6. $GL(2,\mathbb{R})$ action on $\mathbb{H}$

$\psi : \mathbb{C} \to \mathbb{C}$

$z \mapsto \bar{z}$

$$\tau(z) = \frac{x\bar{z} + r^2 - x^2}{1\cdot\bar{z} - x}$$

$$A = \begin{bmatrix} x & r^2 - x^2 \\ 1 & -x \end{bmatrix}$$

$$\tau(z) = -\bar{z} + t = \frac{-\bar{z}+t}{0\cdot\bar{z}+1}$$

$$A = \begin{bmatrix} -1 & t \\ 0 & 1 \end{bmatrix}$$

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $\quad ad - bc \neq 0.$

$$h_A(z) = \frac{az+b}{cz+d} : \mathbb{C}\setminus\mathbb{R} \to \mathbb{C}\setminus\mathbb{R}$$

$\mathbb{C}\setminus\mathbb{R} = \mathbb{H} \cup \mathbb{L}.$

$$\tau = \psi \circ h_A|_{\mathbb{H}}$$

Prop: $h_A : \mathbb{H}\cup\mathbb{L} \to \mathbb{H}\cup\mathbb{L}$

$\det A > 0 \quad h_A(\mathbb{H}) = \mathbb{H} \quad h_A(\mathbb{L}) = \mathbb{L}$

$\det A < 0 \quad h_A(\mathbb{H}) = \mathbb{L} \quad h_A(\mathbb{L}) = \mathbb{H}.$

Proof:

$$h_A(z) - h_A(\bar{z}) = \frac{az+b}{cz+d} - \frac{a\bar{z}+b}{c\bar{z}+d}$$

ψ

$$= \frac{(ac|z|^2+bd) + (adz + bc\bar{z})}{|cz+d|^2} - \frac{(ac|z|^2+bd)+(ad\bar{z}+bcz)}{|cz+d|^2}$$

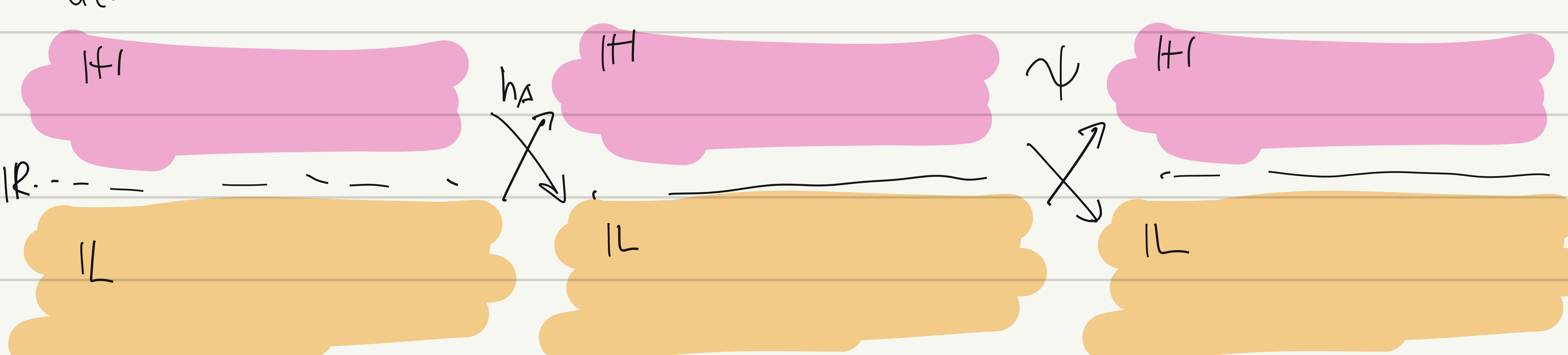
$$= \frac{(ad-bc)(z-\bar{z})}{|cz+d|^2} = \frac{\det A \; 2i\,\mathrm{Im}\,z}{|cz+d|^2}$$

