilon + i v_\epsilon \)

\[
\begin{align*}
    t_j^\epsilon &\leq d_{H_1}^\epsilon(w_\epsilon, z_j^\epsilon) \leq t_j^\epsilon + s_j^\epsilon \\
    t_k^\epsilon &\leq d_{H_1}^\epsilon(w_\epsilon, z_k^\epsilon) \leq t_k^\epsilon + s_k^\epsilon
\end{align*}
\]

We have

\[
(Z_j, Z_k)_w = (Z_j^\epsilon, Z_k^\epsilon)_w
\]

\[
= d_{H_1}^\epsilon(Z_j^\epsilon, W_\epsilon) + d_{H_1}^\epsilon(Z_k^\epsilon, W_\epsilon) - d_{H_1}^\epsilon(Z_j^\epsilon, Z_k^\epsilon)
\]

Hence we have

\[
t_j^\epsilon + t_k^\epsilon - d_{H_1}^\epsilon(Z_j^\epsilon, Z_k^\epsilon) \leq t_j^\epsilon + t_k^\epsilon + \frac{c}{u_\epsilon} - d_{H_1}^\epsilon(Z_j^\epsilon, Z_k^\epsilon)
\]

what we need

Hence it is enough to study \( t_j^\epsilon + t_k^\epsilon - d_{H_1}^\epsilon(Z_j^\epsilon, Z_k^\epsilon) \)

CASE I

\[
C(Z_j^\epsilon, R_k)
\]

\[
R_k = d_{H_1}^\epsilon(Z_j^\epsilon, Z_k^\epsilon)
\]

CASE II

\[
C(Z_k^\epsilon, R_j)
\]

\[
R_k + R_j = d_{H_1}^\epsilon(Z_j^\epsilon, Z_k^\epsilon)
\]

To see how high the points \( Z_j \) and \( Z_k \) could be, we consider the limit case:

CASE I

Biggest Euclidean circle

CASE II

Biggest Euclidean circles
\[ t_j^\varepsilon - t_k^\varepsilon \leq d_{H^1}(Z_j^\varepsilon, Z_k^\varepsilon) \leq \log \frac{u_\varepsilon}{2} + \log \frac{\mathcal{V}_\varepsilon}{\varepsilon^2} \to \infty, \quad \text{as } \varepsilon \to 0 \]

Hence \( (Z_j^\varepsilon, Z_k^\varepsilon)_w \to \infty \), as \( j, k \to \infty \)

\[ \Rightarrow \quad \text{If } (Z_n)_{n \in \mathbb{N}} \text{ does not converge to a point } x \in \mathbb{H} \]

w.r.t. Euclidean metric

either (I) \( \exists \) a bounded subsequence, i.e., \( \exists (Z_{j_n})_{n \in \mathbb{N}} \)

s.t. \( \exists R > 0, \ d_{H^1}(w, Z_j) \leq R \quad \forall j_n \)

or (II) \( d_{H^1}(w, Z_j) \to \infty \) as \( n \to \infty \), and

\( \exists (Z_{j_n})_{n \in \mathbb{N}} \quad Z_{j_n} \to x \), as \( n \to \infty \)

(II) We consider \( Z_{j_n} \) close to \( x \) and \( Z_{k_n} \) close to \( x' \)

w.r.t. Euclidean metric.

\[ (Z_{j_n}, Z_{j_n})_w = d_{H^1}(Z_{j_n}, w) + d_{H^1}(Z_{j_n}, w) - d_{H^1}(Z_{j_n}, Z_{j_n}) \leq 2R \]

Hence \( (Z_{j_n}, Z_{j_n})_w \to \infty \), as \( j, k \to \infty \)

Notice that

\[ d_{E}(Z_{j_n}, x) \to 0, \quad j_n \to \infty \]

\[ d_{E}(Z_{k_n}, x') \to 0, \quad k_n \to \infty \]

\[ \Rightarrow \quad d_{E}(w_n, w') \to 0, \quad \text{as } n \to \infty \]

