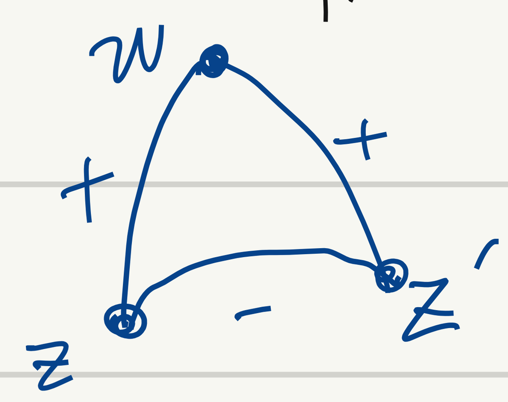
I Topology of $\partial\mathbb{H}$

1. Gromov product on $\mathbb{H}$

Let z, z' and $w \in \mathbb{H}$, the Gromov product of z and z' w.r.t. w (= with respect to) is defined as



$$(z, z')_w := \left(d_{\mathbb{H}}(z, w) + d_{\mathbb{H}}(z', w) - d_{\mathbb{H}}(z, z')\right)$$

Rmk: Drop the $\frac{1}{2}$ in the def to simplify the later discussion

2. Sequences converging at infinity

Let w be a point in $\mathbb{H}$. Let $(z_n)_{n\in\mathbb{N}}$ be a sequence of points in $\mathbb{H}$.
We say that $(z_n)_{n\in\mathbb{N}}$ converges at infinity if

(*) $\lim\limits_{N\to\infty} \inf\{\underbrace{(z_j, z_k)_w}_{= d_{\mathbb{H}}(z_j, w) + d_{\mathbb{H}}(z_k, w) - d_{\mathbb{H}}(z_j, z_k)},\ j, k > N\} = \infty$ (or $(z_j, z_k)_w \to \infty$, as $j, k \to \infty$)

Prop: $(z_n)_{n\in\mathbb{N}}$ converges at infinity iff there exist $x \in \partial\mathbb{H}$ s.t.

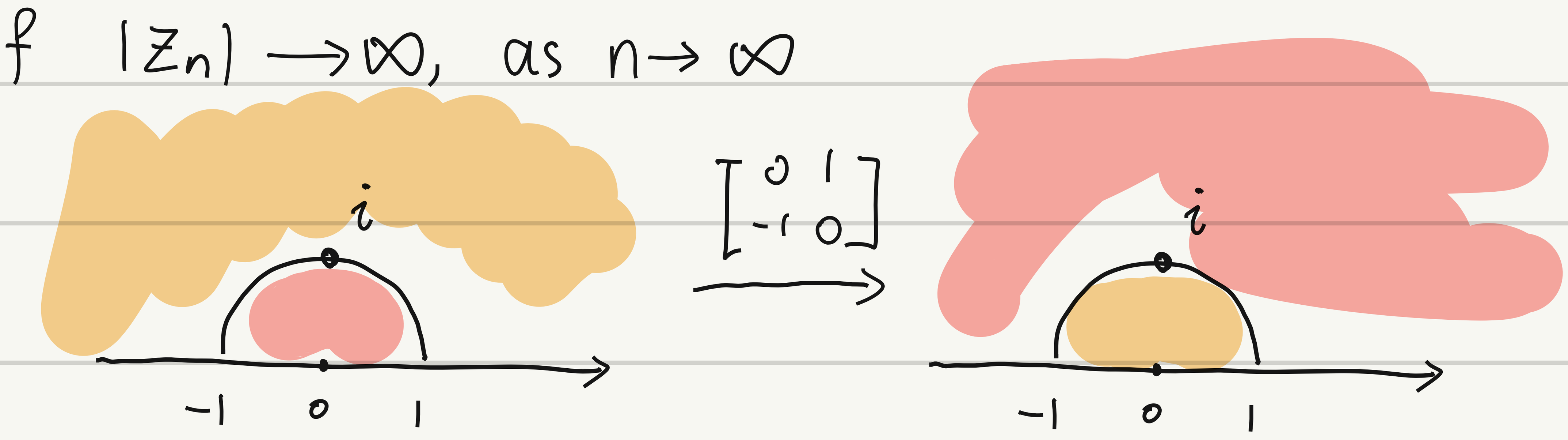
- if $x \in \mathbb{R}$: $|z_n - x| \to 0$, as $n \to \infty$
- if $x = \infty$: $|z_n| \to \infty$, as $n \to \infty$

converges w.r.t. Euclidean metric

$|z_n|$: modulus of complex numbers.

3. Proof of the proposition

"$\Leftarrow$" If $|z_n| \to \infty$, as $n \to \infty$

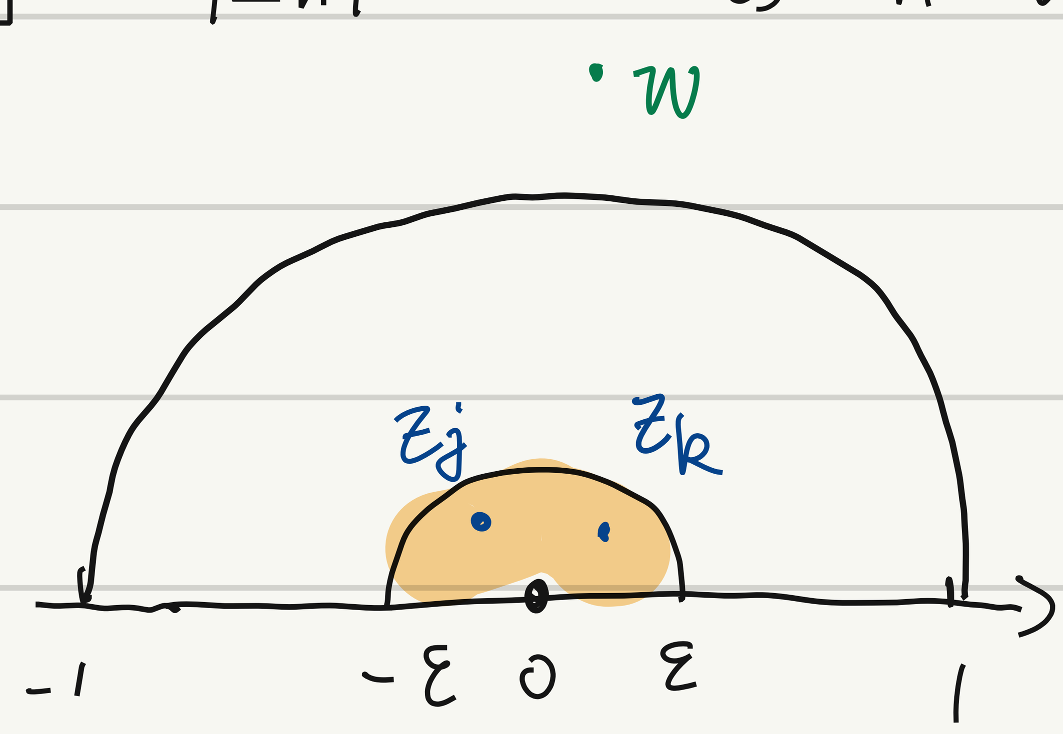


then $\left|-\frac{1}{z_n}\right| \to 0$, as $n \to \infty$.

Hence it is enough to discuss the case when $x \in \mathbb{R}$.

Without loss of generality = WOLG, we may assume that $x = 0$.

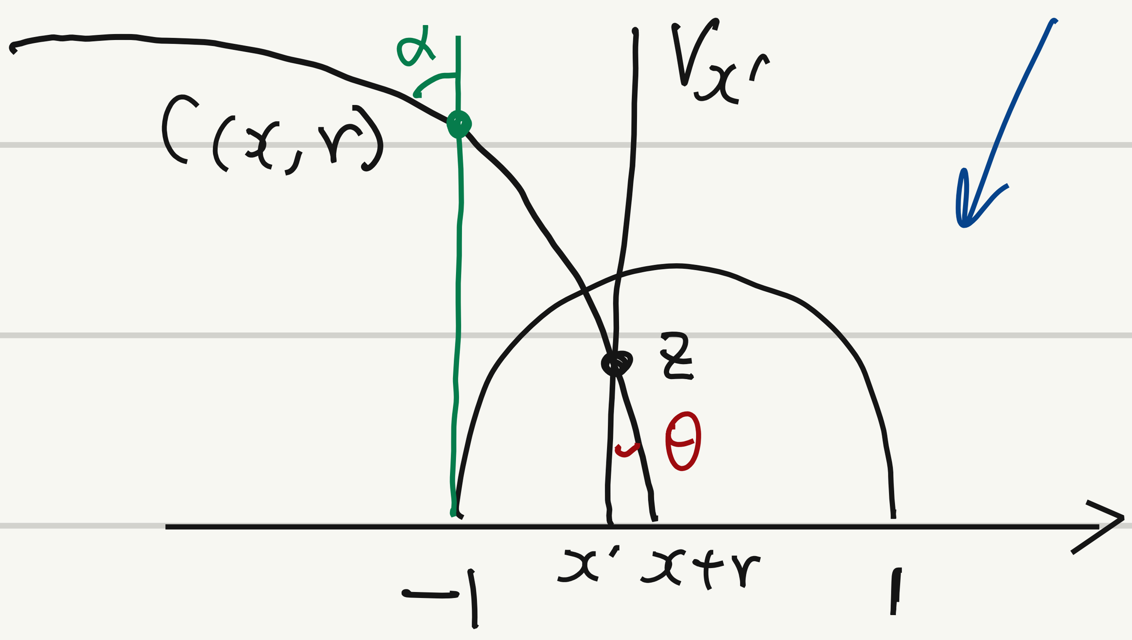
$|z_n| \to 0$ as $n \to \infty$ iff $\forall \varepsilon > 0, \exists N$ s.t. $\forall n > N$, $z_n \in D(0, \varepsilon)$.