If $\mathrm{Im}\, z > 0$

$\det A > 0, \quad \mathrm{Im}(h_A(z)) > 0$

$\det A < 0, \quad \mathrm{Im}(h_A(z)) < 0$ ∎

$\det A < 0$



$$GL(2,\mathbb{R}) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_{2\times 2}(\mathbb{R}) \;\middle|\; ad-bc \neq 0 \right\}.$$

$$GL(2,\mathbb{R}) \xrightarrow{F} \mathcal{F}(\mathbb{H})$$

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \longmapsto f_A = \begin{cases} h_A|_{\mathbb{H}} & \det A > 0 \\ h_A \circ \psi|_{\mathbb{H}} & \det A < 0 \end{cases}$$

$\forall A \in GL(2,\mathbb{R})$

Prop: $h_A \circ \psi = \psi \circ h_A$.

$\mathbb{H} \sqcup \mathbb{L}$

Def: A group G acts on X = set if $\forall g \in G$, $\exists f_g : X \to X$

s.t. ① $f_{id} : X \to X$, $x \mapsto x$

② $\forall g, g' \in G$, $f_{gg'} = f_g \circ f_{g'}$ ($g \cdot g'$)

For $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $f_A(z) = z$

For $A_1 = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}$ $A_2 = \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}$

$$A_1A_2 = \begin{bmatrix} a_1a_2 + b_1c_2 & a_1b_2 + b_1d_2 \\ a_2c_1 + b_2d_1 & b_2c_1 + d_1d_2 \end{bmatrix}$$

$$f_{A_1} \circ f_{A_2}(z) = \frac{a_1 \frac{a_2z+b_2}{c_2z+d_2} + b_1}{c_1 \frac{a_2z+b_2}{c_2z+d_2} + d_1}$$

$$\cdots = \frac{(a_1a_2 + b_1c_2)z + \cdots}{\cdots}$$

$$= f_{A_1A_2}(z) \quad \blacksquare$$

Prop: $GL(2,\mathbb{R})$ acts on $\mathbb{H}$.

Prop: This action is isometric. ($\forall A$, f_A is an isometry.

Proof, Check: $\left| \dfrac{f_A(w) - f_A(z)}{f_A(\bar{w}) - f_A(z)} \right| = \left| \dfrac{w-z}{\bar{w}-z} \right|$ (ψ)

$$\text{LHS} = \left| \frac{\frac{aw+b}{cw+d} - \frac{az+b}{cz+d}}{\frac{a\bar{w}+b}{c\bar{w}+d} - \frac{az+b}{cz+d}} \right| = \left| \frac{(aw+b)(cz+d) - (cw+d)(az+b)}{(a\bar{w}+b)(cz+d) - (c\bar{w}+d)(az+b)} \right|$$

$c\bar{w} + d = \overline{cw+d}$

$$= \left| \frac{(ad-bc)(w-z)}{(ad-bc)(\bar{w}-z)} \right| \Rightarrow d_{\mathbb{H}}(f_A(w), f_A(z)) = d_{\mathbb{H}}(w,z) \quad \blacksquare$$

Hence $GL(2,\mathbb{R}) \xrightarrow{F} \underline{Isom(\mathbb{H})}$

$A \longmapsto \underline{f_A}$

- F is a homomorphism. ✓
- F is surjective. ✓

$\forall f \in Isom(\mathbb{H})$, $\exists A \in GL(2,\mathbb{R})$ s.t. $f_A = f$.

Hint: $\tau(z) = f_A(z)$ $A = \begin{bmatrix} x & r^2-x^2 \\ 1 & -x \end{bmatrix}$ $\det A = -r^2 < 0$

$h_{A_1A_2} = h_{A_1} \circ h_{A_2}$ $f_{A_1A_2} = (\psi \circ h_{A_1}) \circ (\psi \circ h_{A_2}) = \underbrace{\psi^2}_{= id.} \circ h_{A_1A_2} = h_{A_1A_2}$

Prop: $\ker(F) = \{\lambda I_2 = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \mid \lambda \neq 0\}$

Proof: $A = \lambda Id$ iff $A \in \ker(F)$ (i.e. $F(A) = f_A = \underline{id}: \mathbb{H} \to \mathbb{H}$)

"$\Leftarrow$" $x_1 = 0$ $x_2 = \infty$ $x_3 = 1$ $f(x_1)$ $f(x_2)$ $f(x_3)$ determine f.

