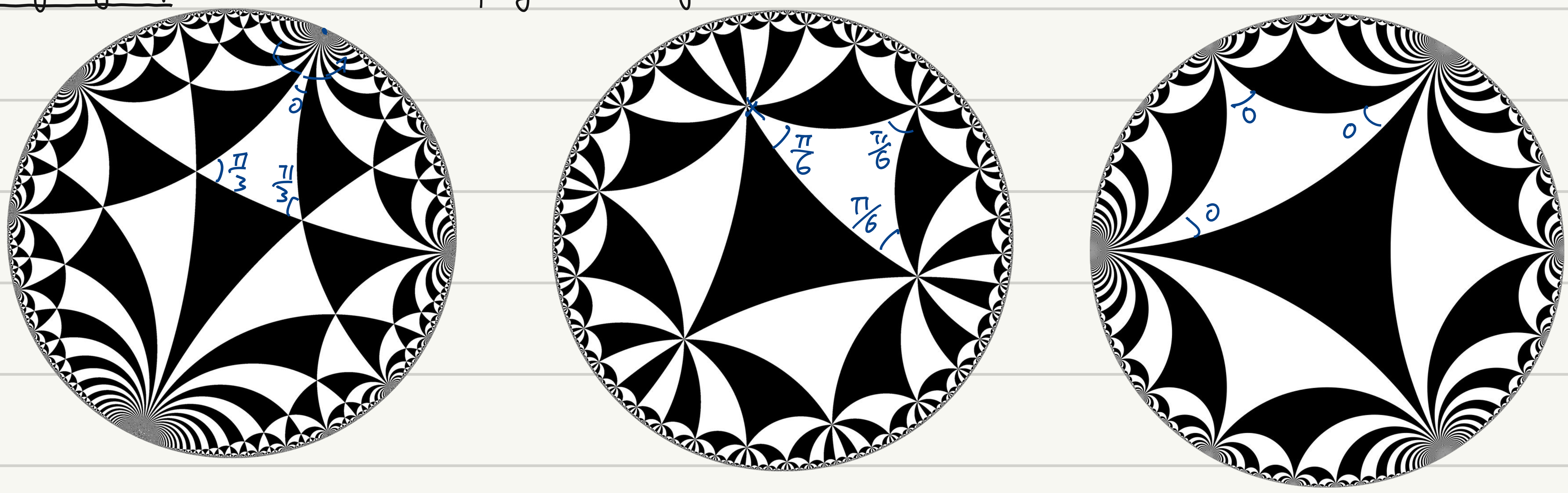
Triangle group (From Wiki page: Triangle Group)



$\Delta(3,3,\infty)$ $\quad$ $\Delta(6,6,6)$ $\quad$ $\Delta(\infty,\infty,\infty)$

$\Delta(n,\infty,\infty)$ $n>1$

geodesic

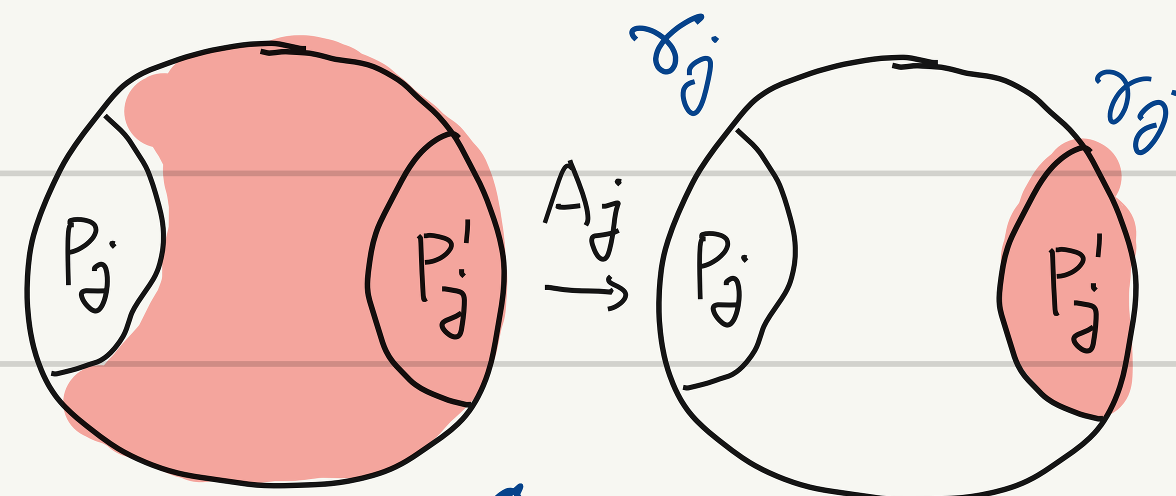
Schottky group $\gamma_1 \dots \gamma_n, \gamma_1' \dots \gamma_n'$ s.t. $\forall j,k$ $P_j \cap P_k = \phi$ $\quad P_j \cap P_k' = \phi$ $\quad P_j' \cap P_k' = \phi$

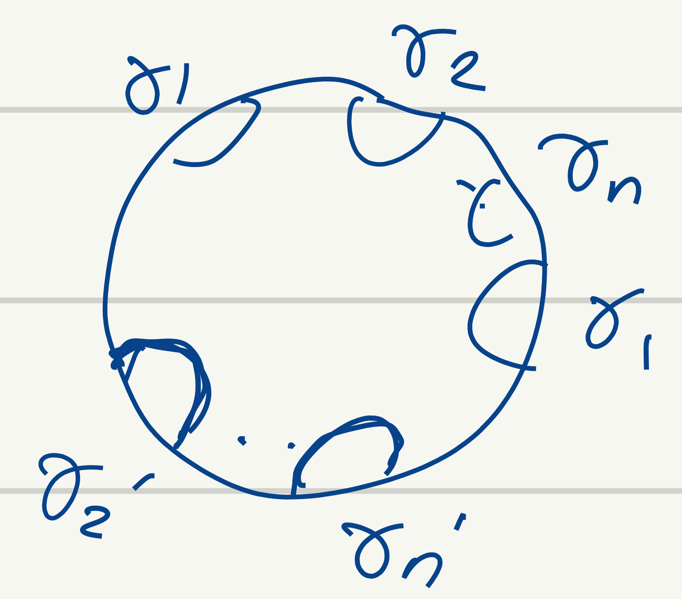
$\qquad = \partial P_1 \dots \partial P_n, \partial P_1', \dots, \partial P_n'$

half plane

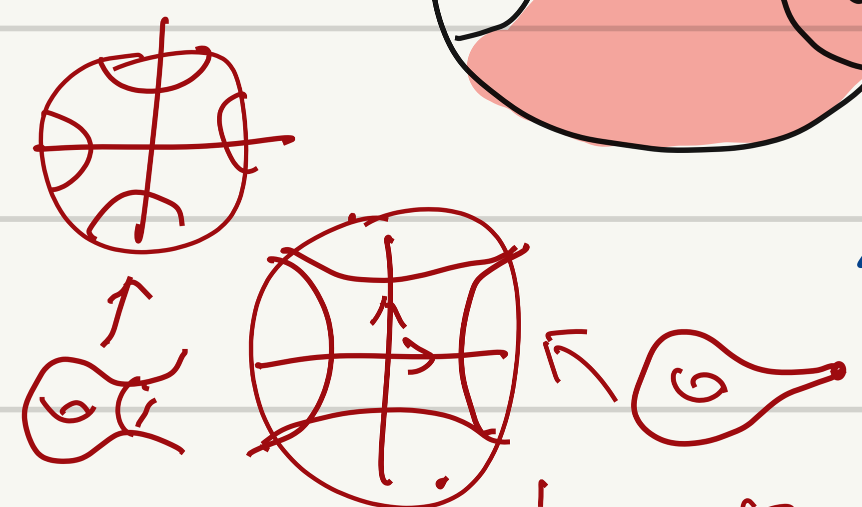
Consider $A_j(P_j^c) = P_j'$

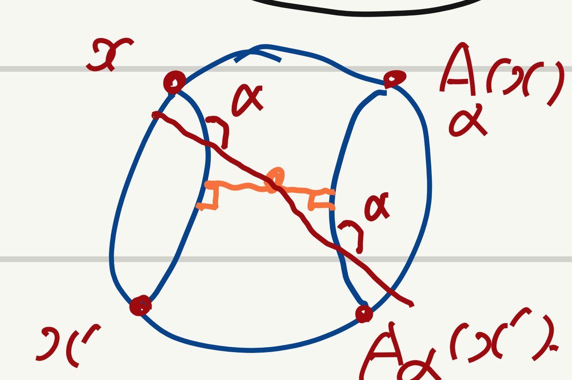
$\in PSL(2,\mathbb{R})$



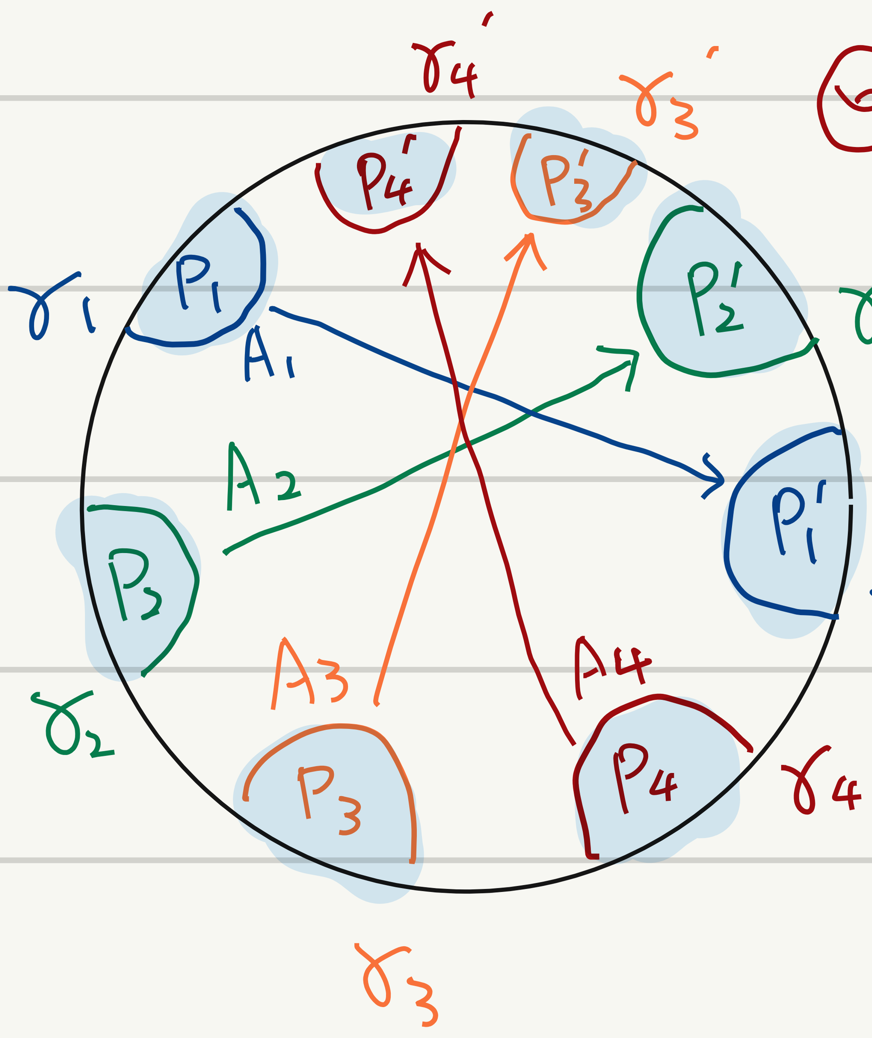


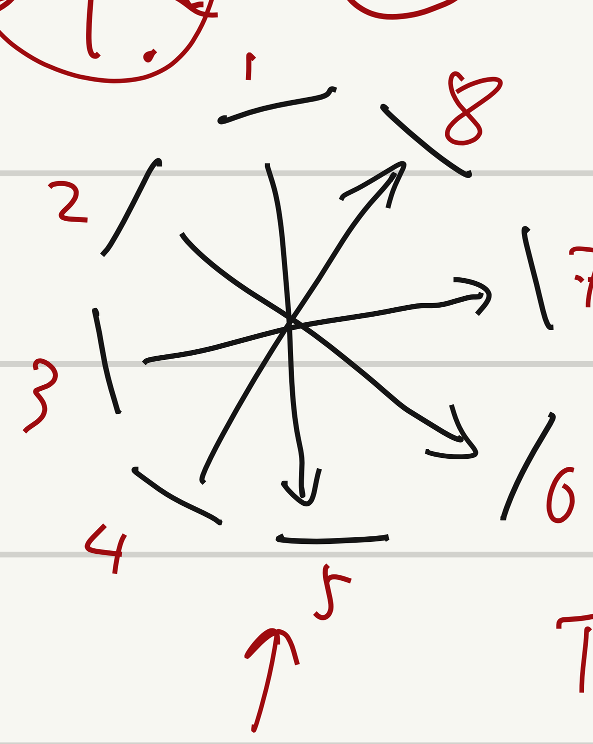
$\Gamma = \langle A_1, \dots, A_n \rangle$