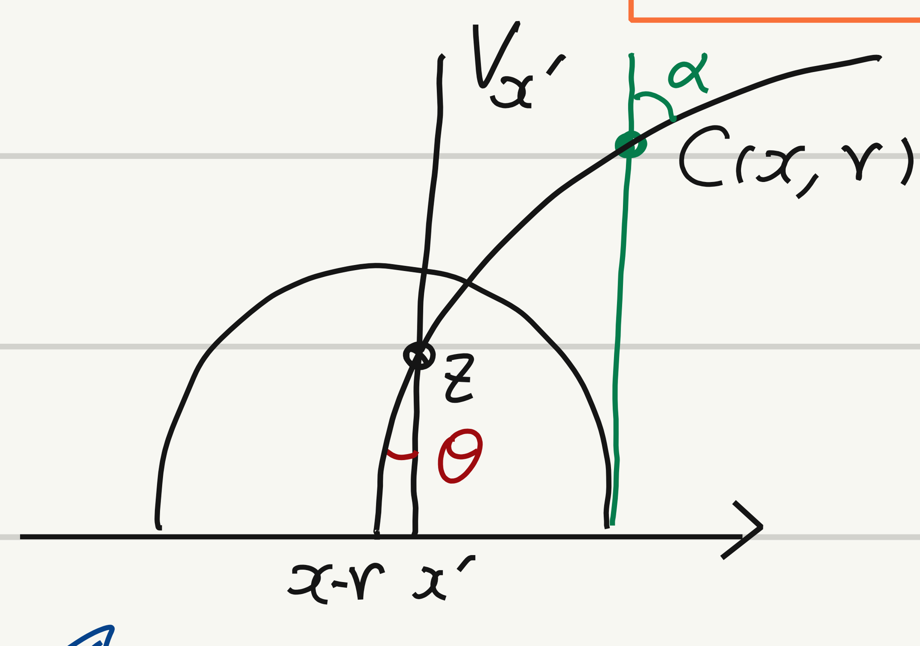


Lemma: • $\forall\, C(x,r)$ with $x+r\in[-1,1]$, $\forall z=x'+it \in C(x,r)\cap D(0,1)$

We consider the angle θ between $C(x,r)$ and $V_{x'}$

$\forall \delta>0\ \ \exists M>0$, s.t. $\theta<\delta$ if $r>M$ $\boxed{\text{independant of } x}$

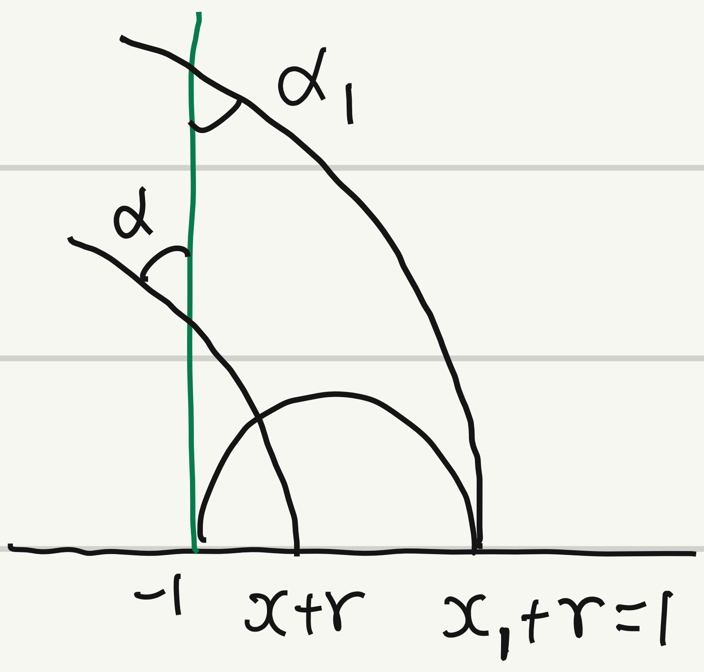




• $\forall\, C(x,r)$ with $x-r\in[-1,1]$, ... (the same)

Proof: $\theta<\alpha$. For $x+r\in[-1,1]$, we consider $C(x_1,r)$ s.t. $x_1+r=1$

We denote the intersection angle between $C(x_1,r)$ and V_{-1} by α_1.

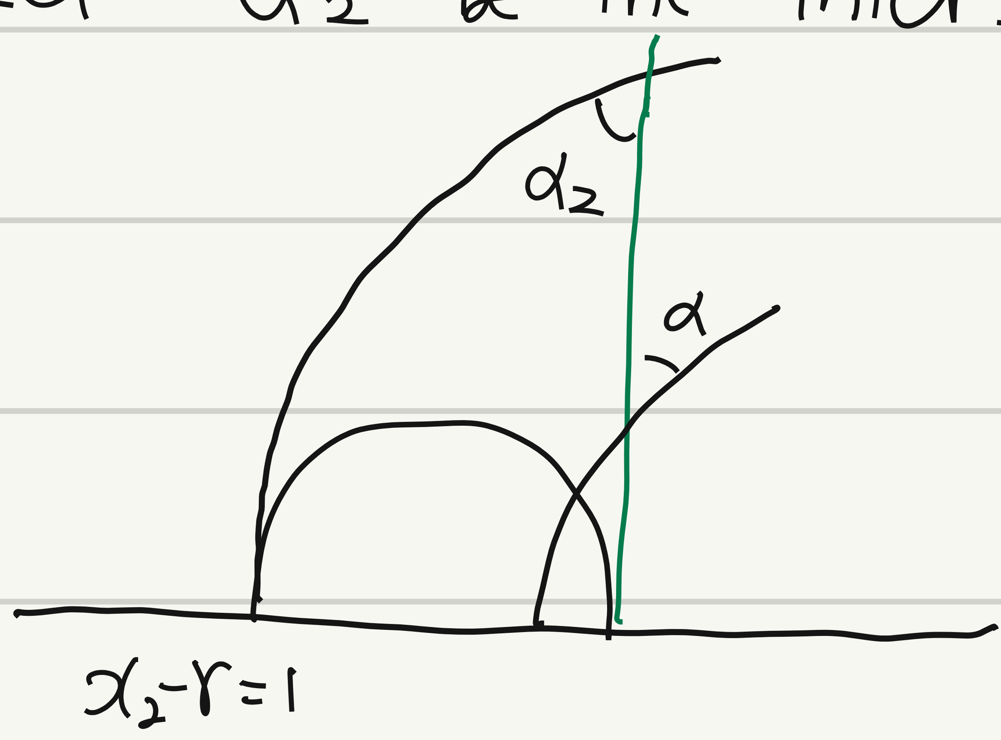


• $\alpha_1 \longrightarrow 0$, as $r\to\infty$

• $\theta<\alpha<\alpha_1$ Hence the lemma.

For $x-r\in[-1,1]$, we consider $C(x_2,r)$ with $x_2-r=-1$

Let α_2 be the intersection angle between $C(x_2,r)$ and V_1



• $\alpha_2\to 0$, as $r\to\infty$

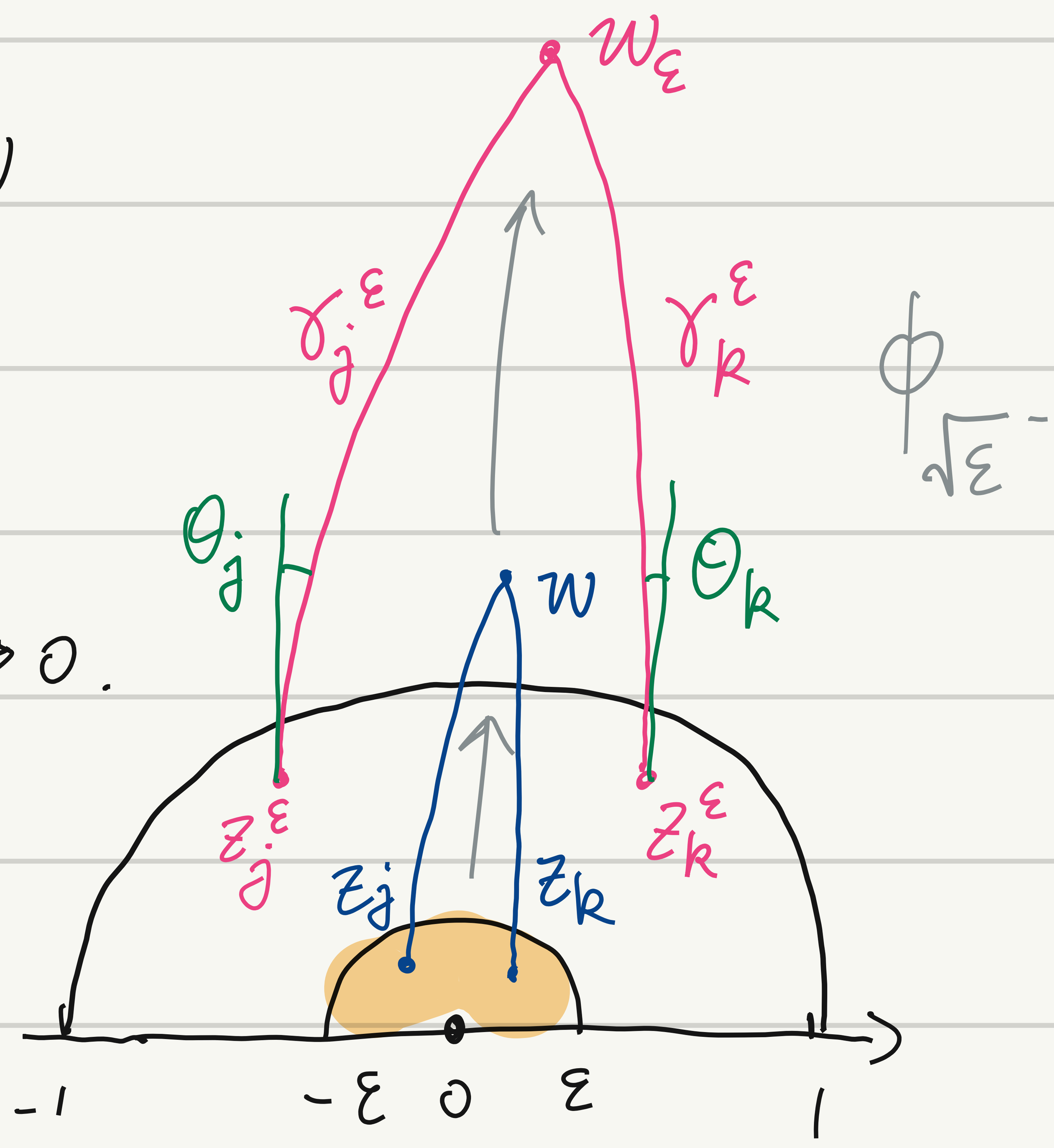
• $\theta<\alpha<\alpha_2$ Hence the lemma