$f(z) = \frac{az+b}{cz+d}$ $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$\begin{cases} f(0) = 0 \\ f(1) = 1 \\ f(\infty) = \infty \end{cases} \overset{?}{\Rightarrow} A = \lambda Id.$

$f(0) = \frac{b}{d}$ $f(1) = \frac{a+b}{c+d}$ $f(\infty) = \frac{a}{c}$.

$\frac{b}{d} = 0$ $\frac{a+b}{c+d} = 1$ $\frac{a}{c} = \infty$

$\Downarrow$ $\Downarrow$

$\begin{cases} d \neq 0 \\ b = 0 \end{cases}$ + + $\begin{cases} a \neq 0 \\ c = 0 \end{cases}$

$\Downarrow$

$a = d \neq 0 \Rightarrow \underline{A = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}}$

"$\Rightarrow$" $A = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$ $f_A(z) = h_A(z) = \frac{\lambda z + 0}{0 \cdot z + \lambda} = z$ ✓

$\det A > 0$

$\ker(F) = \{\lambda I_2 \mid -\} \subset GL(2,\mathbb{R})$.

Prop: $Isom(\mathbb{H}) \cong GL(2,\mathbb{R}) / \{\lambda I_2 \mid -\} =: PGL(2,\mathbb{R})$

$\frac{az+b}{cz+d} = \frac{\lambda az + \lambda b}{\lambda cz + \lambda d}$

$Z(GL(2,\mathbb{R})) = \{\lambda I_2 \mid \lambda \neq 0\}$

$\begin{bmatrix}\lambda & 0\\ 0 & \lambda^{-1}\end{bmatrix}\begin{bmatrix}a & b\\ c & d\end{bmatrix}\begin{bmatrix}\lambda^{-1} & 0\\ 0 & \lambda\end{bmatrix}$ $\quad \begin{bmatrix}\lambda & 0\\ 0 & \lambda\end{bmatrix}$.

$\begin{bmatrix}a & \lambda^2 b\\ \lambda^{-2} c & d\end{bmatrix} = \begin{bmatrix}a & b\\ c & d\end{bmatrix} \iff \lambda = \pm 1.$

$\begin{bmatrix}p & q\\ r & s\end{bmatrix} = \begin{bmatrix}\lambda & \\ & \lambda^{-1}\end{bmatrix}\begin{bmatrix}1 & t\\ & 1\end{bmatrix}\begin{bmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{bmatrix}.$

$\det = 1$

$\begin{bmatrix}1 & t\\ 0 & 1\end{bmatrix}\begin{bmatrix}a & b\\ c & d\end{bmatrix}\begin{bmatrix}1 & -t\\ 0 & 1\end{bmatrix} = \begin{bmatrix}a+ct & b+dt\\ c & d\end{bmatrix}\begin{bmatrix}1 & -t\\ 0 & 1\end{bmatrix}$

$\begin{bmatrix}-1 & 0\\ 0 & 1\end{bmatrix}$ $\qquad = a+ct$

KAN 分解.