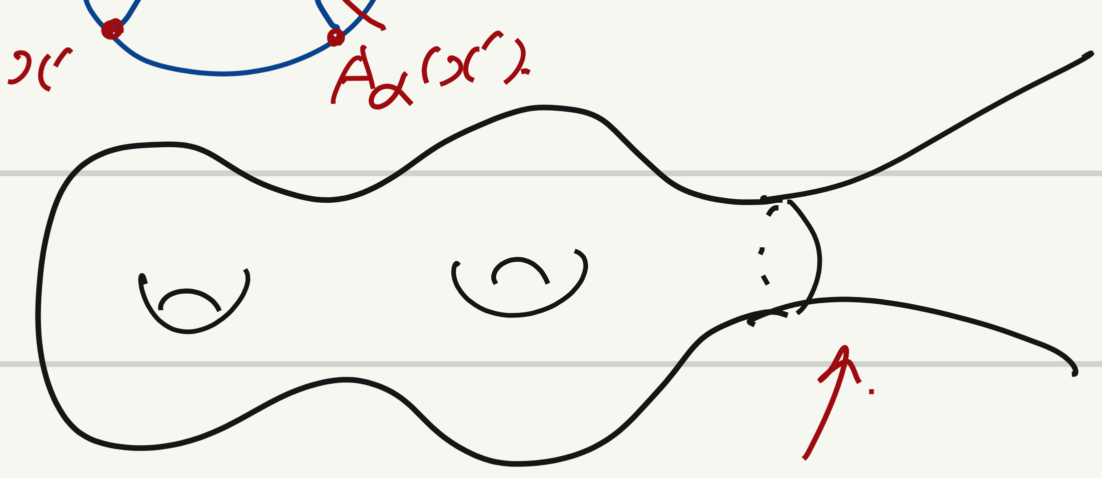


Ex: $(n=4)$

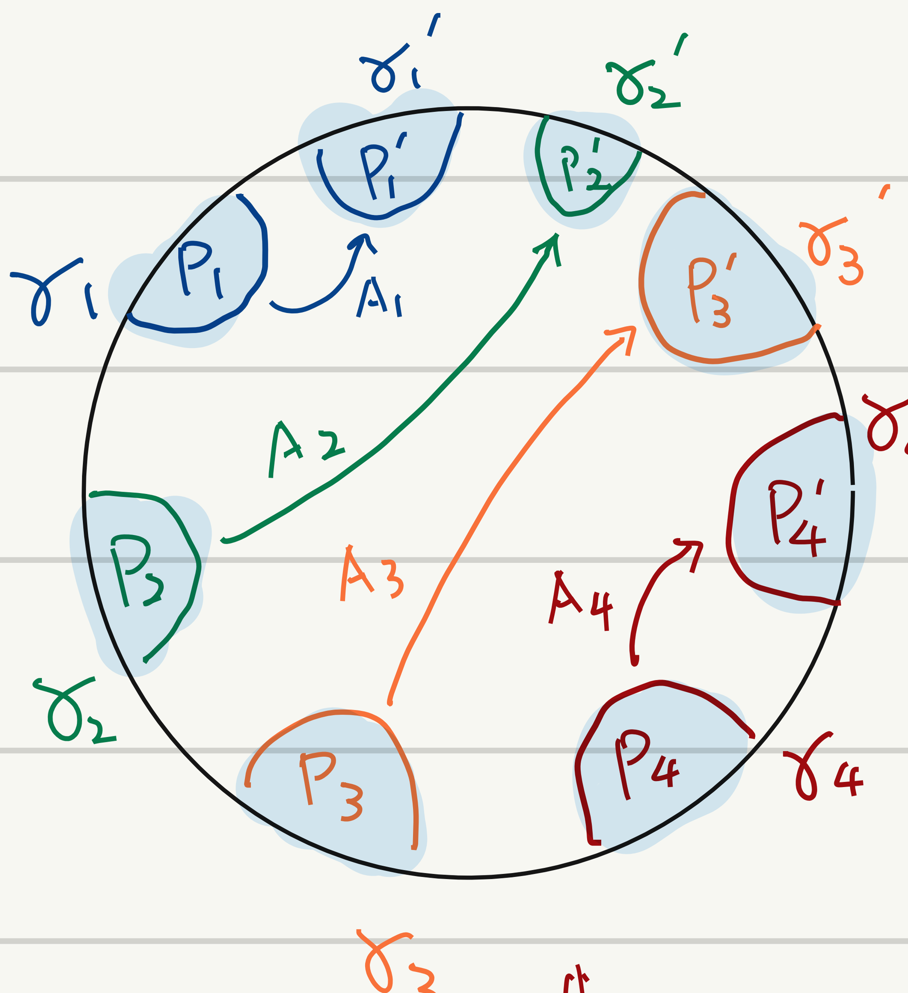


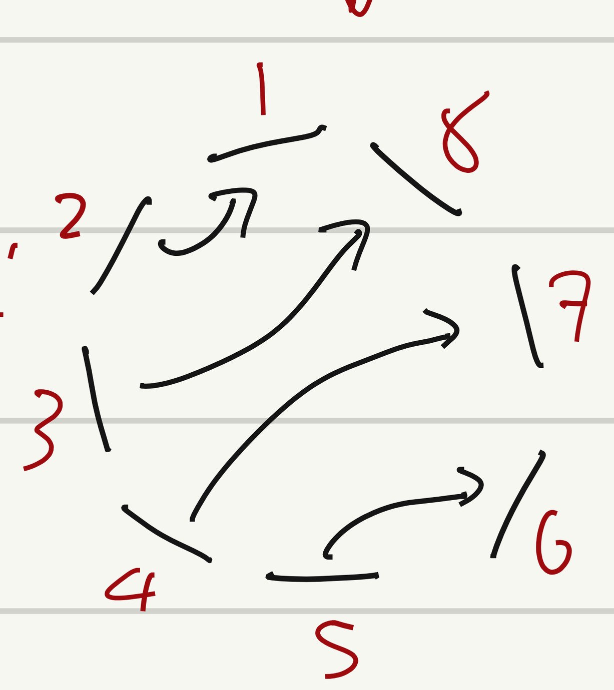


Γ_1

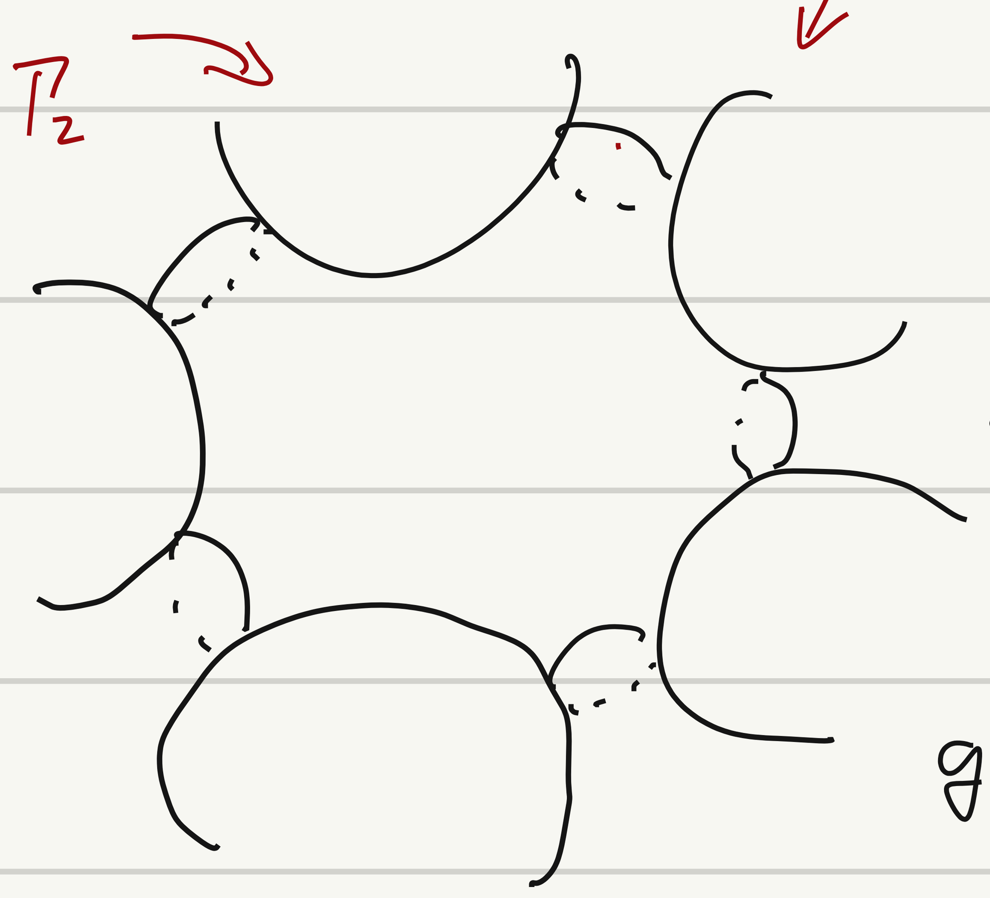


$g=2$ $\quad$ $\#\partial = 1$

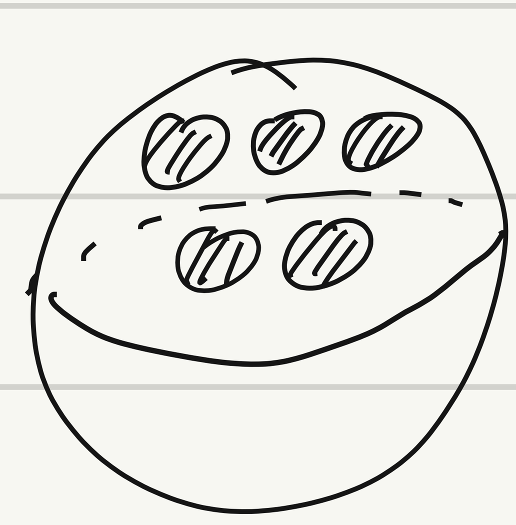




Γ_2

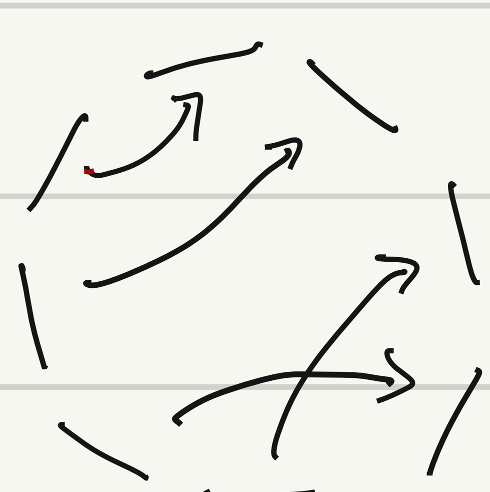


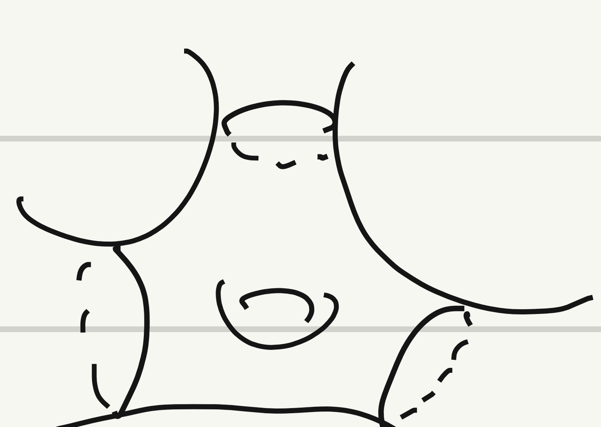
$\cong$



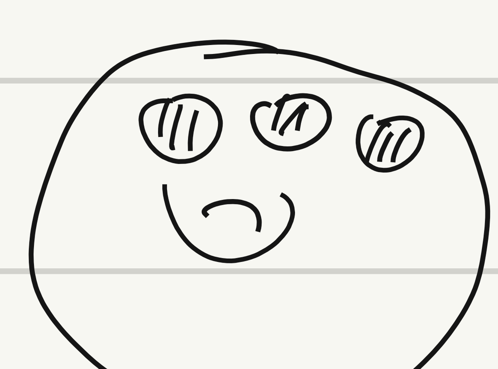
$g=0$ $\quad$ $\partial\# = 5$

Also





$\cong$

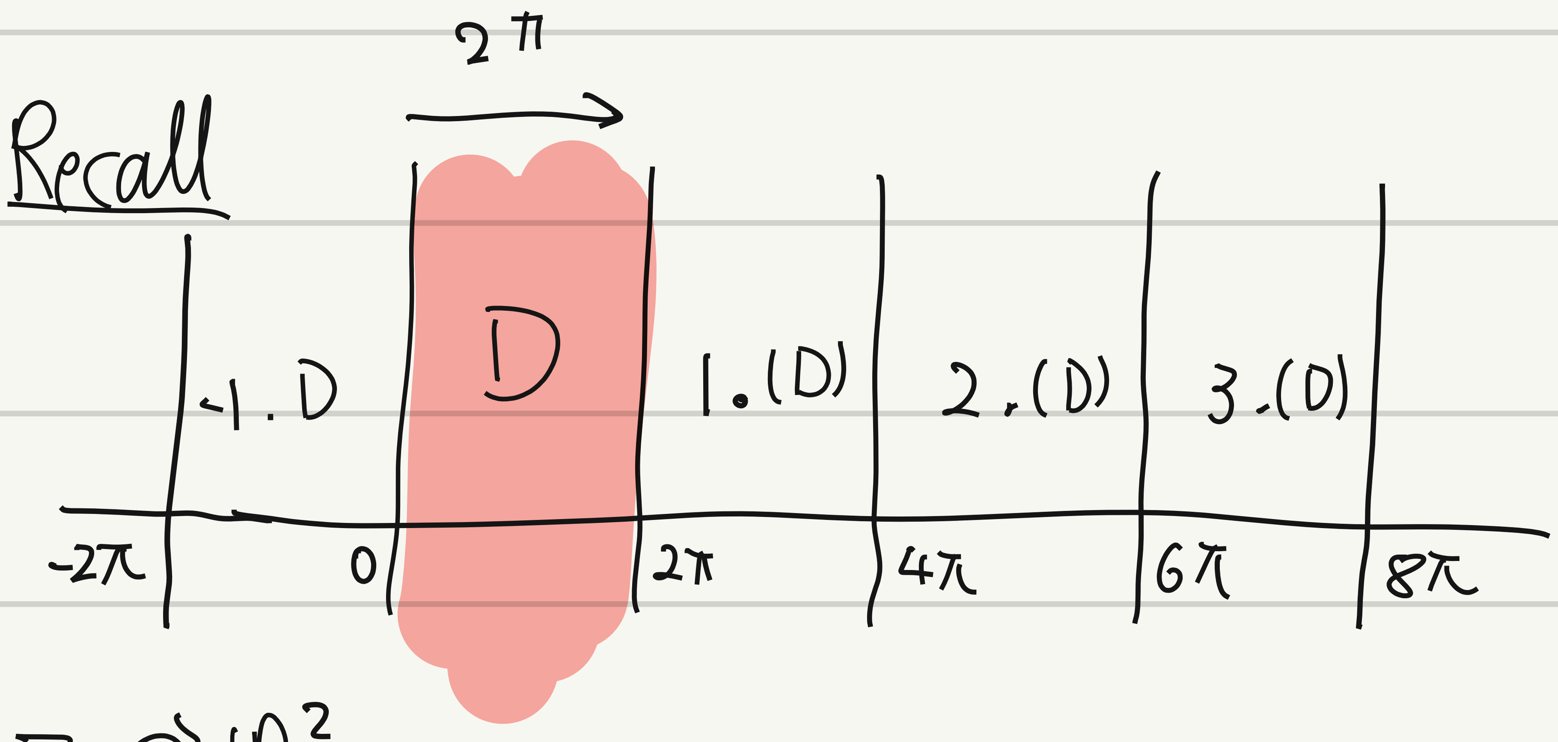


$g=1$ $\quad$ $\partial\# = 3$

$\boxed{\text{Prop}}$ $\forall n$, $\exists\, S_{g,k}$ with $(g,k) = (0, n+1), (1, n-1), (2, n-3), \dots$ $\quad (2g+k-1=n)$