∎

For each ε, we use $\phi_{\sqrt{\varepsilon}^{-1}}$ and get

$$w_\varepsilon=\phi_{\sqrt{\varepsilon}^{-1}}(w)=\varepsilon^{-1}w$$

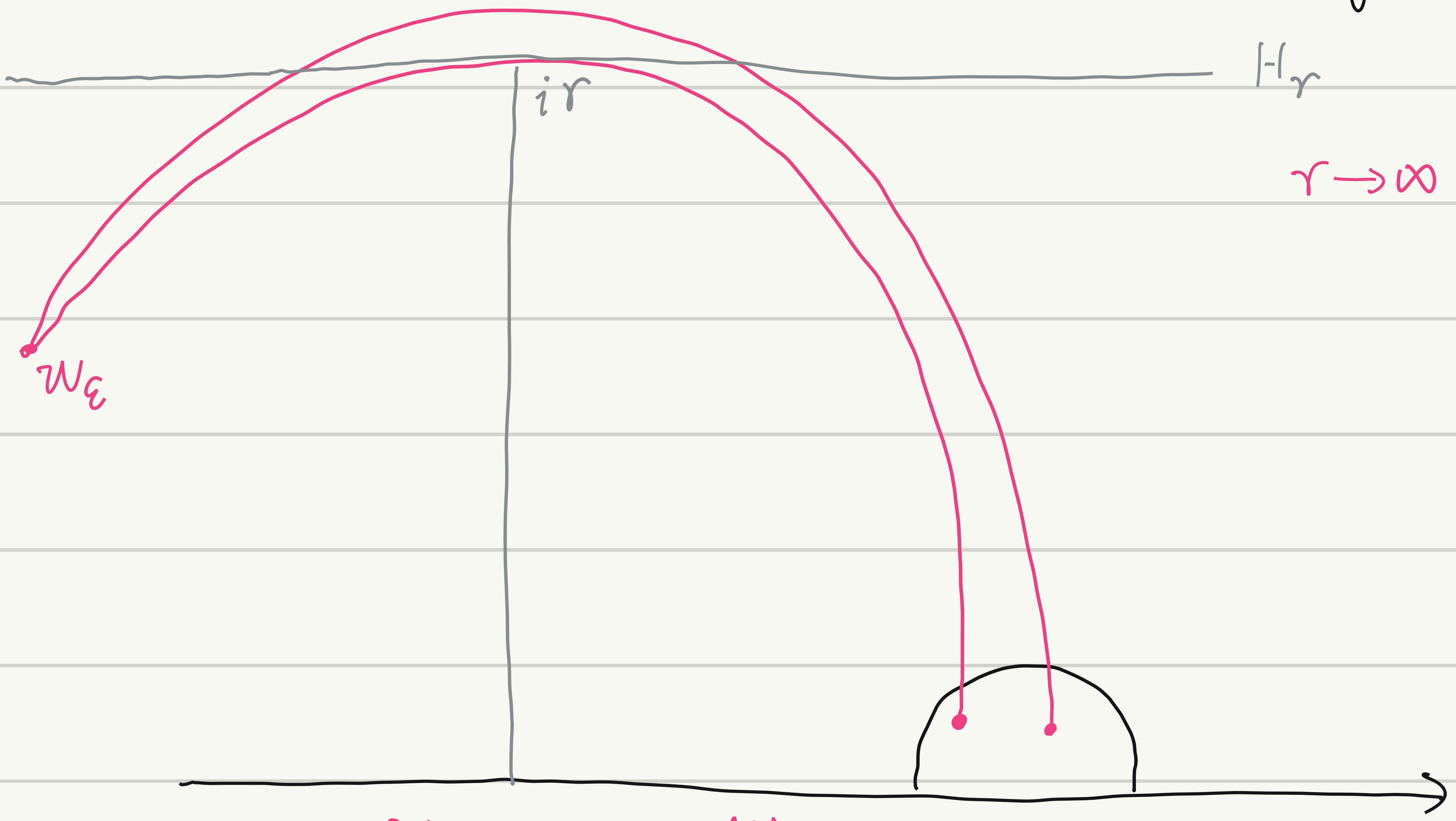
$$z_n^\varepsilon=\phi_{\sqrt{\varepsilon}^{-1}}(z_n)$$



Then we have $|w_\varepsilon|\to\infty$, as $\varepsilon\to 0$.

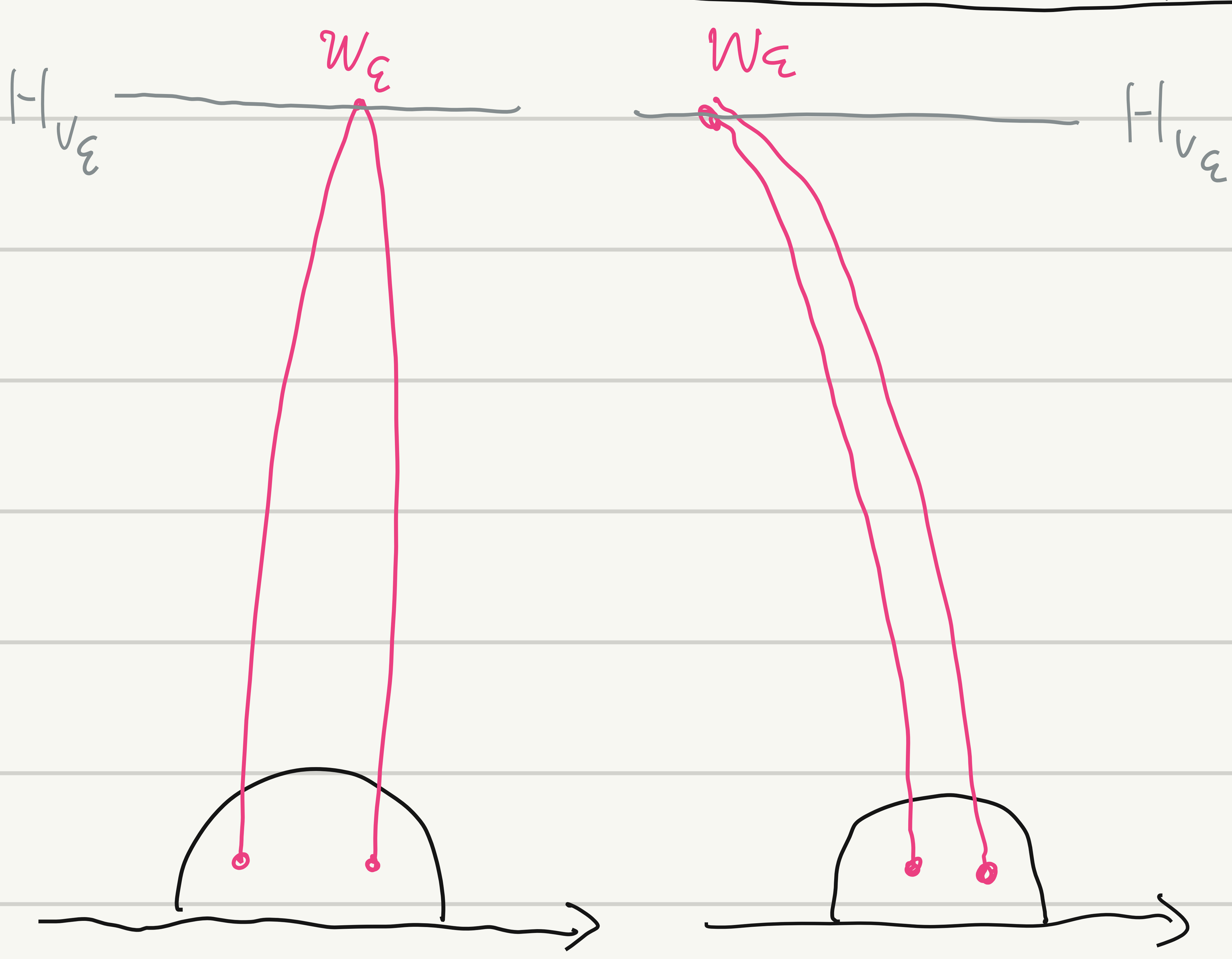
Then by lemma, $\forall \varepsilon'>0$, $\exists\delta>0$ s.t. $\forall\varepsilon<\delta$, we have $\forall n$, $\theta_n<\varepsilon'$.

There are several cases after rescaling



$r \to \infty$, as $|w_\varepsilon| \to \infty$

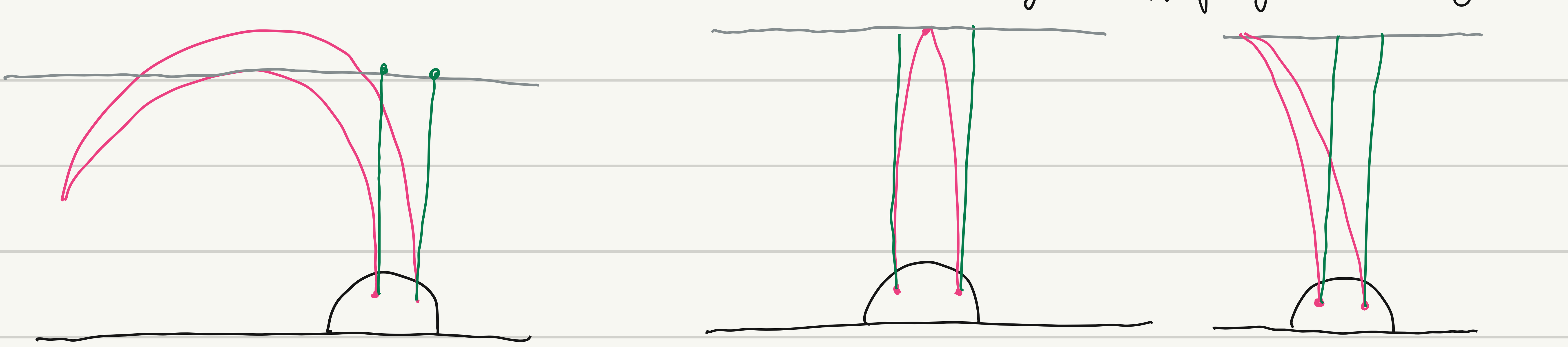
or



$w_\varepsilon = u_\varepsilon + i v_\varepsilon$

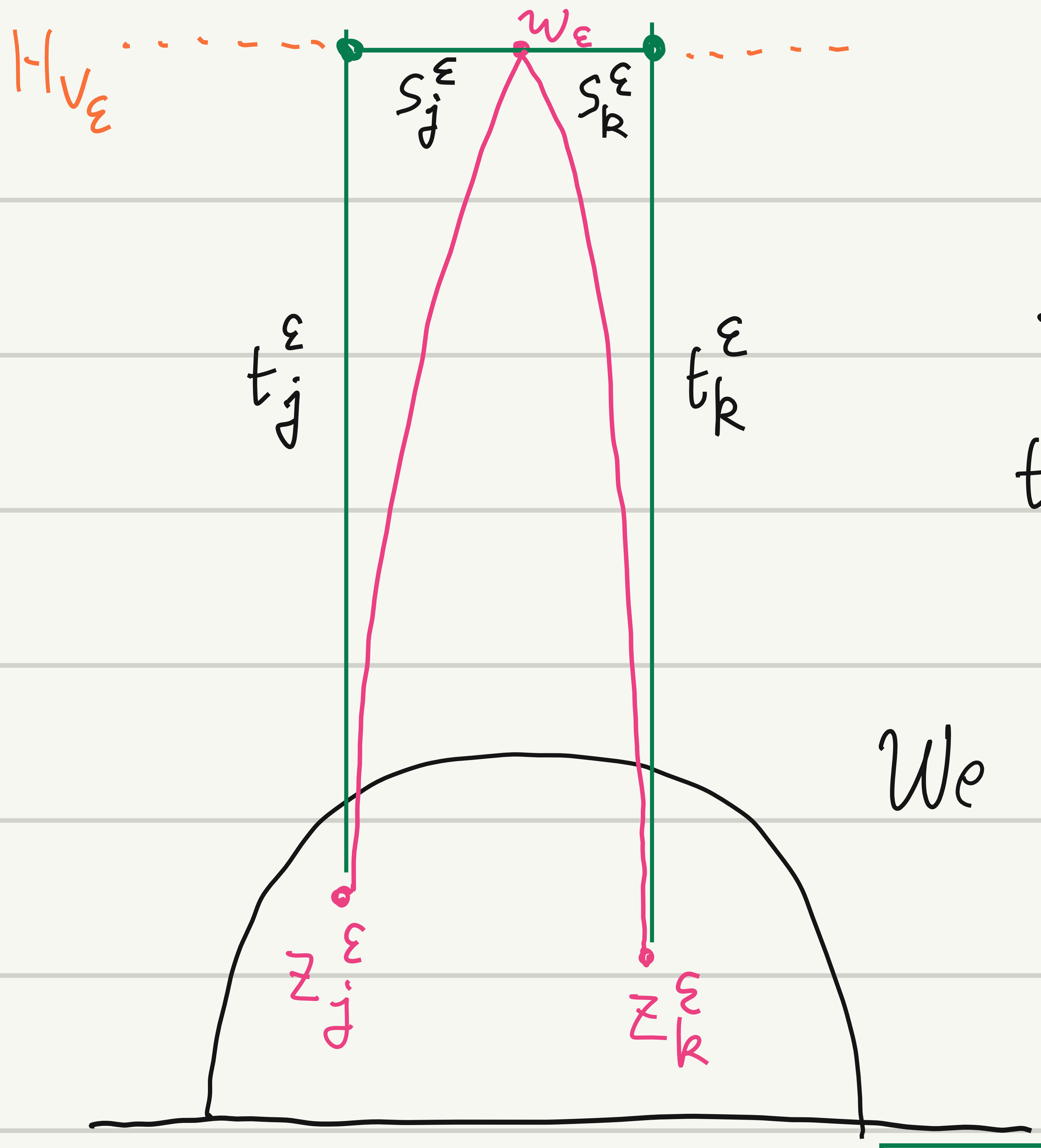
$v_\varepsilon \to \infty$, as $|w_\varepsilon| \to \infty$

The idea is to find some lower bound on $(z_j, z_k)_w$ going to ∞, as $j,k \to \infty$.