$AB_1A^{-1} \cdot AB_2A^{-1} = AB_1B_2A^{-1}$

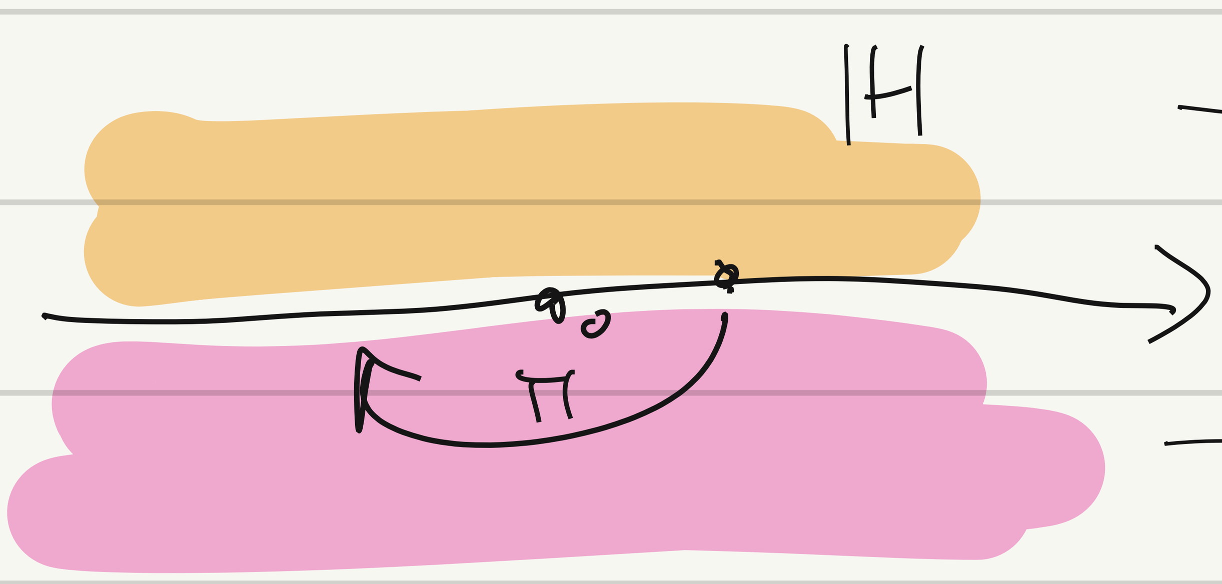
$\;\; \| \qquad\qquad \| \qquad\qquad \|$

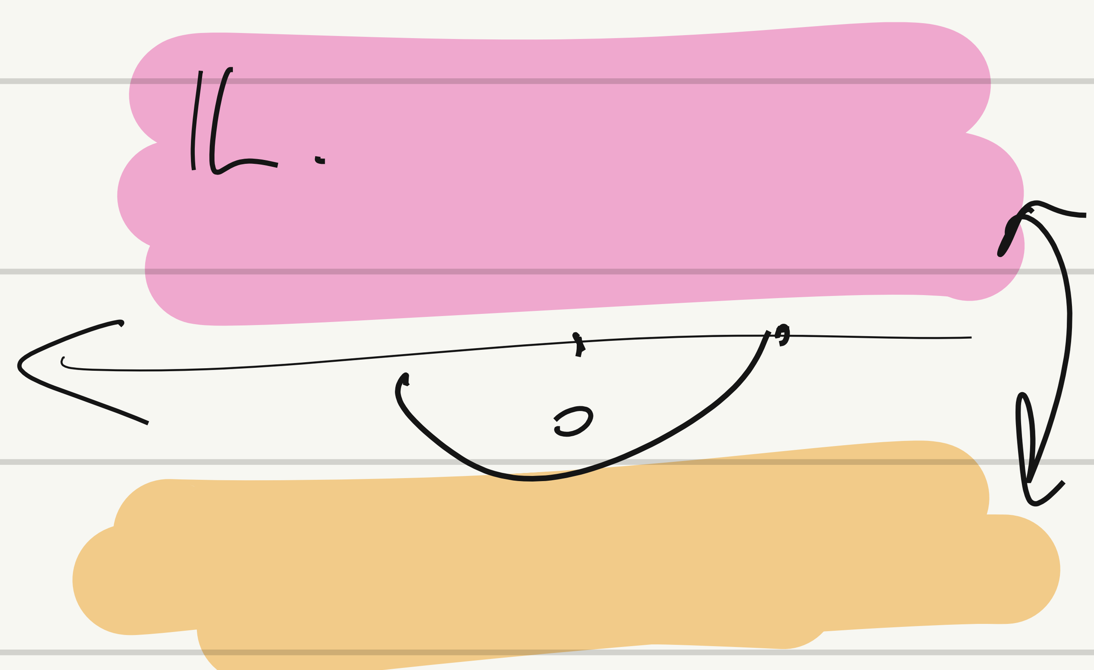
$\;\; B_1 \qquad\quad B_2 \;\Rightarrow\; B_1B_2.$