5. Fundamental domain of a discrete subgroup $\Gamma < \mathrm{Isom}^+(\mathbb{H}) \cong PSL(2,\mathbb{R})$

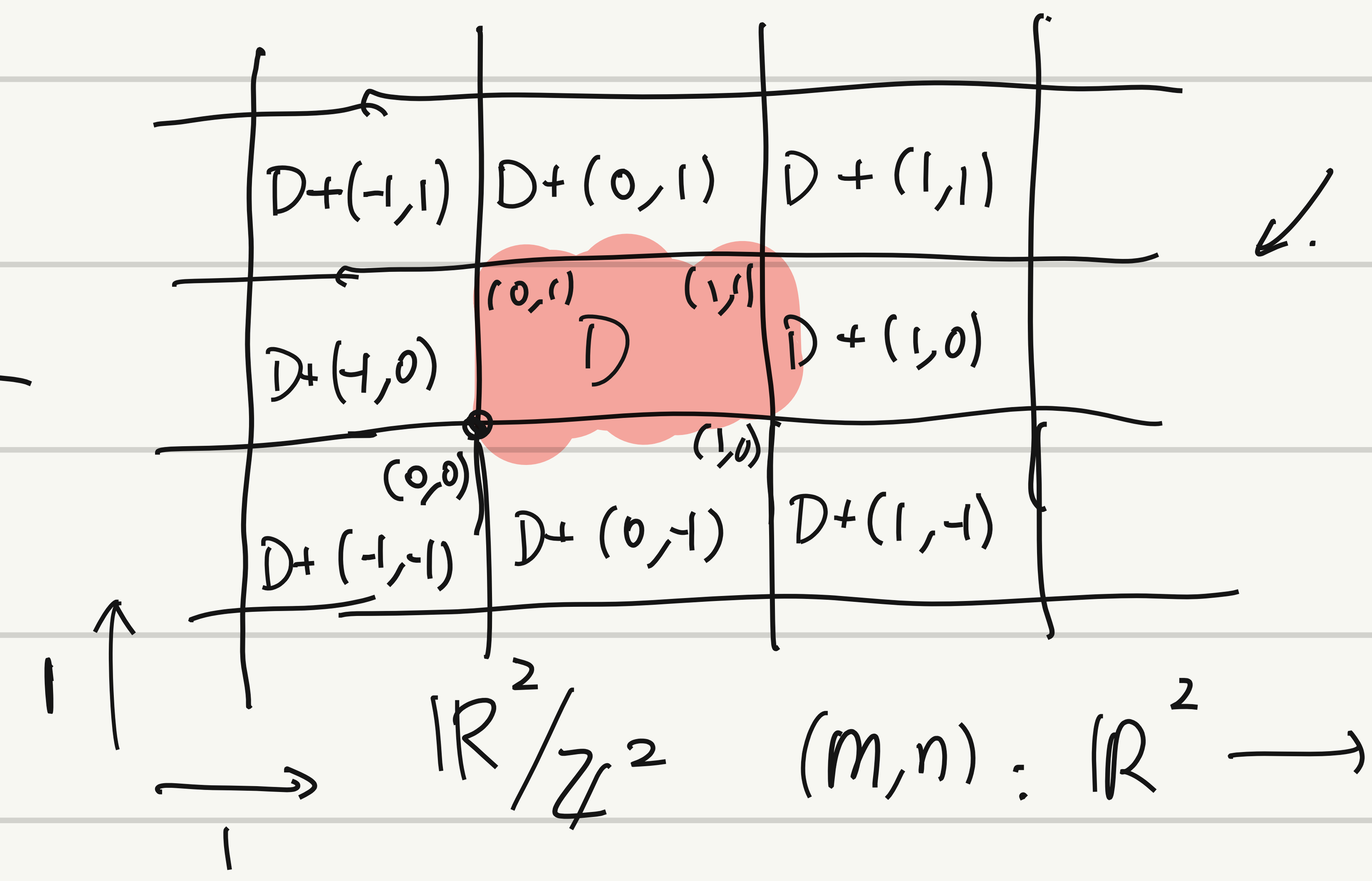
Recall



$\mathbb{Z} \circlearrowright \mathbb{R}^2$

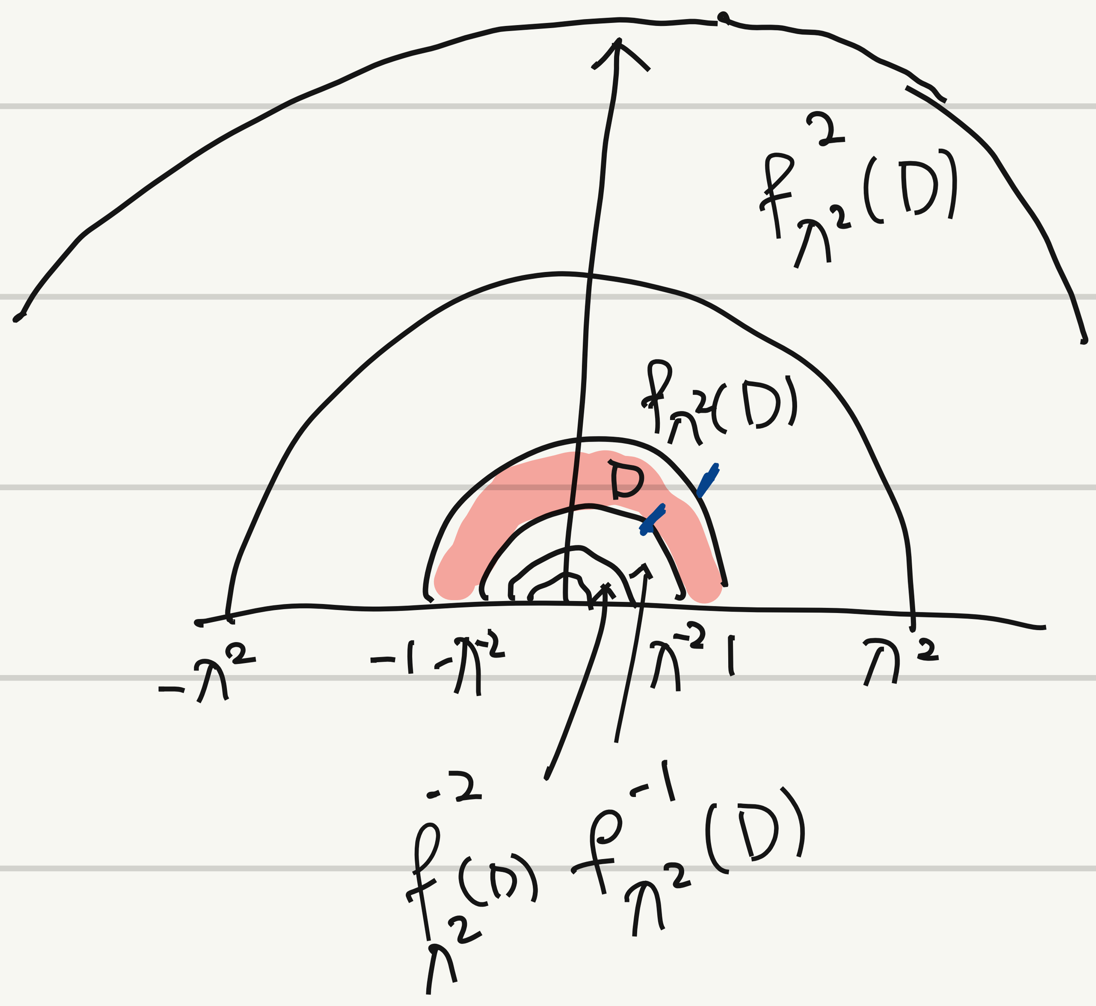
$n: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

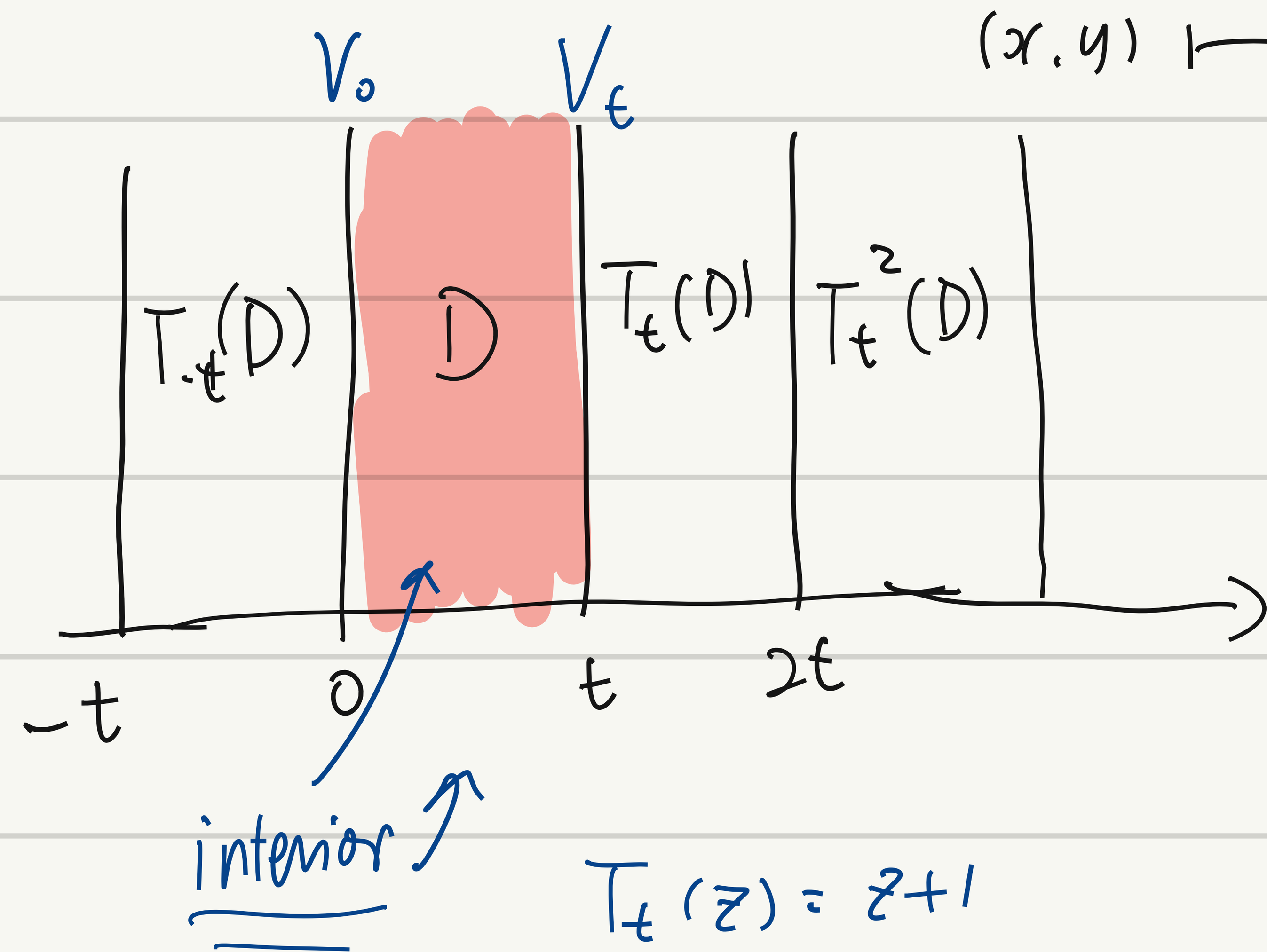
$(x,y) \mapsto (x+2\pi n, y)$



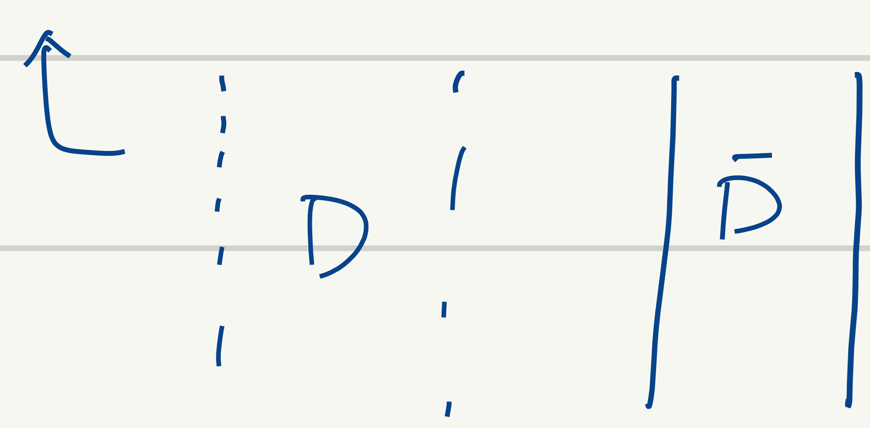
$\mathbb{R}^2/\mathbb{Z}^2$ $\quad (m,n): \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

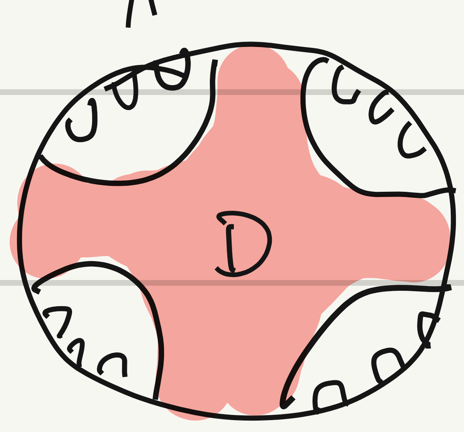
$(x,y) \longmapsto (x,y)+(m,n)$





$T_t(z) = z+1$





Schottky group.