This lower bound is obtained by replacing the length of pink geodesic by the length of green geodesic

We will discuss only the second case. The first one and the last one can be treated similarly.



Let $w_\varepsilon = u_\varepsilon + i v_\varepsilon$

$$t_j^\varepsilon \le d_{\mathbb{H}}(w_\varepsilon, z_j^\varepsilon) \le t_j^\varepsilon + s_j^\varepsilon \le t_j^\varepsilon + \frac{2}{v_\varepsilon}$$

$$t_k^\varepsilon \le d_{\mathbb{H}}(w_\varepsilon, z_k^\varepsilon) \le t_k^\varepsilon + s_k^\varepsilon \le t_k^\varepsilon + \frac{2}{v_\varepsilon}$$

Only for this case

We have $(z_j, z_k)_w = (z_j^\varepsilon, z_k^\varepsilon)_{w_\varepsilon}$

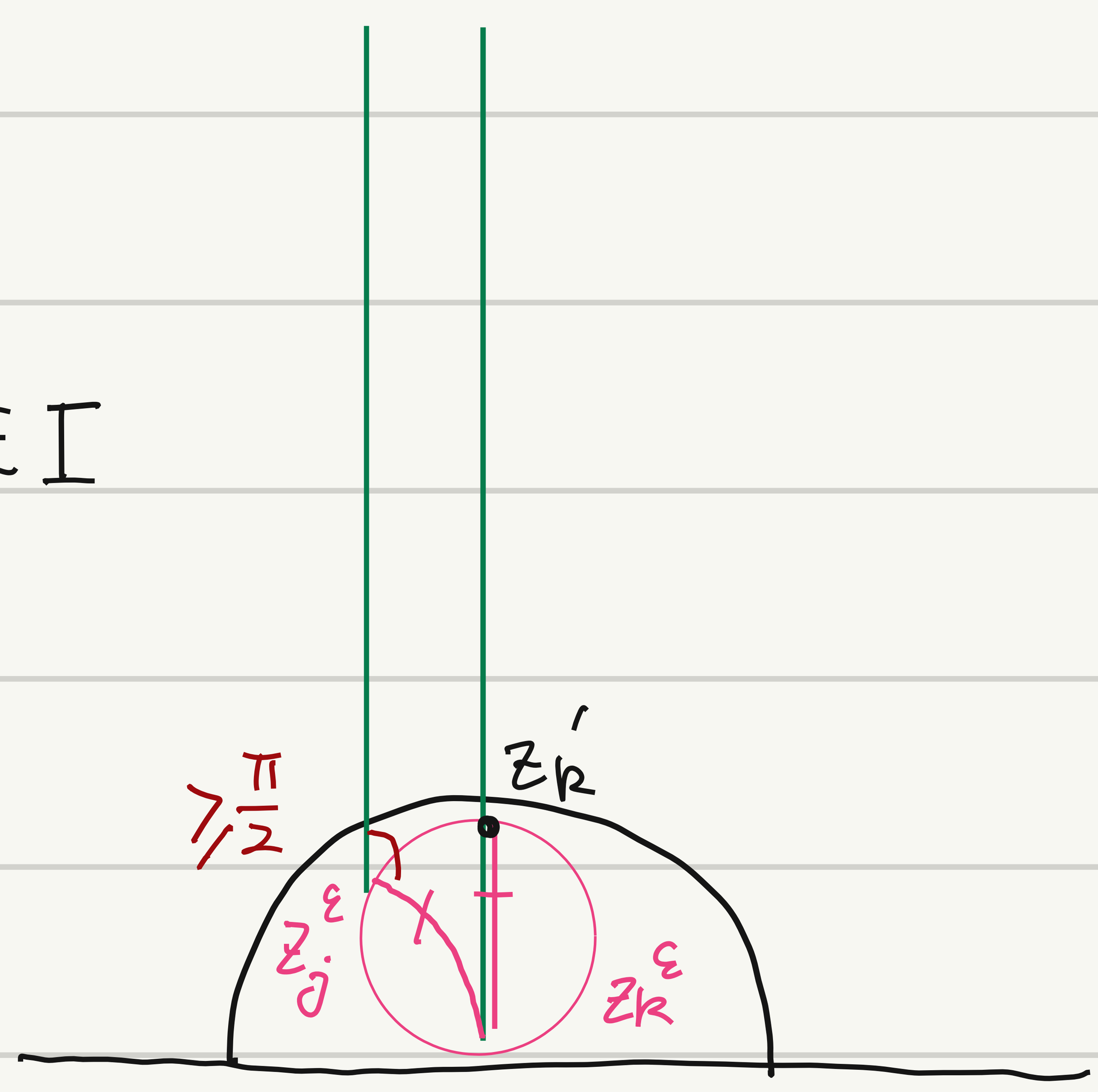
$$= d_{\mathbb{H}}(z_j^\varepsilon, w_\varepsilon) + d_{\mathbb{H}}(z_k^\varepsilon, w_\varepsilon) - d_{\mathbb{H}}(z_j^\varepsilon, z_k^\varepsilon)$$

Hence we have $\boxed{t_j^\varepsilon + t_k^\varepsilon - d_{\mathbb{H}}(z_j^\varepsilon, z_k^\varepsilon) \le (z_j, z_k)_w} \le t_j^\varepsilon + t_k^\varepsilon + \frac{4}{v_\varepsilon} - d_{\mathbb{H}}(z_j^\varepsilon, z_k^\varepsilon)$

what we need

Hence it is enough to study $t_j^\varepsilon + t_k^\varepsilon - d_{\mathbb{H}}(z_j^\varepsilon, z_k^\varepsilon)$