Hence \( d_{H^1}(w_n, w') \to 0, \quad \text{as } n \to \infty \)

And \( C(Z_{j_n}, R_n) \to H(x, w') \), \( C(Z_{k_n}, R'_n) \to H(x', w') \), as \( n \to \infty \)
Hence \((Z_{j_n}, Z_{k_n})_W \to s + t\), as \(j_n, k_n \to \infty\)

Hence \((Z_j, Z_k) \to \infty\), as \(j, k \to \infty\).

Rmk: This means \((Z_j, Z_k) \to \infty\), as \(j, k \to \infty\) is equivalent to the convergence w.r.t. Euclidean metric.

Rmk: The proposition is independent of the choice of \(W\).

4. Equivalence relation among sequences converging to infinity.

For two sequences converging to infinity \((Z_n)_{new} and (W_n)_{new}\).

\((Z_n)_{new} and (W_n)_{new}\) are equivalent \([(Z_n)_{new}, (W_n)_{new}]\) if

\((Z_j, W_k)_W \to \infty\), as \(j, k \to \infty\)

\((\text{Gromov}) \ \forall H \ni \sigma := \{(Z_n)_{new} \mid (Z_j, Z_k)_W \to \infty, as j, k \to \infty\}\)

Let \(\sigma^+\) be a ray in \(H\) with \(\sigma^+(0) = W\) ending at \(x\).

Then \((\sigma^+(n))_{new}\) is a sequence converging to \(x\).

- \((\sigma^+(j), \sigma^-(k))_W \to \infty\), as \(j, k \to \infty\).

\[\sigma^+ \cap \sigma^- \Rightarrow (\sigma^+(j), \sigma^-(k))_W \to \infty, as j, k \to \infty \Rightarrow (\sigma^+(n))_{new} \cap (\sigma^-(n))_{new} \to \infty, as j, k \to \infty \Rightarrow (\sigma^+(n))_{new} \cap (\sigma^-(n))_{new} \to \infty\]

Hence, \([\sigma^+] = x = [(\sigma^+(n))_{new}]\) by the construction in the proof of the previous proposition.

- A sequence \((Z_n)_{new}\) converges to \(x\) \(\iff (Z_n)_{new} \in [(\sigma^+(n))_{new}]\)

5. Metric on \(\Theta H\)

Let \((Z_n)_{new}\) and \((W_n)_{new}\) be two sequences converging to different points in \(\Theta H\).

\(Z_n \to x\), as \(n \to \infty\) in the sense of \(x\)

\(W_n \to x'\)

Hence \((Z_n)_{new} \not\sim (W_n)_{new}\).

The Gromov product between \(x\) and \(x'\) is \(\langle x, x' \rangle_W := \lim_{n \to \infty} (Z_n, W_n)_W\)
$d_w(x, x') := e^{-\frac{(x-x')_w}{2}}$

Prop: $(H, d_w)$ is a metric space.

Proof:
1. $d_w(x, x') \geq 0$, "0" iff $x = x'$ ($e^{-t} \to 0$ as $t \to \infty$)
2. $d_w(x, x') = d_w(x', x)$ (Symmetry) ($z, z', w = (z', z)w$)
3. $d_w(x, x') + d_w(x', x'') \geq d_w(x, x'')$

To show 3, we use isometry to put $w, x, x'$ in a standard position

$x = 0 \quad \text{Re}(w) = \frac{1}{2}$
$x' = 1$

Let $w = \frac{1}{2} + iy$. To compute $d_{w-1}(H(x, i), w)$

We use the $\pi$-rotation at $i$

$[0 \ 1]$ exchanges $H(x, i)$ and $H_1$

$0$ and $\infty$

$d_{w-1}(H(x, i), w) = d_{w-1}(H_1, -\frac{i}{w}) = |\log \text{Im}(-\frac{i}{w})| = l$

radius become vertical geodesics.