Ob: ① $\forall f \in \Gamma$, $f(D) \cap D \neq \phi$, then $f = id \in \Gamma$

② $\bigcup_{f \in \Gamma} \overline{f(D)} = \mathbb{R}^2$ or $\mathbb{H}$

Fuchsian group: $PSL(2,\mathbb{R})$ $(\mathrm{Isom}^+(\mathbb{H}))$ discrete subgroup.

Let Γ be a discrete subgroup of $\mathrm{Isom}^+(\mathbb{H})$

Fuchsian group

Def: A (open) domain $D \subset \mathbb{H}$ is called a fundamental domain of Γ, if

① $\forall f \in \Gamma \setminus \{id\}$, $f(D) \cap D = \phi$

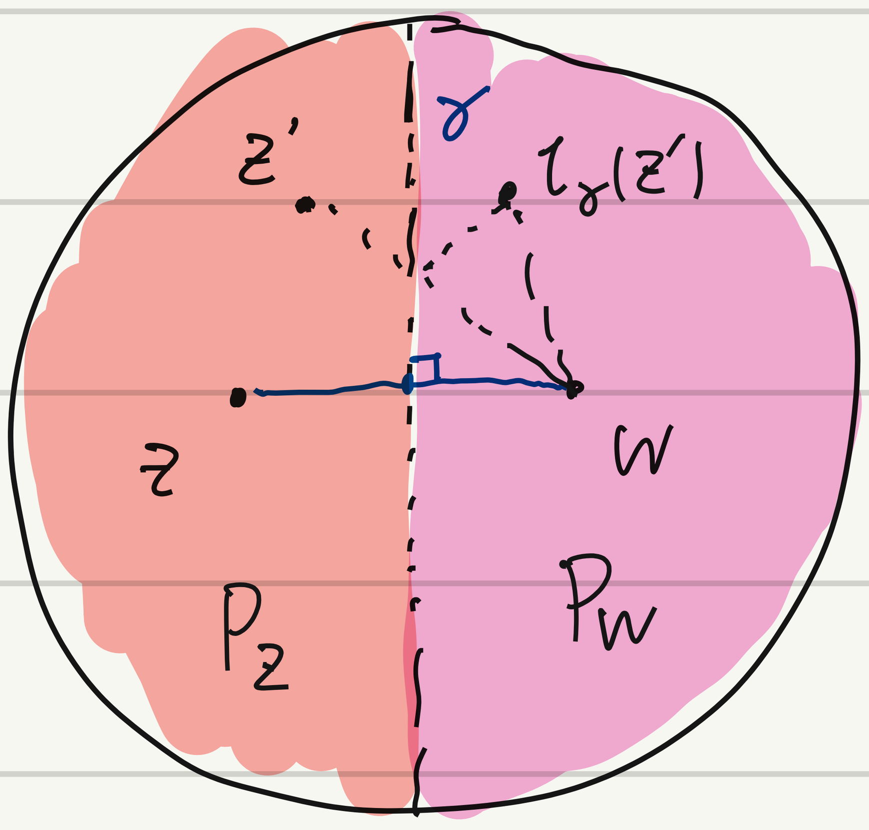
② $\bigcup_{f \in \Gamma} f(\bar{D}) = \mathbb{H}$

Prop: $\forall \Gamma < \mathrm{Isom}^+(\mathbb{H})$, discrete $\exists$ a fund. domain D for Γ.