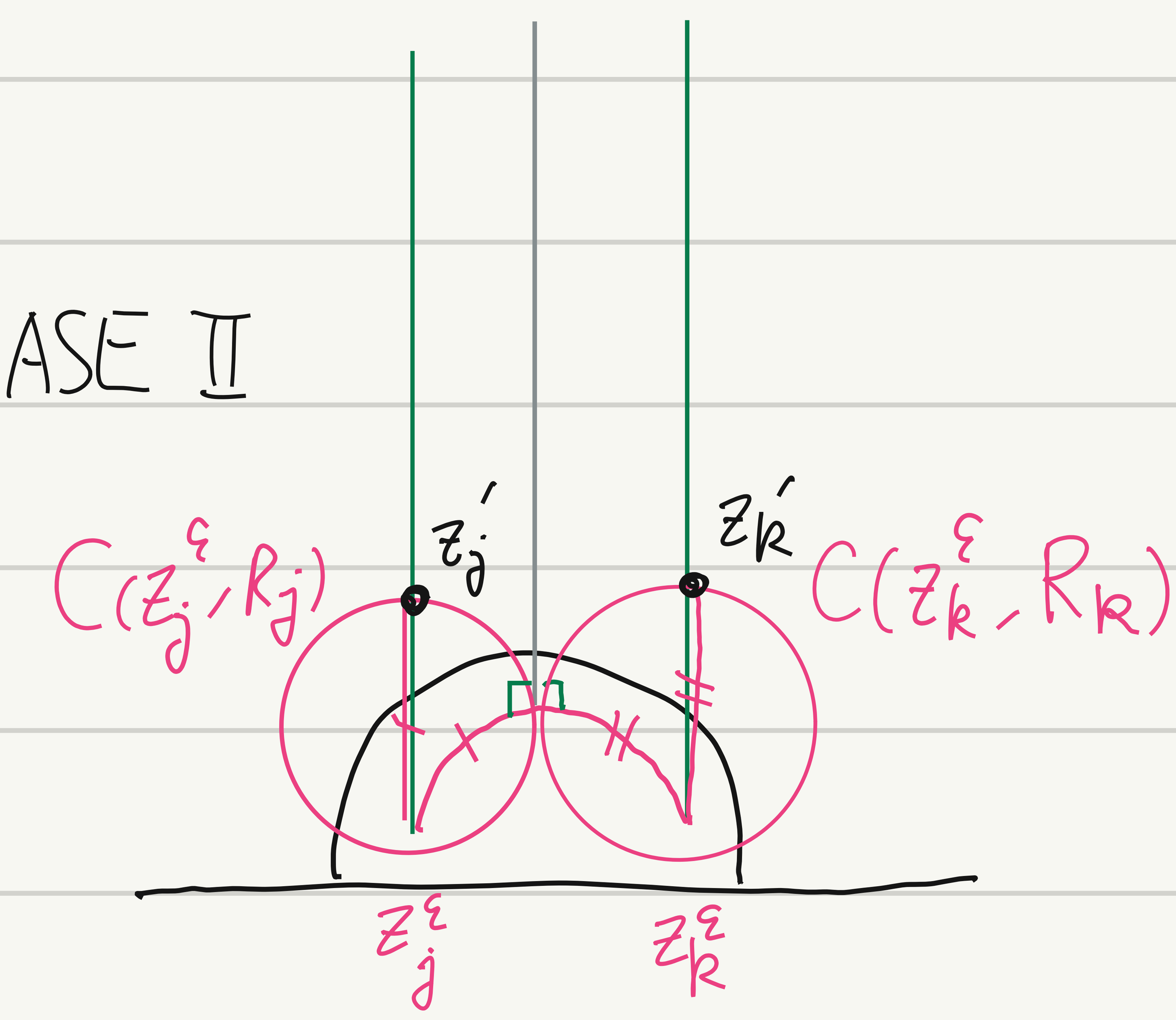
CASE I



$C(z_k^\varepsilon, R_k)$

$R_k = d_{\mathbb{H}}(z_j^\varepsilon, z_k^\varepsilon)$

CASE II

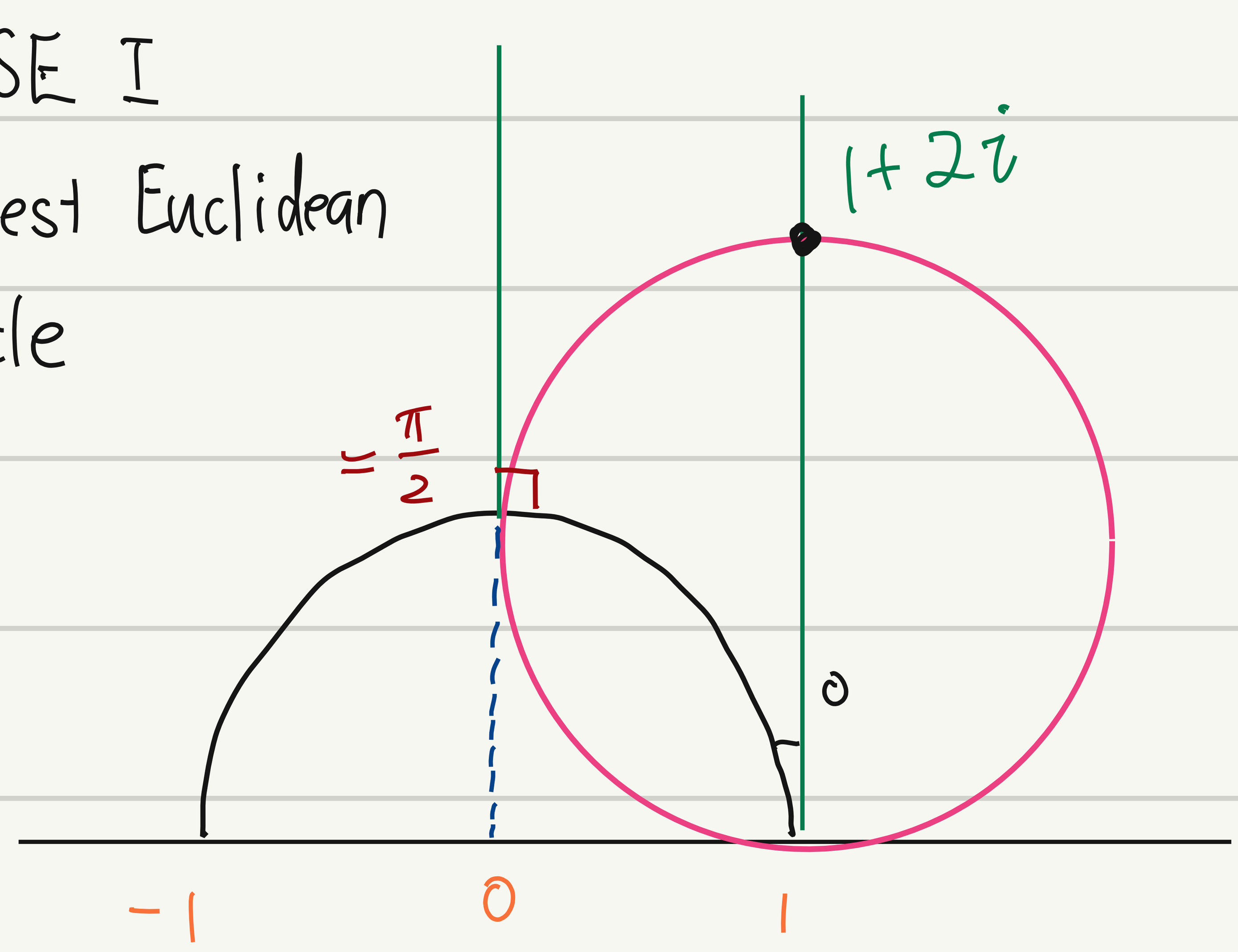


$R_k + R_j = d_{\mathbb{H}}(z_j^\varepsilon, z_k^\varepsilon)$

To see how high the points z_j' and z_k' could be, we consider the limit case :

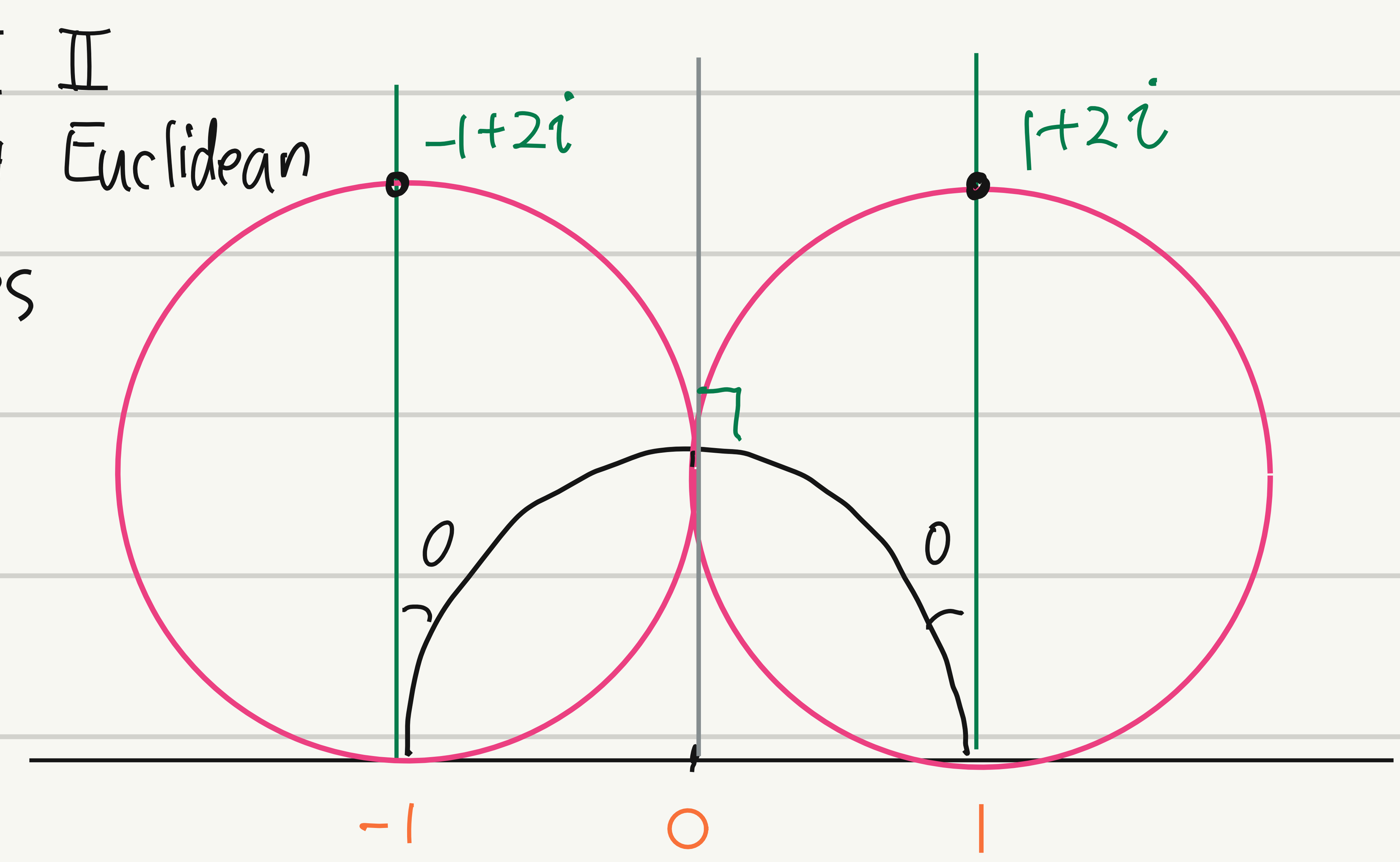
CASE I

Biggest Euclidean circle



CASE II

Biggest Euclidean circles



Hence : $t_j^\varepsilon + t_k^\varepsilon - d_{\mathbb{H}}(z_j^\varepsilon, t_k^\varepsilon)$

$$\geq \log \frac{v_\varepsilon}{2} + \log \frac{v_\varepsilon}{2} \to \infty, \quad \text{as } \varepsilon \to \infty$$

Hence $(z_j, z_k)_w \to \infty$, as $j, k \to \infty$

"$\Rightarrow$" If $(z_n)_{n\in\mathbb{N}}$ does not converge to a point $x \in \partial\mathbb{H}$ w.r.t. Euclidean metric

either (I) $\exists$ a bounded subsequence, i.e. $\exists (z_{j_n})_{n\in\mathbb{N}}$ s.t. $\exists R>0$, $d_{\mathbb{H}}(w, z_{j_n}) \leq R \ \forall j_n$

or (II) $d_{\mathbb{H}}(w, z_n) \to \infty$ as $n \to \infty$, and

$\exists$ $(z_{j_n})_{n\in\mathbb{N}}$ $z_{j_n} \to x$, as $n \to \infty$

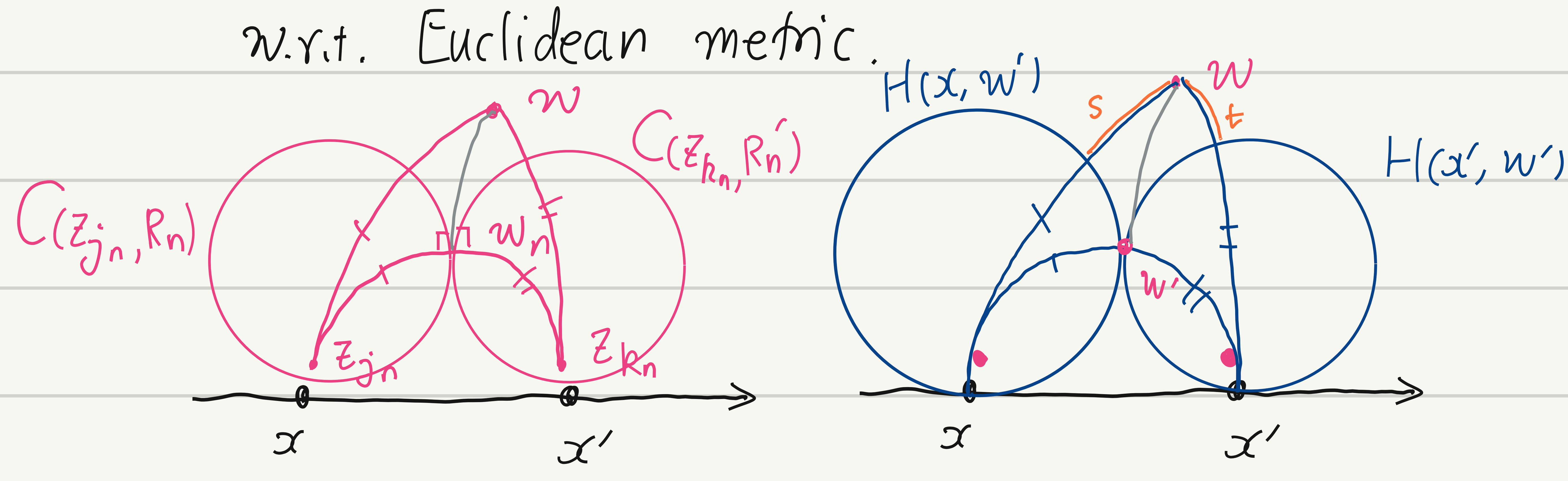
$(z_{k_n})_{n\in\mathbb{N}}$ $z_{k_n} \to x'$, as $n \to \infty$

(I) $$(z_{j_n}, z_{j_{n'}})_w = \underbrace{d_{\mathbb{H}}(z_{j_n}, w)}_{\leq R} + \underbrace{d_{\mathbb{H}}(z_{j_{n'}}, w)}_{\leq R} - \underbrace{d_{\mathbb{H}}(z_{j_n}, z_{j_{n'}})}_{\leq 2R}$$

$$\leq 4R$$

Hence $(z_j, z_k)_w \not\to \infty$, as $j, k \to \infty$

(II) We consider z_{j_n} close to x and z_{k_n} close to x' w.r.t. Euclidean metric.