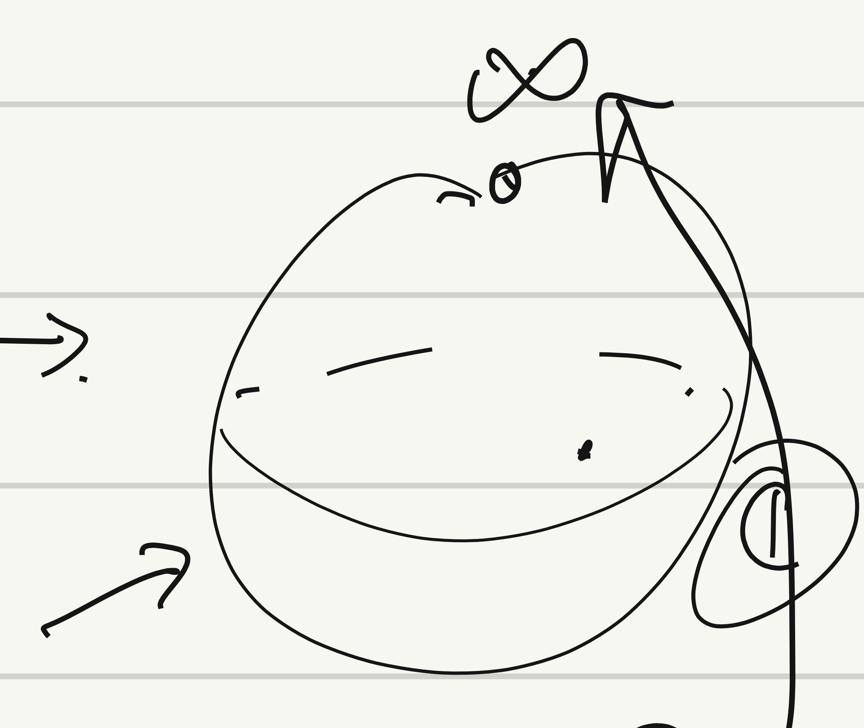
$GL(2,\mathbb{R}) \ni A$: $\det A > 0 \rightarrow SL(2,\mathbb{R})/\pm Id = PSL(2,\mathbb{R}) \cong Isom^+(\mathbb{H})$