$\frac{-i}{w} = \frac{-1}{\frac{1}{2} + iy} = \frac{-2 + 4iy}{1 + 4y^2}$

$d_{w-1}(x, x', w) = d = \log \frac{2y}{\sqrt{2}} = \log 2y \Rightarrow 2y = e^d$

Hence $\text{Im}(-\frac{i}{w}) = \frac{2e^d}{1 + e^{2d}} = (\cosh d)^{-1}$

Hence $(x, x')_w = 2d = 2\log(\cosh d)$ and $d_w(x, x') = \frac{1}{\cosh d}$

Lemma.

$\cosh \sin \theta = 1$

Trigonometry formula for triangles with interior angle 0, $\frac{\pi}{2}$ and $\theta$

Hence $d_w(x, x') = \sin \theta = \sin \frac{d_w(x, x')}{2}$

Now we use a different renormalization.

WOLG, let $x = 0$, $x' = 1$, $x'' = \infty$. Then $w$ could be anywhere.
We assume that 
\[ x = 0 \quad \text{and} \quad x' = \infty \]

We have 
\[ d + \theta + \theta = \pi \]

Moreover, we have 
\[ d_w(x, x') = \sin \theta \]
\[ d_w(x, x') = \sin \alpha \]
\[ d_w(x', x'') = \sin \beta \]

Hence the goal is to show 
\[ d_w(x, x'') = \sin \alpha \leq \sin \beta + \sin \theta = d_w(x, x') + d_w(x', x'') \]

\[ \sin(\pi - \alpha - \beta) = \sin(\alpha + \theta) \]

Notice that 
\[ 0 < \alpha < \frac{\pi}{2}, \quad 0 < \beta < \frac{\pi}{2}, \quad 0 < \theta < \frac{\pi}{2} \]

We consider 
\[ \sin(\alpha + \beta) + \sin \beta - \sin \alpha = f_\theta(\alpha) \]

For any \( \beta \), 
\[ f'_\theta(\alpha) = \frac{d f_\theta(\alpha)}{d \alpha} = \cos(\alpha + \beta) - \cos \alpha \]

\[ f_\theta(\alpha) = 0 \iff \cos \alpha = \cos(\alpha + \beta) \iff \beta = 0 \quad (\text{i.e. } W = X \text{ impossible}) \]

\[ \in [0, \frac{\pi}{2}] \quad \in [0, \frac{\pi}{2} + \beta] \]

Since \( \cos \alpha > \cos(\alpha + \beta) \) we have 
\[ f'_\theta(\alpha) < 0 \iff \cos \text{ is monotonically decreasing in } [0, \pi] \]

Hence 
\[ \min f_\theta(\alpha) = f_\theta(\frac{\pi}{2}) = \cos \beta + \sin \beta - 1 > 0 \]

\[ \beta = \frac{\pi}{2} \quad \text{i.e. } w \in [x, x''] \]

Hence 
\[ d_w(x, x'') \leq d_w(x, x') + d_w(x', x'') \]

The proof for 
\[ \beta = \alpha + \theta \]

\[ \text{Prop:} \quad \forall x, x' \in \Theta \mathcal{H}, \quad d_w(x, x') \leq 1 \]

\[ d_w(x, x') = 1 \iff w \in \mathcal{G}[x, x'], \]

\[ \text{Prop: The topology induced by this metric is the same as the one in the lecture.} \]
Prop: \((\Theta H^1, d_w')\) is compact.