$R_n = d_{\mathbb{H}}(z_{j_n}, w_n)$
$R_n' = d_{\mathbb{H}}(z_{k_n}, w_n)$

$R_n + R_n' = d_{\mathbb{H}}(z_{j_n}, z_{k_n})$

Notice that $d_E(z_{j_n}, x) \to 0$, $j_n \to \infty$

$d_E(z_{k_n}, x') \to 0$, $k_n \to \infty$

$\Rightarrow$ $d_E(w_n, w') \to 0$, as $n \to \infty$

Hence $d_{\mathbb{H}}(w_n, w') \to 0$, as $n \to \infty$.

And $C(z_{j_n}, R_n) \xrightarrow[\text{Euclidean}]{} H(x, w')$, $C(z_{k_n}, R_n') \xrightarrow[\text{Euclidean}]{} H(x', w')$, as $n \to \infty$

Hence $(z_{j_n}, z_{k_n})_w \longrightarrow s+t$, as $j_n, k_n \to \infty$

Hence $(z_j, z_k) \not\to \infty$, as $j,k \to \infty$. ∎

<u>Rmk</u>: This means $(z_j, z_k) \to \infty$, as $j,k\to\infty$ is equivalent to the convergence w.r.t. Euclidean metric.

<u>Rmk</u>: The proposition is independent of the choice of w.

4. Equivalence relation among sequences converging to infinity.

For two sequences converging to infinity $(z_n)_{n\in\mathbb{N}}$ and $(w_n)_{n\in\mathbb{N}}$.

(*) $(z_n)_{n\in\mathbb{N}}$ and $(w_n)_{n\in\mathbb{N}}$ are <u>equivalent</u> $\left((z_n)_{n\in\mathbb{N}} \sim (w_n)_{n\in\mathbb{N}}\right)$ if

$$(z_j, w_k)_w \longrightarrow \infty, \quad \text{as } j,k\to\infty$$

(Gromov) $\partial\mathbb{H} := \{(z_n)_{n\in\mathbb{N}} \mid (z_j, z_k)_w \to \infty, \text{ as } j,k\to\infty\} / \sim$

Let γ^+ be a ray in $\mathbb{H}$ with $\gamma^+(0) = w$ ending at x

Then $(\gamma^+(n))_{n\in\mathbb{N}}$ is a sequence converging to x.

- $(\gamma^+(j), \gamma^+(k))_w \to \infty$, as $j,k\to\infty$.

$\gamma^+ \sim \eta^+ \Rightarrow \boxed{(\gamma^+(j), \eta^+(k))_w \to \infty, \text{ as } j,k\to\infty} \Rightarrow (\gamma^+(n))_{n\in\mathbb{N}} \sim (\eta^+(n))_{n\in\mathbb{N}}$

$\gamma^+ \not\sim \eta^+ \Rightarrow \boxed{(\gamma^+(j), \eta^+(k))_w \to M<\infty, \text{ as } j,k\to\infty} \Rightarrow (\gamma^+(n))_{n\in\mathbb{N}} \not\sim (\eta^+(n))_{n\in\mathbb{N}}$

by the construction in the proof of the previous proposition.

<u>Hence</u>: $\underbrace{[\gamma^+]}_{\text{Equivalence class of rays}} = x = \underbrace{[(\gamma^+(n))_{n\in\mathbb{N}}]}_{\text{Equivalence class of sequence}}$

- A sequence $(z_n)_{n\in\mathbb{N}}$ <u>converges to x</u> (**) if $(z_n)_{n\in\mathbb{N}} \in [(\gamma^+(n))_{n\in\mathbb{N}}]$

5. Metric on $\partial\mathbb{H}$

Let $(z_n)_{n\in\mathbb{N}}$ and $(w_n)_{n\in\mathbb{N}}$ be two sequences converges to different points in $\partial\mathbb{H}$. $(x\neq x')$

$z_n \to x$, $w_n \to x'$ as $n\to\infty$ in the sense of (**)

Hence $(z_n)_{n\in\mathbb{N}} \not\sim (w_n)_{n\in\mathbb{N}}$.

The <u>Gromov product</u> between x and x' is $(x, x')_w := \liminf_{n\to\infty} (z_n, w_n)_w$

$$d_w(x,x') := e^{-\frac{(x,x')_w}{2}}$$

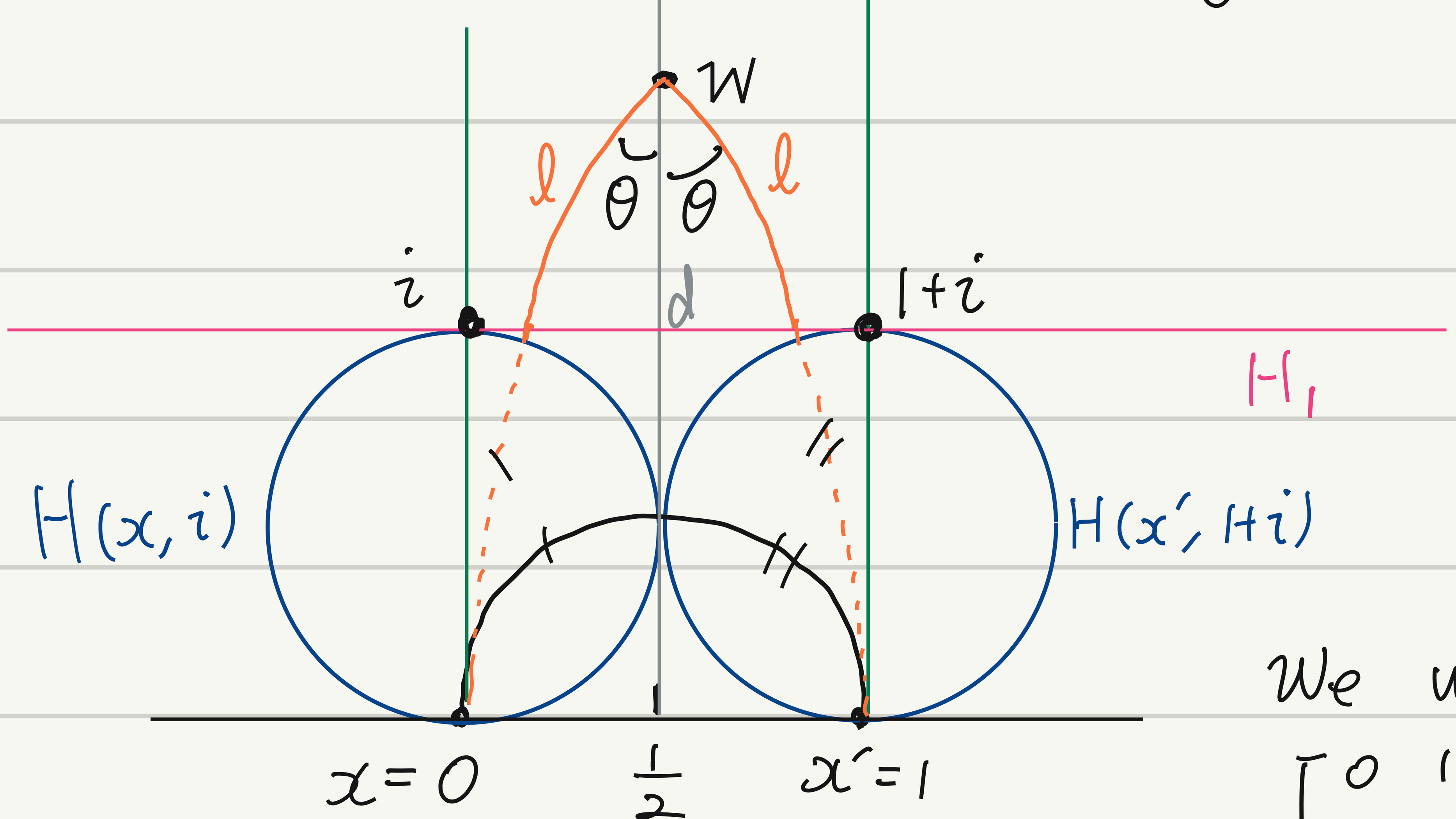
Prop: $(\partial\mathbb{H}, d_w)$ is a metric space.

Proof:
① $d_w(x,x') \geq 0$, "$=0$" iff $x=x'$ $\left(e^{-t} \to 0 \text{ as } t \to +\infty\right)$

② $d_w(x,x') = d_w(x',x)$ (Symmetry $\quad (z,z')_w = (z',z)_w$)

③ $d_w(x,x') + d_w(x',x'') \geq d_w(x,x'')$

To show ③, we use isometry to put w, x, x' in a standard position



$x=0 \qquad Re(w) = \frac{1}{2}$.

$x'=1$

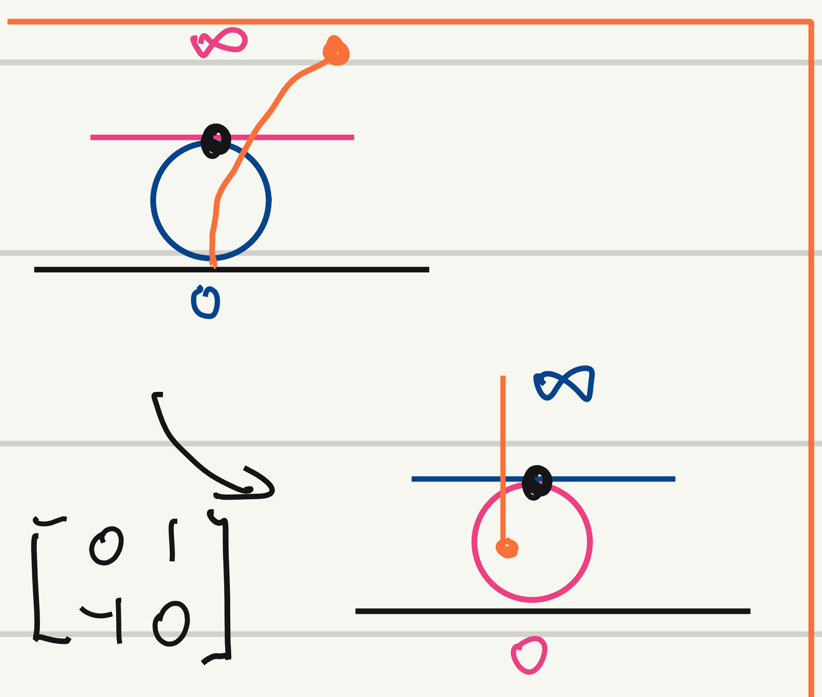
Let $w = \frac{1}{2} + iy$.

To compute $d_{\mathbb{H}}(H(x,i), w)$

We use the π-rotation at i

$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ exchanges $H(x,i)$ and H_1

0 and ∞



$$d_{\mathbb{H}}(H(x,i), w) = d_{\mathbb{H}}(H_1, -\tfrac{1}{w}) = \left|\log \operatorname{Im}(-\tfrac{1}{w})\right| = \ell$$

radius become vertical geodesics.