Def: The Dirichlet fund. domain of Γ with center z_0

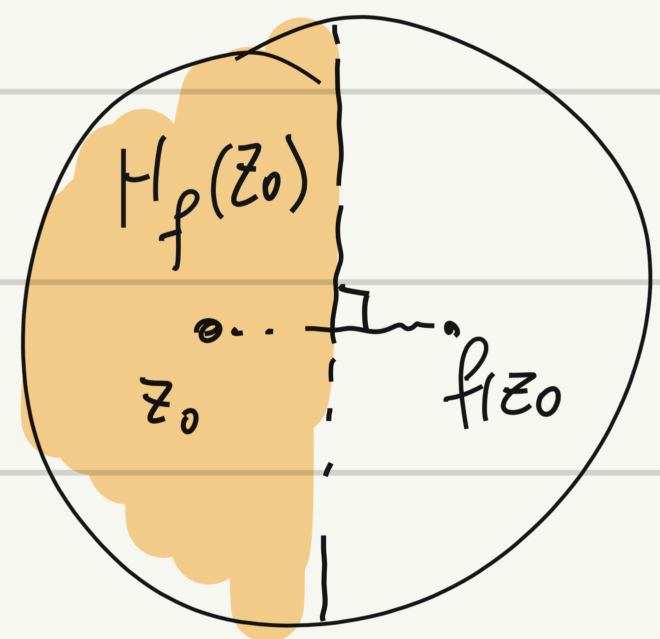
is $R_{z_0} = \bigcap_{f \in \Gamma \setminus \{id\}} \mathbb{H}_f(z_0)$ $\quad \mathbb{H}_f(z_0) = \{ z \in \mathbb{H} \mid d_{\mathbb{H}}(z, z_0) < d_{\mathbb{H}}(z, f(z_0)) \}$

(z_0 is not a fix pt of any $f \in \Gamma$) $\forall f \neq id \in \Gamma$

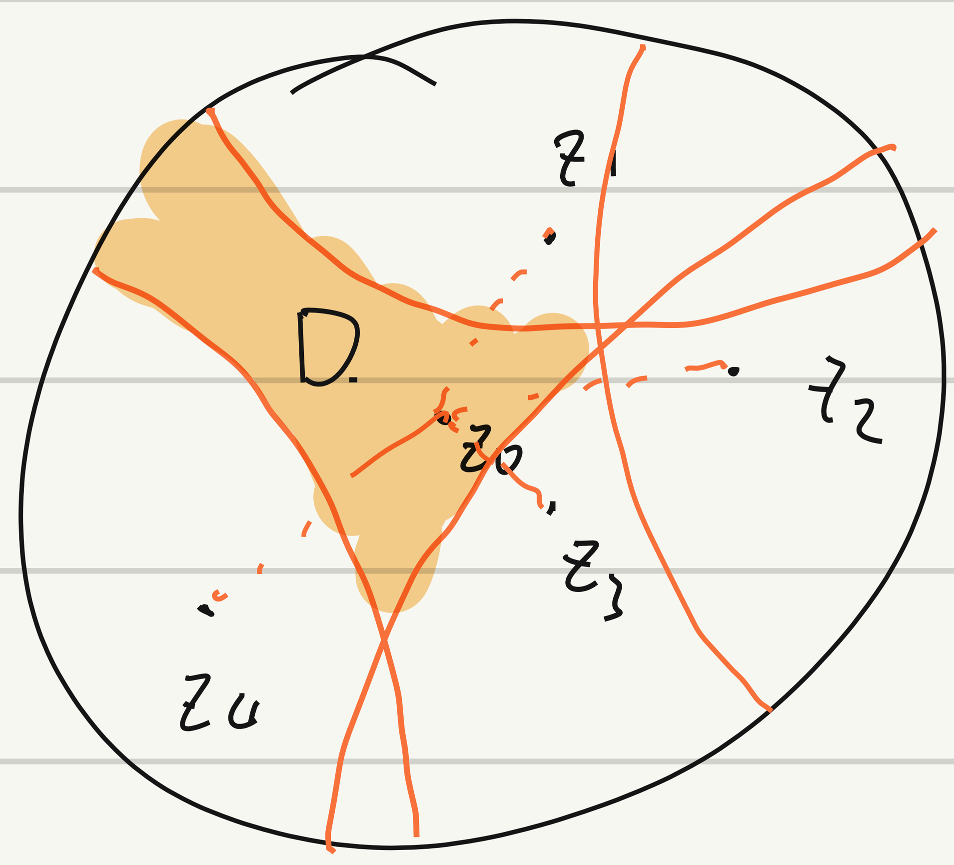


$P_z = \{ z' \in \mathbb{H} \mid d_{\mathbb{H}}(z', z) < d_{\mathbb{H}}(z', w) \}$.

$P_w = \{ z' \in \mathbb{H} \mid$ ——— > ——— $\}$



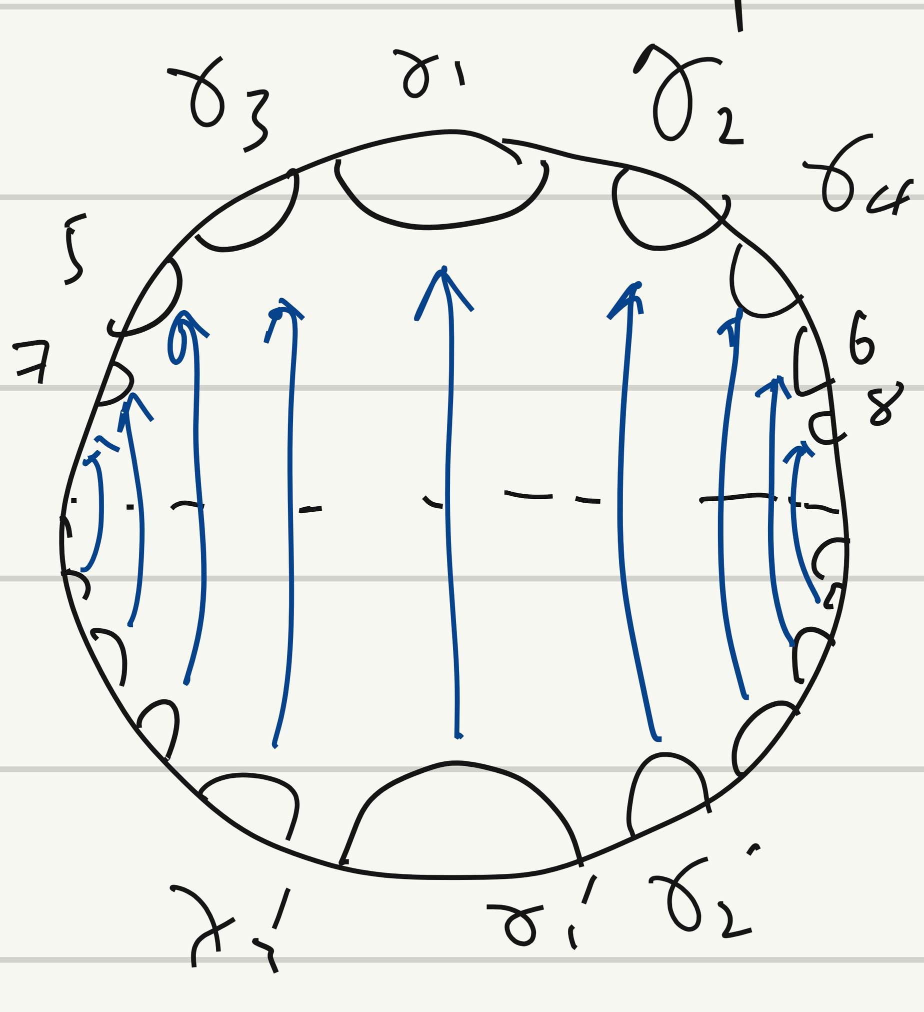
$$R_{z_0} = \bigcap_{f \in \Gamma \setminus \{id\}} H_f(z_0)$$

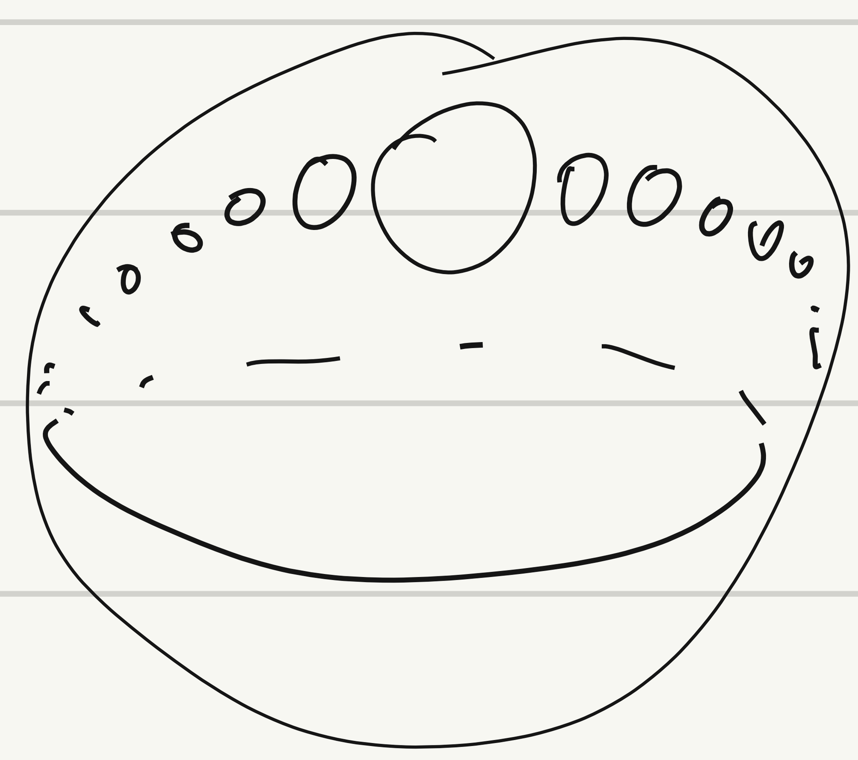


$f_i(z_0) = z_i$

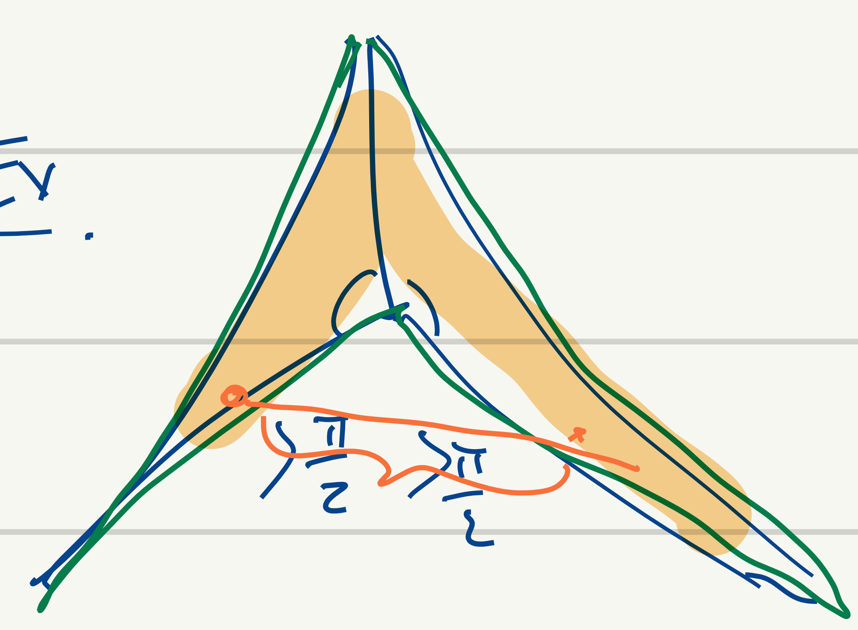
finite type

Prop: D finite sides polygone if Γ finitely generated
infinite sides polygone if Γ infinitely generated