Prop: \((\Theta H^1, d_w) \rightarrow (\Theta H^1, d_w')\) is a bi-Lipschitz map, i.e. \(C_1, C_2 \text{ s.t. } \forall x, x' \in \Theta H^1, \text{ we have} \)
\[ C_1 d_w'(x, x') \leq d_w(x, x') \leq C_2 d_w'(x, x') \]

**Topology of \( \overline{H^1} \)**

1. Open sets

The basis of \( \overline{H^1} \) is given by \((\phi \text{ and } \overline{H^1})\)

1. Disk in \( H^1 \)
2. Open half plane with boundary at infinity.

Some discussion:

Let \( P \) denote a half plane, i.e. a connected component of \( H^1 \setminus \gamma \).

Similarly to \( \Theta H^1 \), we can ask what is the DP.

For all \( \eta_0 \in [\eta^+], \) eventually it is contained in \( P \) (up to taking away the starting point).

Ex: \( \eta^+ \) is not in \( P \)

The ray ending at \( x \) should contain all rays ending at \( \eta^+ \).

\[ x = [\eta^+] \in \Theta H^1 \]
Let $\sigma^+$ and $\sigma^-$ be two rays in $\sigma$ ending at $x^+$ and $x^-$ respectively. If $\eta^+ \in \partial \sigma$, then $\eta^+ \prec \sigma^+$ and $\eta^+ \prec \sigma^-$.

Let $\Theta$ be the intersection angle. $\Rightarrow \Theta \neq 0$ or $\pi$.

$\exists \epsilon > 0 \text{ s.t. } \eta^+ \text{ having intersection angle } \epsilon \in ]-\epsilon, \epsilon[ \text{ with } \eta^+ \exists \eta^+ \subset \eta \text{ s.t. } [\eta^+] \cap \partial \sigma$

Let $w$ be the intersection point.

Consider $(\sigma_1 \cap \sigma_2, d_0)$. $\forall x \in \sigma_1 \cap \sigma_2, x^+ \subset \partial \sigma$

$\forall x \in \sigma_1 \cap \sigma_2, x - \sin \frac{\epsilon}{2}, x^+ \subset \partial \sigma$

$\forall x \in \sigma_1 \cap \sigma_2, x - \sin \frac{\epsilon}{2}, x^+ \subset \partial \sigma$

**An open set in $\overline{\mathbb{H}}$** is given by

- Finite intersection of basis open sets
- Any union of basis open sets

$\forall z \in \overline{\mathbb{H}}$, a neighborhood of $z$ is an open set $U$ with $z \in U$

Let $U$ be a neighborhood of $z$.

**Prop:** $\forall z \in \overline{\mathbb{H}}$, $\exists D(z, r) \subset U$ for some $r$

$\forall z \in \overline{\mathbb{H}}$, $\exists D(z, r) \subset U$ for some $r$

$\forall z = \pm \infty$, $\exists D(\pm \infty, (a, b)) \subset U$ for some $a < b$

**Rmk:** $\exists D(x, r) = C(x, r)$ geodesic

$\exists D(\pm \infty, (a, b)) = C(\pm \infty, (a, b))$ geodesic.

$\exists D(\pm \infty, (a, b)) = \overline{D}(a, \infty) = V_a$ geodesic.

2. Convergence of sequence in $\overline{\mathbb{H}}$

$(Z_n)_{n \in \mathbb{N}}$ converges to $z \in \overline{\mathbb{H}}$ if:

- $z = \infty$, $|Z_n| \rightarrow \infty$, as $n \rightarrow \infty$
- $z \neq \infty$, $|Z_n - z| \rightarrow 0$, as $n \rightarrow \infty$

**Rmk:** For $z \in \partial \mathbb{H}$, we can also use Gromov product to describe the convergence as previously discussed.
3. Extend an isometry to a continuous map $\overline{H} \to \overline{H}$.

Prop: $\forall f \in \text{Isom}(\mathbb{H})$, $\exists \vartheta f: \mathbb{H} \to \mathbb{H}$.