$\det A < 0 \searrow f_A$

$\det A < 0.$







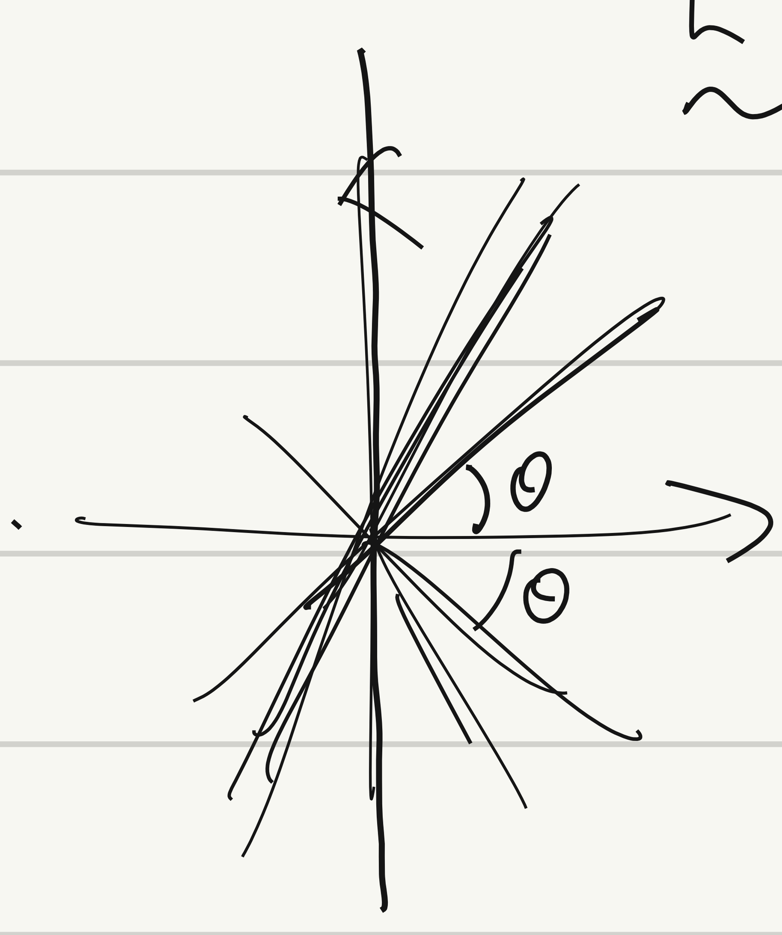
$\mathbb{C}^2.$ $\quad \begin{bmatrix}z_1\\ z_2\\ 1\end{bmatrix} \quad \begin{bmatrix}1\\ 0\end{bmatrix}$

$\begin{bmatrix}a & b\\ c & d\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix} = \begin{bmatrix}ax+by\\ cx+dy\end{bmatrix}$

$\in \mathbb{R}^2 \qquad \in \mathbb{R}^2.$

$\mathbb{C}P^1$

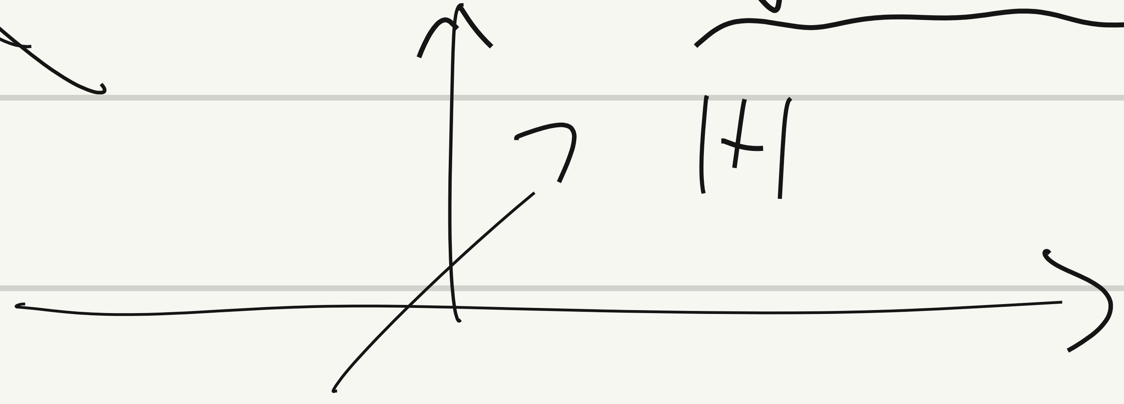
$GL(2,\mathbb{C})$: $\begin{bmatrix}a & b\\ c & d\end{bmatrix}\begin{bmatrix}z_1\\ z_2\end{bmatrix} = \begin{bmatrix}az_1+bz_2\\ cz_1+dz_2\end{bmatrix}$



$\infty \qquad \dfrac{x}{y}. \quad \mathbb{R}P^1 \qquad \dfrac{ax+by}{cx+dy}$

$\mathbb{H}$

$\dfrac{z_1}{z_2} = \dfrac{az_1+bz_2}{cz_1+dz_2}$



$\partial \mathbb{H} \cong \mathbb{R}P^1$