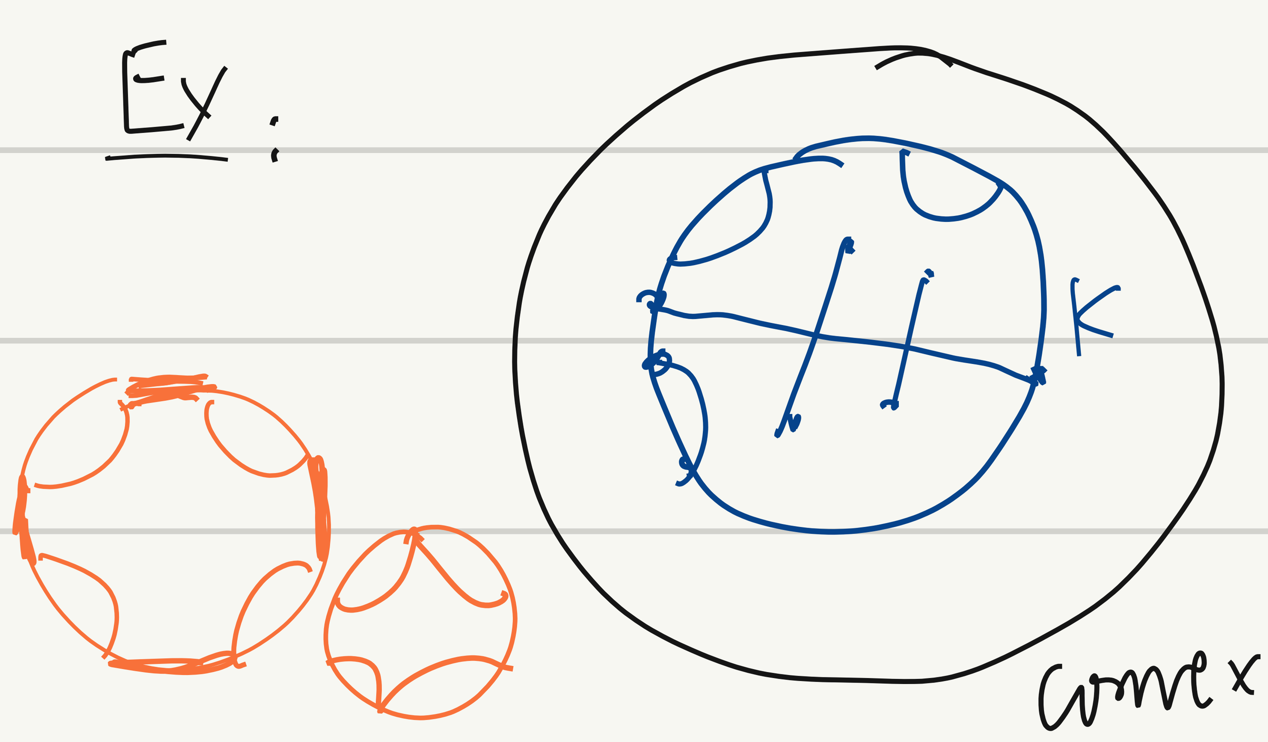


Ex.

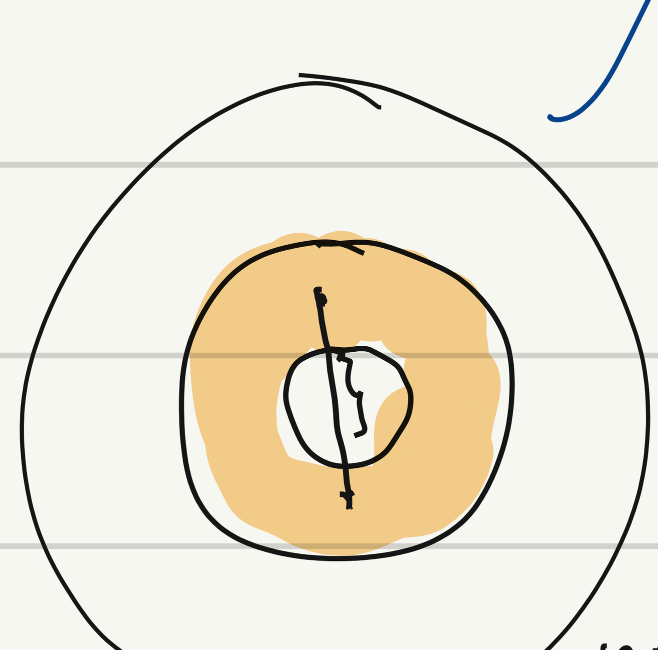


Def: A subset $K \subset \mathbb{H}$ is (geodesic) convex $\forall z, w \in K$, we have
$[z, w] \subset K$.
↳ connecting z and w geodesic segment

Ex:



convex

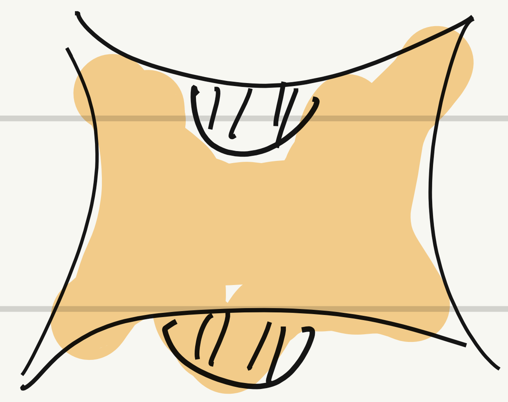


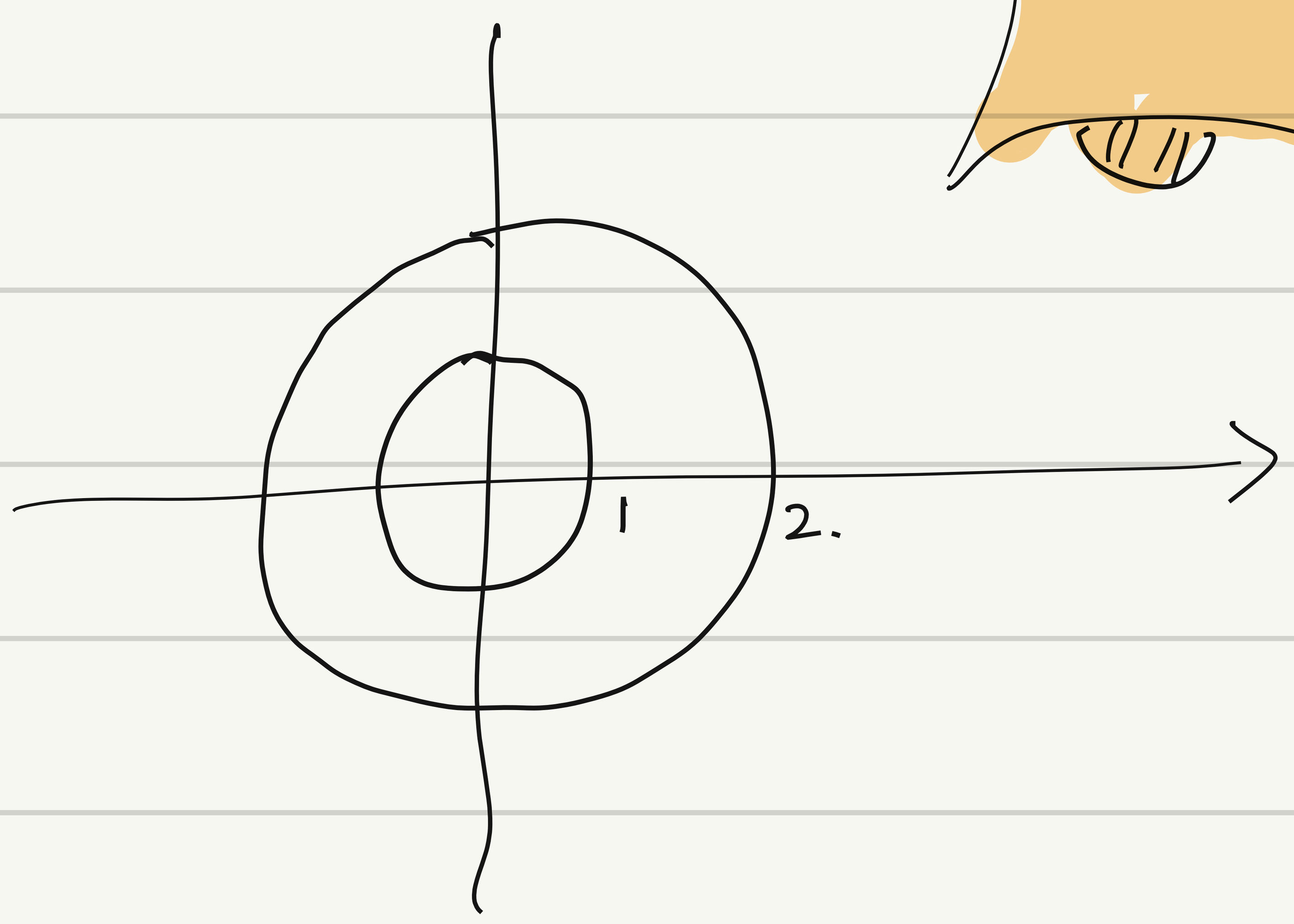
non convex.

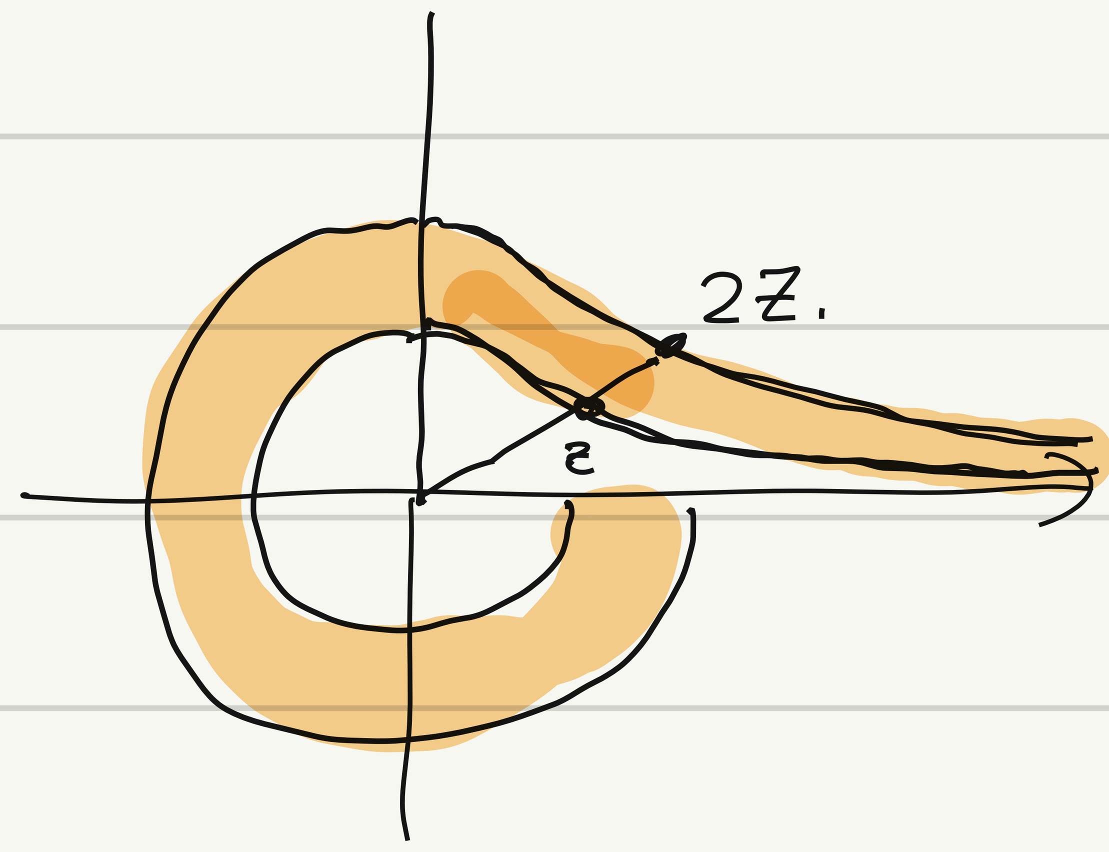
Prop: If Γ is finitely gene.
R_{z_0} is a convex polygone with fintely many sides.
(possibly with part of $\partial\mathbb{H}$)