Proof: $f$ preserves the equivalence relation among rays
- the equivalence relation among sequence $\ldots$
  
  i.e. $\sigma^+ \sim_\theta \eta^+ \iff f(\sigma^+) \sim f(\eta^+)$

  $(\sigma_n)_{n \in \mathbb{N}} \sim_\theta (\eta_n)_{n \in \mathbb{N}} \iff (f(\sigma_n))_{n \in \mathbb{N}} \sim_\theta (f(\eta_n))_{n \in \mathbb{N}}$

  $(z_j, w_k)_j \to \infty$, as $j, k \to \infty$

  $\iff (f(z_j), f(w_k))_{j} \to \infty$, as $j, k \to \infty$

  $\iff (f(z_j), f(w_k))_j \to \infty$, as $j, k \to \infty$

Cor: $\vartheta f$ is bijective.

Proof: $\varnothing$

Let $C(w, R)$ be the circle centered at $w$ of radius $R$.

Prop: $\exists$ a bi-Lipschitz map $f: (C(w, R), d_c) \overset{\vartheta_w}{\to} (\mathbb{H}, d_w)$

Proof: Consider $\sigma_0^+$ with $\sigma_0(0) = w$.

All radius can be parametrized by $[0, 2\pi)$ by considering rotation from $\sigma_0^+$ along the positive rotation direction, denoted by $\sigma_0^+$.

Let $\{z_0\} = \sigma_0^+ \cap C(w, R)$, and $x_0$ be the endpoint of $\sigma_0^+$.

$d_c(z_0, z_0^*) = |\theta - \theta^*| \sinh R = \frac{|\theta - \theta^*|}{2} (2 \sinh R)$

$d_w(x_0, x_0^*) = \sin \frac{|\theta - \theta^*|}{2}$

Lemma: $\frac{\alpha}{2} < \sin \alpha < \alpha$ for $\alpha \in [0, \frac{\pi}{2}]$

Hence we have $C_1 = C_2 = \frac{1}{2} \sinh R$ s.t.

$\frac{C_1}{2} d_c(z_0, z_0^*) \leq d_w(x_0, x_0^*) \leq C_1 d_c(z_0, z_0^*)$

Prop: $\vartheta f$ is a bi-Lipschitz map.

Proof: $\vartheta f: \mathbb{H} \overset{\vartheta_w}{\to} C(w, R) \overset{f}{\to} C(f(w), R) \overset{\vartheta_w}{\to} \mathbb{H}$ bi-Lip isom bi-Lip
Prop.: \( \tilde{f}: \mathbb{H} \to \mathbb{H} \) is continuous

Proof:
1. \( \tilde{f}^{-1} \) sends \( \text{D}(w, R) \) to \( \text{D}(\tilde{w}, R) \)
2. \( \tilde{f}^{-1} \) preserves the finite intersections and arbitrary unions among basis open sets.
   \[ \tilde{f}^{-1}(B_1 \cap \ldots \cap B_n) = \text{D}(\tilde{b}_1, R) \cap \ldots \cap \text{D}(\tilde{b}_n, R) \]
   where still a basis open set.
3. \( \tilde{f}^{-1}(\bigcup_{\alpha} B_{\alpha}) = \bigcup_{\alpha} \text{D}(\tilde{b}_{\alpha}, R) \) \( \Rightarrow \) open set.

Hence \( \tilde{f} \) is continuous.

Remark: A map \( f: X \to Y \) between topological space is continuous if \( \forall \) open set in \( Y \), \( f^{-1}(\text{open set}) \) is an open set in \( X \).

4. Another way to see that \( \tilde{f} \) is continuous is to notice that \( \tilde{f} \) is the restriction of a Möbius transformation \( F \) on \( \mathbb{C} \) which is continuous and orientation preserving.
Since the topology on \( \mathbb{C} \) restricted to \( \mathbb{H} \) is equivalent to what we described, by taking \( \tilde{f} = F|_{\mathbb{H}} \), we get a continuous map on \( \mathbb{H} \).