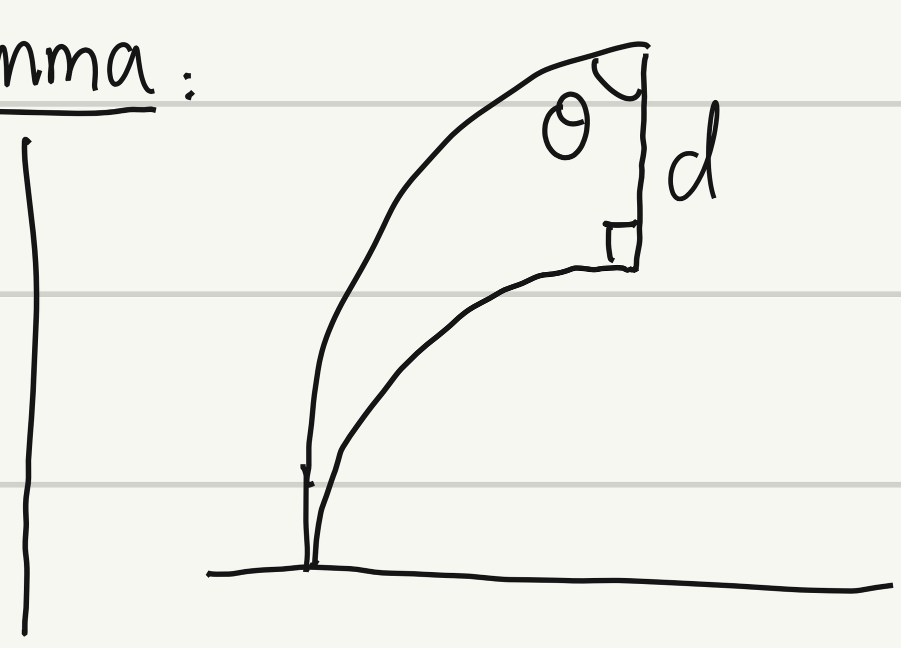
$$-\frac{1}{w} = -\frac{1}{\frac{1}{2}+iy} = \frac{-2+4yi}{1+4y^2}$$

$$d_{\mathbb{H}}(\gamma_{[x,x']}, w) = d = \log \frac{y}{\frac{1}{2}} = \log 2y \Rightarrow 2y = e^d$$

Hence $\operatorname{Im}(-\frac{1}{w}) = \frac{2e^d}{1+e^{2d}} = (\operatorname{ch} d)^{-1}$

Hence $(x,x')_w = 2\ell = 2\log(\operatorname{ch} d)$ and $d_w(x,x') = \frac{1}{\operatorname{ch} d}$

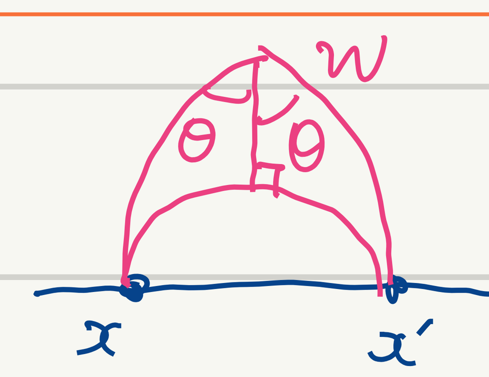
Lemma:



$\operatorname{ch} d \cdot \sin\theta = 1$

Trigonometry formula for triangles with interior angle 0, $\frac{\pi}{2}$ and θ

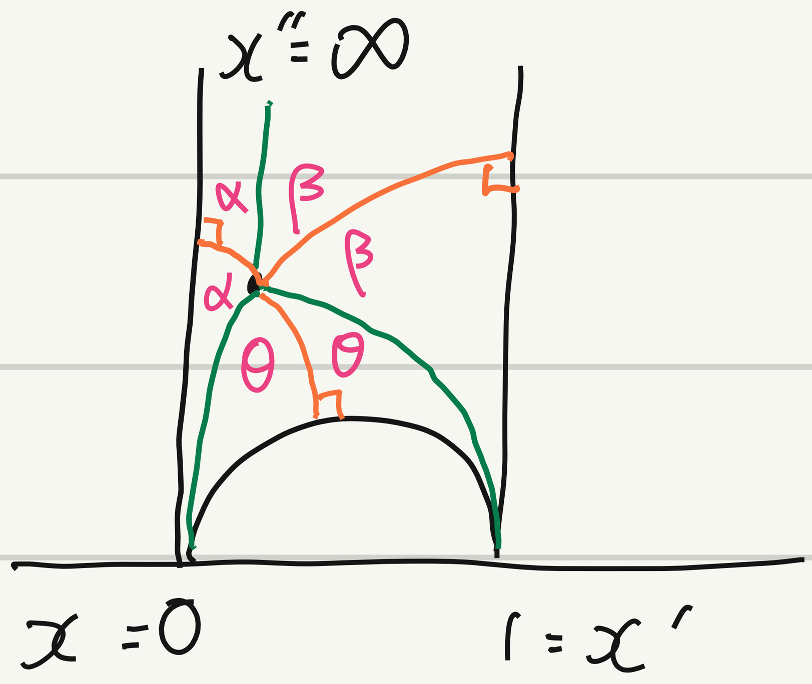
Hence $d_w(x,x') = \sin\theta = \sin\frac{\angle xwx'}{2}$



Now we use a different renormalization.

WOLG, let $x=0$, $x'=1$, $x''=\infty$. Then w could be anywhere.

We assume that



We have $\alpha+\beta+\theta=\pi$

Moreover, we have

$d_w(x,x') = \sin\theta$

$d_w(x,x'') = \sin\alpha$

$d_w(x',x'') = \sin\beta$

Hence the goal is to show

$$d_w(x,x'') = \sin\alpha \le \sin\beta + \underbrace{\sin\theta}_{\substack{\parallel \\ \sin(\pi-\alpha-\beta) = \sin(\alpha+\beta)}} = d_w(x,x') + d_w(x',x'')$$

Notice that $0 \le \alpha \le \frac{\pi}{2}$, $0 \le \beta \le \frac{\pi}{2}$, $0 \le \theta \le \frac{\pi}{2}$

We consider $\sin(\alpha+\beta) + \sin\beta - \sin\alpha = f_\beta(\alpha)$

For any β, $f_\beta'(\alpha) = \dfrac{d f_\beta(\alpha)}{d\alpha} = \cos(\alpha+\beta) - \cos\alpha$

$f_\beta'(\alpha) = 0 \iff \underbrace{\cos\alpha}_{\in[0,\frac{\pi}{2}]} = \underbrace{\cos(\alpha+\beta)}_{\in[0,\frac{\pi}{2}+\beta]} \iff \beta = 0$ (i.e. $w = x$ impossible)

Since $\cos\alpha > \cos(\alpha+\beta)$ we have $f_\beta'(\alpha) < 0$ ← Cos is monotonically decreasing in $[0,\pi]$

Hence $\min f_\beta(\alpha) = f_\beta(\frac{\pi}{2}) = \cos\beta + \sin\beta - 1 > 0$

↑ $\alpha = \frac{\pi}{2}$ i.e. $w \in \gamma[x,x'']$

Hence $d_w(x,x'') \le d_w(x,x') + d_w(x',x'')$.

The proof for

is the same. This time we have $\beta = \alpha + \theta$

<u>Rmk.</u> $d_w(x,x') = 0 \iff \sin\theta = 0$ for $\theta \in [0,\frac{\pi}{2}]$

$\iff \theta = 0 \iff x = x'$.

<u>Prop</u>: $\forall x, x' \in \partial\mathbb{H}$, $d_w(x,x') \le 1$

$d_w(x,x') = 1 \iff w \in \gamma[x,x']$.

↑ geodesic with end point x and x'.

<u>Prop</u>: The topology induced by this metric is the same as the one in the lecture.

Prop: $(\partial\mathbb{H}, d_w)$ is compact.

$\boxed{\text{Induced topologys are equivalent}}$

Prop: $(\partial\mathbb{H}, d_w) \xrightarrow{id} (\partial\mathbb{H}, d_{w'})$ is a bi-Lipschitz map

i.e. $\exists\, C_1, C_2$ s.t. $\forall x, x' \in \partial\mathbb{H}$, we have

$$C_1 d_{w'}(x, x') \le d_w(x, x') \le C_2 d_{w'}(x, x')$$

II. Topology of $\overline{\mathbb{H}}$

1. Open sets

The basis of $\overline{\mathbb{H}}$ is given by (+ ϕ and $\overline{\mathbb{H}}$)

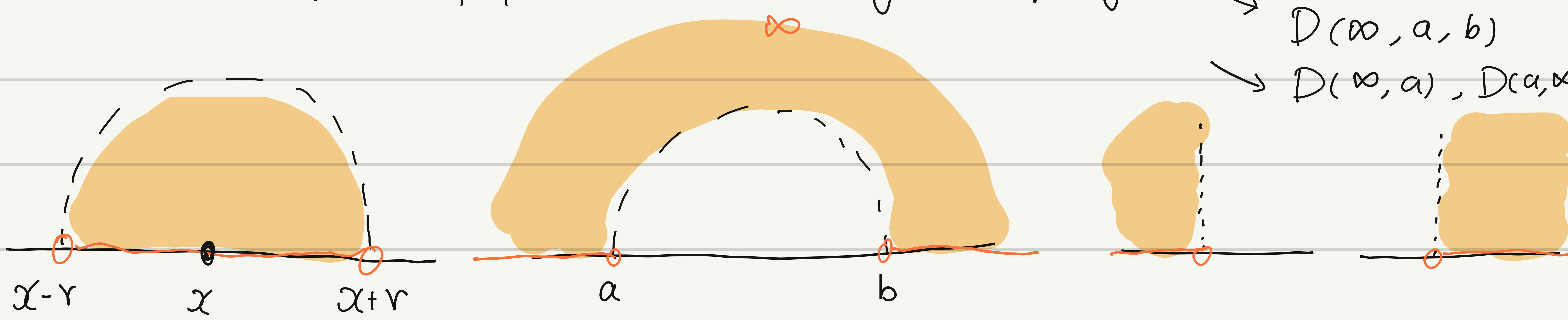
① disk in $\mathbb{H}$

② Open half plane with boundary at infinity.

$D(x, r)$

$D(\infty, a, b)$

$D(\infty, a)$, $D(a, \infty)$



Some discussion:

Let P denote a half plane, i.e. a connected component of $\mathbb{H}\setminus\gamma$

geodesic.

Similarly to $\partial\mathbb{H}$, we can ask what is the ∂P.