$\mathbb{C}^* \longrightarrow \mathbb{C}^*$

$z \longmapsto 2z$



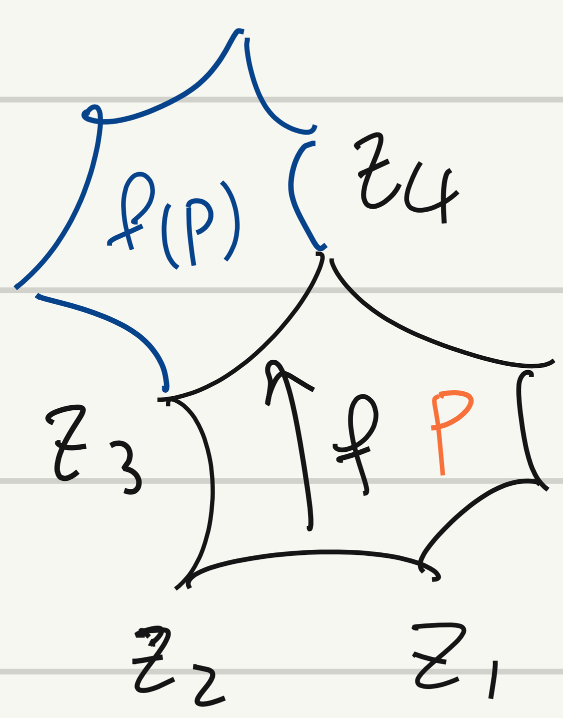




6. Poincaré polygone theorem

Let $P = [z_1, z_2] \cup \cdots \cup [z_n, z_1]$ convex.

Suppose $\ell([z_1, z_2]) = \ell([z_3, z_4])$, then $\exists!\, f \in \mathrm{Isom}^+(\mathbb{H})$ s.t. $f(z_1) = z_4$ $f(z_2) = z_3$

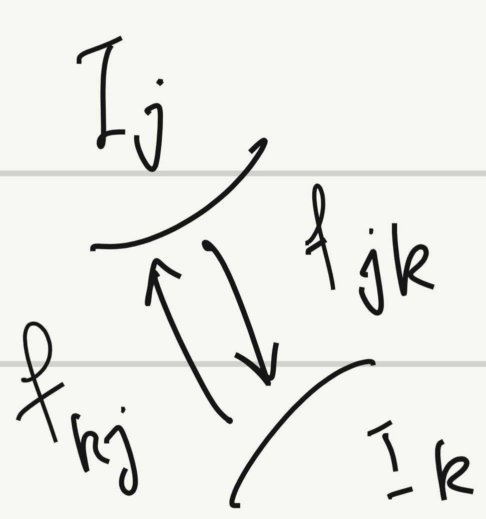


Def: f is a side pairing of P

Let $I_1, \dots, I_{2n}$ denote $[z_1, z_2], \dots, [z_{2n}, z_1]$.

Assume $\exists$ partition of $I_1 \dots I_{2n}$ into pairs.

(I_{j_1}, I_{j_1}'), ..., (I_{j_n}, I_{j_n}')



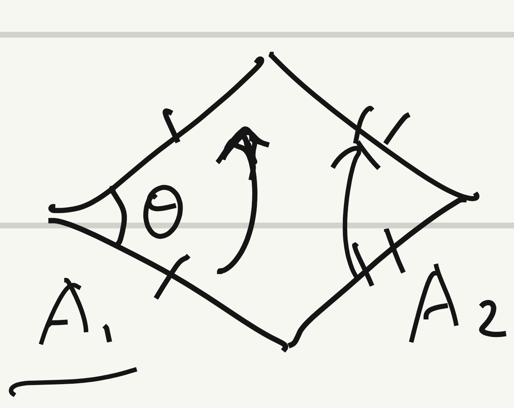
$\exists$ f_1 ... f_n. side pairings.

A_1 ... A_n

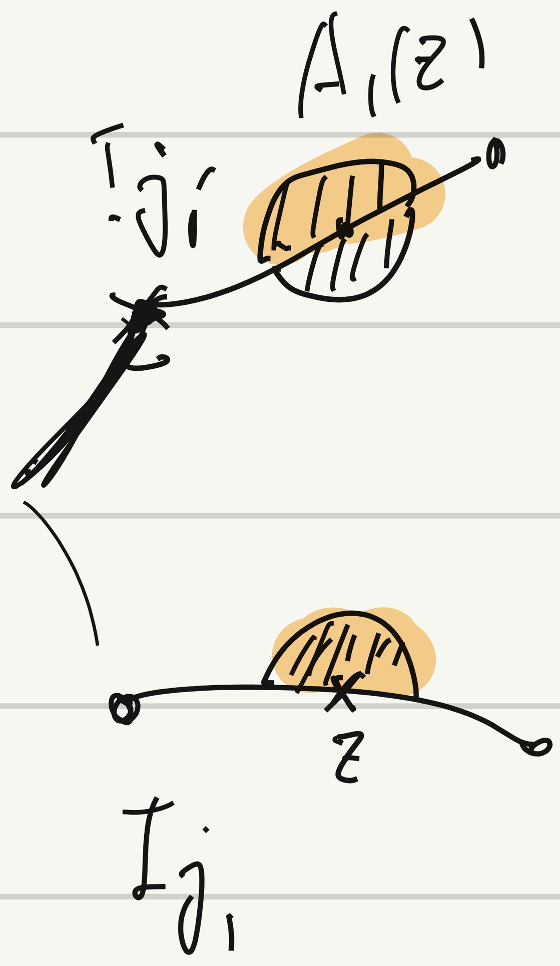
Q: $\Gamma = \langle A_1 \dots A_n \rangle$ is discrete?

$\theta = \alpha\pi$

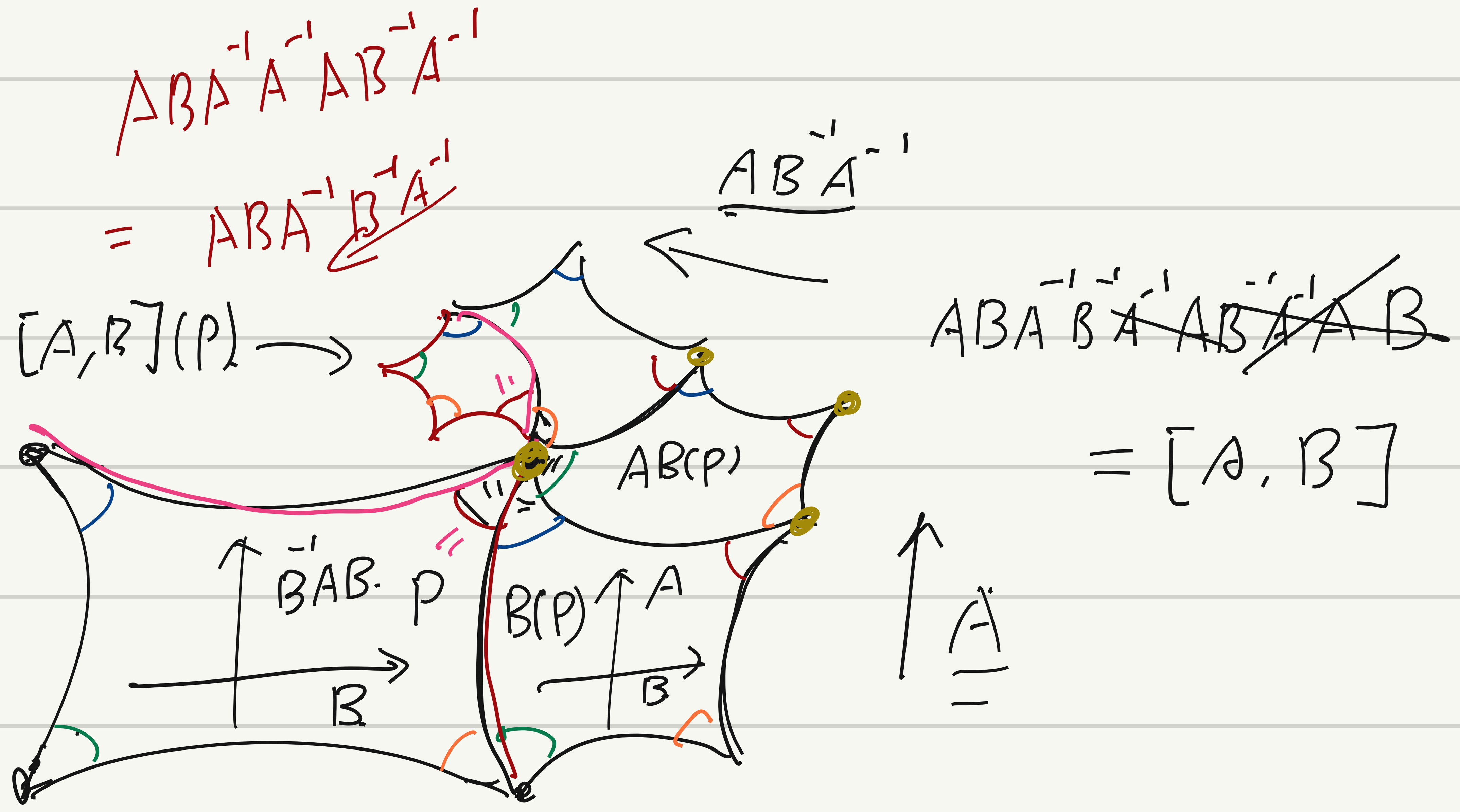
$\alpha \in \mathbb{R} \setminus \mathbb{Q}$



$\Gamma = \langle A_1, A_2 \rangle$ is not discrete.



$\dfrac{2\pi}{n}$



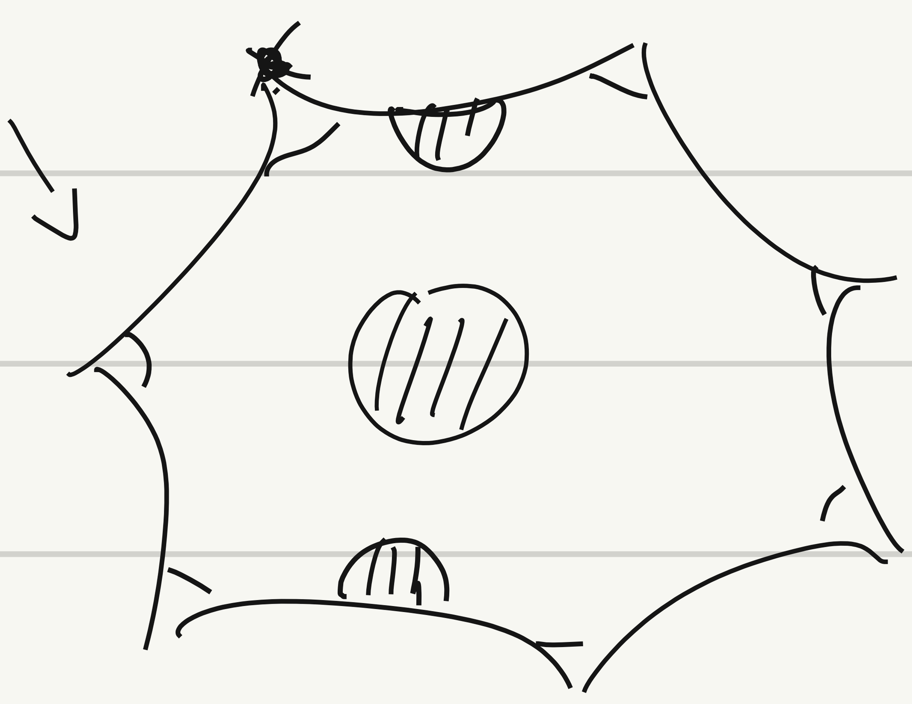
$v_1 \ \ldots \ v_{2n}$ vertices of P.