We consider rays in $\mathbb{H}$. Let η^+ be a ray

Prop: η^+ is one of the following type

① $\eta^+ \subseteq P$

② $\eta^+ \cap \gamma \neq \phi$ with $\eta^+(0) \in \overline{P}^c$

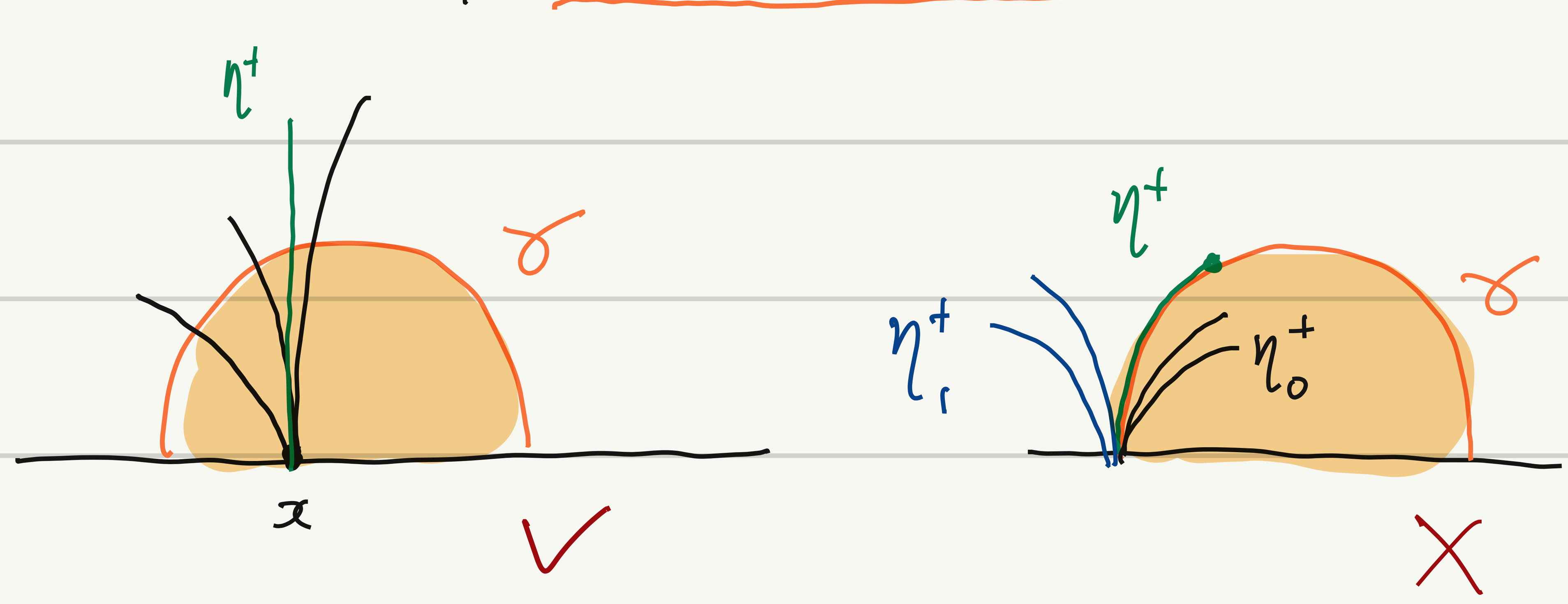
③ $\eta^+ \cap \gamma \neq \phi$ with $\eta^+(0) \in \overline{P}$

④ $\eta^+ \subset \gamma$

For all $\eta_0^+ \in [\eta^+]$, eventually it is contained in P (up to taking away the starting part)

$$\partial P = \{[\eta^+] \mid \forall \eta_0^+ \sim \eta^+, \exists t_0 \text{ s.t. } \forall t > t_0, \eta_0^+(t) \in P\}.$$

Ex:

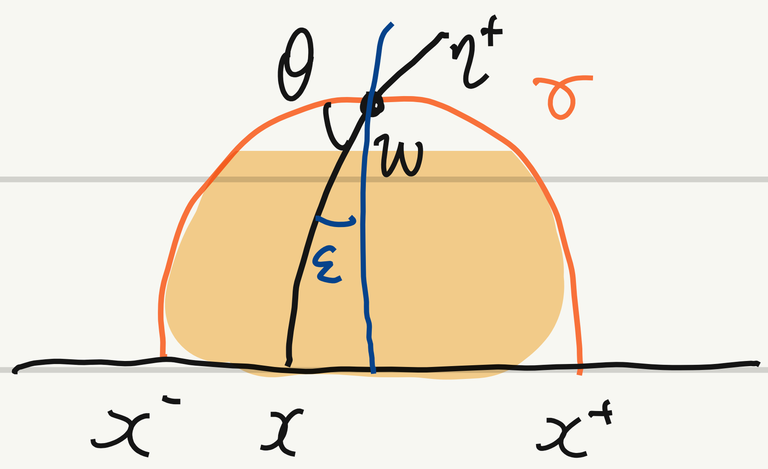


η_1^+ is not in P

$\eta_1^+ \sim \eta^+$.

$\boxed{\begin{array}{l} x = [\eta^+] \in \partial\mathbb{H} \\ \text{Should contain all} \\ \text{rays ending at } x \end{array}}$

Let γ^+ and γ^- be two rays in γ ending at x^+ and x^- respectively.



If $\eta^+ \in \partial P$, then $\eta^+ \not\propto \gamma^+$ and $\eta^+ \not\propto \gamma^-$

$\eta^+ \not\subset \gamma$.

Let θ be the intersection angle. $\Rightarrow \theta \neq 0$ or π.

$\exists \varepsilon > 0$ s.t. η' having intersection angle $\alpha \in]-\varepsilon, \varepsilon[$ with η^+,

$\exists \eta'^+ \subset \eta$ s.t. $[\eta'^+] \in \partial P$

Let w be the intersection point.

Consider $(\partial \mathbb{H}, d_w)$. ① $x \in]x^-, x^+[= \partial P$

② $\forall x' \in]x - \sin\frac{\varepsilon}{2}, x + \sin\frac{\varepsilon}{2}[$, $x' \in]x^-, x^+[. = \partial P$. ∎

- An open set in $\overline{\mathbb{H}}$ is given by
  - finite intersection of basis open sets
  - any union of basis open sets.
- $\forall z \in \overline{\mathbb{H}}$, a (open) neighborhood of z is an open set U with $z \in U$
- Let U be a neighborhood of z.

Prop: If $z \in \mathbb{H}$, $\exists D(z, R) \subset U$ for some R

If $z = x \in \mathbb{R}$, $\exists D(x, r) \subset U$ for some r

If $z = \infty$, $\exists D(\infty, a, b) \subset U$ for some $a < b$

Rmk: $\partial D(x, r) = C(x, r)$ geodesic

$\partial D(\infty, a, b) = C(\frac{a+b}{2}, \frac{b-a}{2})$ geodesic.

$\partial D(\infty, a) = \partial D(a, \infty) = V_a$ geodesic.

2. Convergence of sequence in $\overline{\mathbb{H}}$

$(z_n)_{n \in \mathbb{N}}$ converges to $z \in \overline{\mathbb{H}}$ if

- $z = \infty$, $|z_n| \to \infty$, as $n \to \infty$
- $z \neq \infty$, $|z_n - z| \to 0$, as $n \to \infty$

Rmk: For $z \in \partial \mathbb{H}$, we can also use Gromov product to describe the convergence as previously discussed.

3. Extend an isometry to a continuous map $\overline{\mathbb{H}} \to \overline{\mathbb{H}}$.

Prop: $\forall f \in \mathrm{Isom}(\mathbb{H})$, $\exists\, \partial f : \partial\mathbb{H} \to \partial\mathbb{H}$.

Proof: f preserve - the equivalence relation among rays

- the equivalence relation among sequence ....

i.e. $\gamma^+ \sim \eta^+ \Leftrightarrow f(\gamma^+) \sim f(\eta^+)$

$$(z_n)_{n\in\mathbb{N}} \sim (w_n)_{n\in\mathbb{N}} \overset{\circledast}{\Longleftrightarrow} (f(z_n))_{n\in\mathbb{N}} \sim (f(w_n))_{n\in\mathbb{N}}$$

$$(z_j, w_k)_w \to \infty, \text{ as } j,k \to \infty$$

$$\Leftrightarrow (f(z_j), f(w_k))_{f(w)} \to \infty, \text{ as } j,k \to \infty$$

$$\Leftrightarrow (f(z_j), f(w_k))_w \to \infty, \text{ as } j,k \to \infty$$

Equivalence relation is independant of the base point w.

Cor: ∂f is bijective.

Proof: $\circledast$

Let $C(w,R)$ be the circle centered at w of radius R.

Prop: $\exists$ a bi-Lipschitz map : $\left(C(w,R), d_{\mathbb{H}}\big|_{C(w,R)}\right) \xrightarrow{\partial_w} (\partial\mathbb{H}, d_w)$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad {}^{\prime\prime}$ d_C

Proof: Consider γ_0^+ with $\gamma_0^+(0) = w$

All radius can be parametrized by $[0, 2\pi)$ by considering rotation from γ_0^+ along the positive rotation direction, denoted by γ_θ^+

Let $\{z_\theta\} = \gamma_\theta^+ \cap C(w,R)$, and x_θ be the endpoint of γ_θ^+.

$$d_C(z_\theta, z_{\theta'}) = |\theta - \theta'| \sinh R = \frac{|\theta-\theta'|}{2}(2\sinh R)$$

$$d_w(x_\theta, x_{\theta'}) = \sin\frac{|\theta-\theta'|}{2}$$

Lemma: $\frac{\alpha}{2} < \sin\alpha < \alpha$ for $\alpha \in [0, \frac{\pi}{2}]$

Hence we have $C_1 = C_2 = \frac{1}{2\sinh R}$ s.t.

$$\frac{C_1}{2} d_C(z_\theta, z_{\theta'}) \le d_w(x_\theta, x_{\theta'}) \le C_1 d_C(z_\theta, z_{\theta'})$$

Prop: ∂f is a bi-Lipschitz map.

Proof: $\partial f : \partial\mathbb{H} \underset{\text{bi-Lip}}{\xrightarrow{\partial_w}} C(w,R) \underset{\text{isom}}{\xrightarrow{f}} C(f(w),R) \underset{\text{bi-Lip}}{\xrightarrow{\partial_{f(w)}}} \partial\mathbb{H}$

Prop: $\tilde{f} : \overline{\mathbb{H}} \to \overline{\mathbb{H}}$ is continuous

Proof: ① $\tilde{f}^{-1}$ sends · $D(w, R)$ to $D(f(w), R)$ $\quad \tilde{f}^{-1} = f^{-1} \cup \partial f^{-1}$

· $P \cup \partial P$ to $Q \cup \partial Q$

② $\tilde{f}^{-1}$ preserves the finite intersections and arbitrary unions among basis open set.

$$\tilde{f}^{-1}(B_1 \cap \cdots \cap B_n) = \tilde{f}^{-1}(B_1) \cap \cdots \cap \tilde{f}^{-1}(B_n)$$

still a basis open set. $\quad$ open set.

$$\tilde{f}^{-1}\left(\bigcup_\alpha B_\alpha\right) = \bigcup_\alpha \tilde{f}^{-1}(B_\alpha)$$

open set.

Hence $\tilde{f}$ is continuous. ∎

Rmk: A map $f : X \to Y$ between topological space is continuous if $\forall V$ open set in Y, $f^{-1}(V)$ is an open set in X.

4. Another way to see that $\tilde{f}$ is continuous

$F(\mathbb{H}) = \mathbb{H}$ or $F \circ \psi(\mathbb{H}) = \mathbb{H}$

(orientation preserving)

Notice that f is the restriction of a Möbius transformation F on $\hat{\mathbb{C}}$ which is continuous

or $F \circ \psi$ (orientation reversing)

Since the topology on $\hat{\mathbb{C}}$ restricted to $\overline{\mathbb{H}}$ is equivalent to what we described, by taking $\tilde{f} = \begin{cases} F|_{\overline{\mathbb{H}}} \\ F \circ \psi|_{\overline{\mathbb{H}}} \end{cases}$, we get a continuous map on $\overline{\mathbb{H}}$.