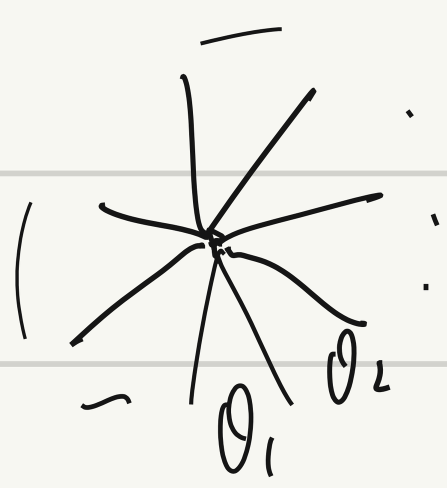
$\theta_1 \ \ldots \ \theta_{2n} \rightsquigarrow \{\theta_{11}, \ldots, \theta_{1s_1}\} \sqcup \{\theta_{21}, \ldots, \theta_{2s_2}\} \sqcup \cdots \sqcup \{\theta_{k1}, \ldots, \theta_{ks_k}\}$

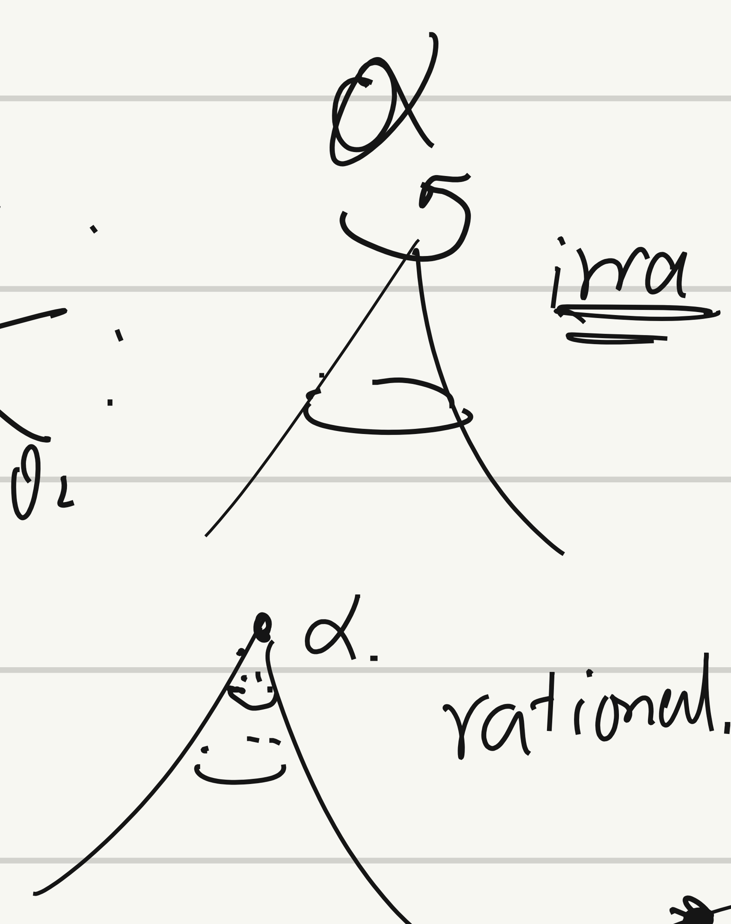
$= \{\theta_1 \ldots \theta_{2n}\}$

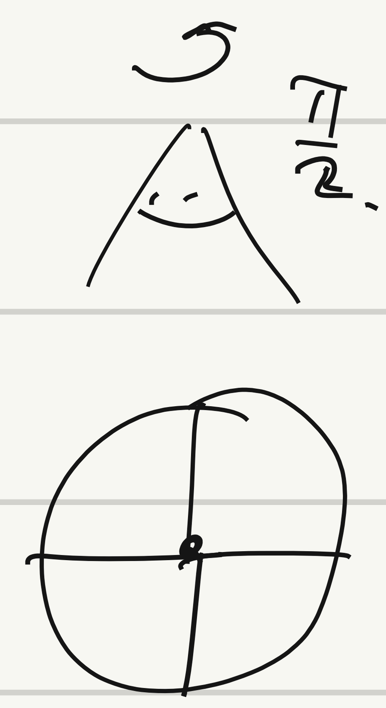
Thm (Poincaré Polygone theorem) (not the full version)

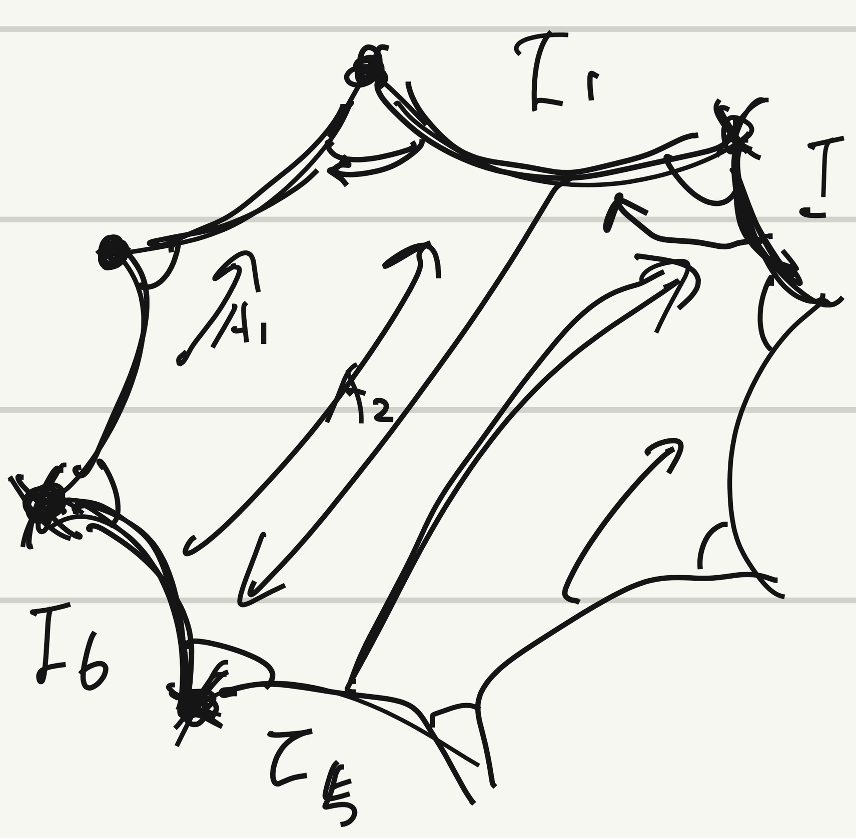
- Γ is discrete group iff
  $\forall j \in \{1, \ldots, k\}, \quad \alpha_j = \sum_{l=1}^{s_j} \theta_{jl} \in \left\{ \frac{2\pi}{n} \;\middle|\; n \in 1, 2, \ldots \right\}$
- If Γ is discrete.
  $\Gamma = \langle A_1 \ldots A_n \mid R_1 \ldots R_k \rangle$
  one relation for each $j \in \{1, \ldots, k\}$
- P is a fund. domain of Γ.

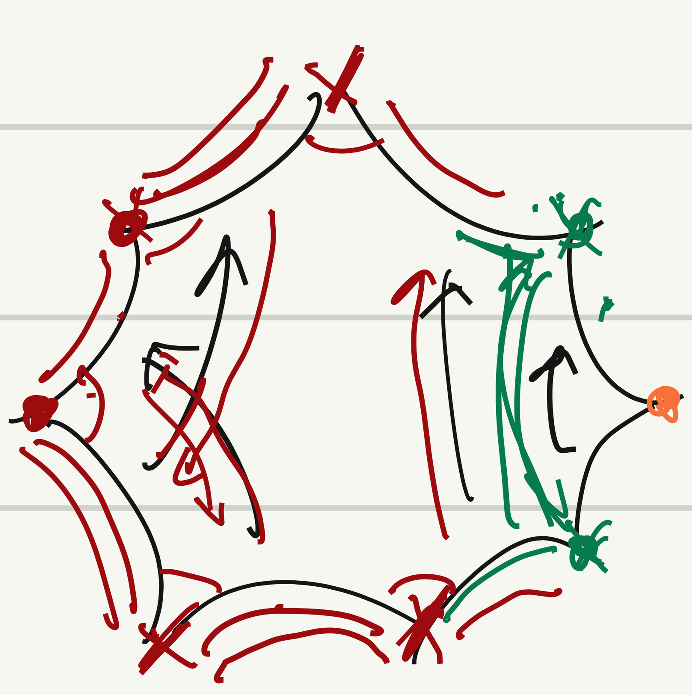


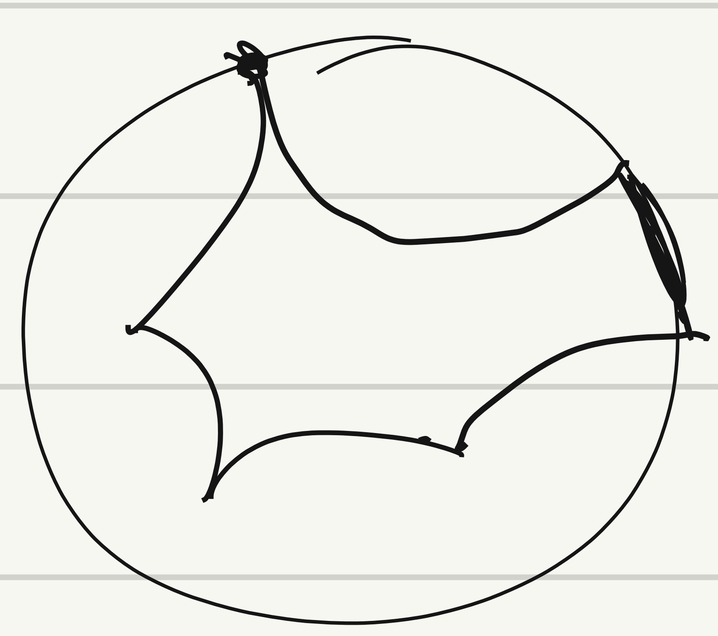








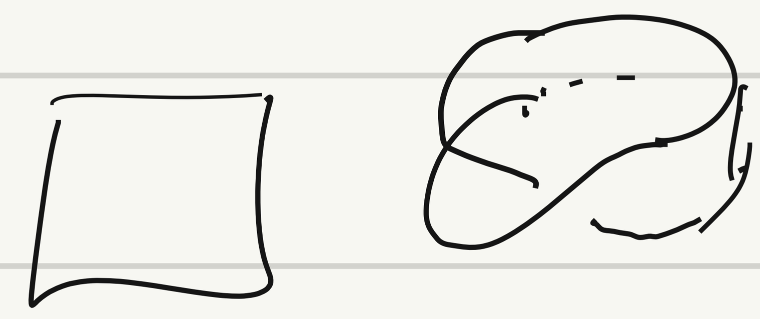




Prop: If D is a fund. domain of Γ,

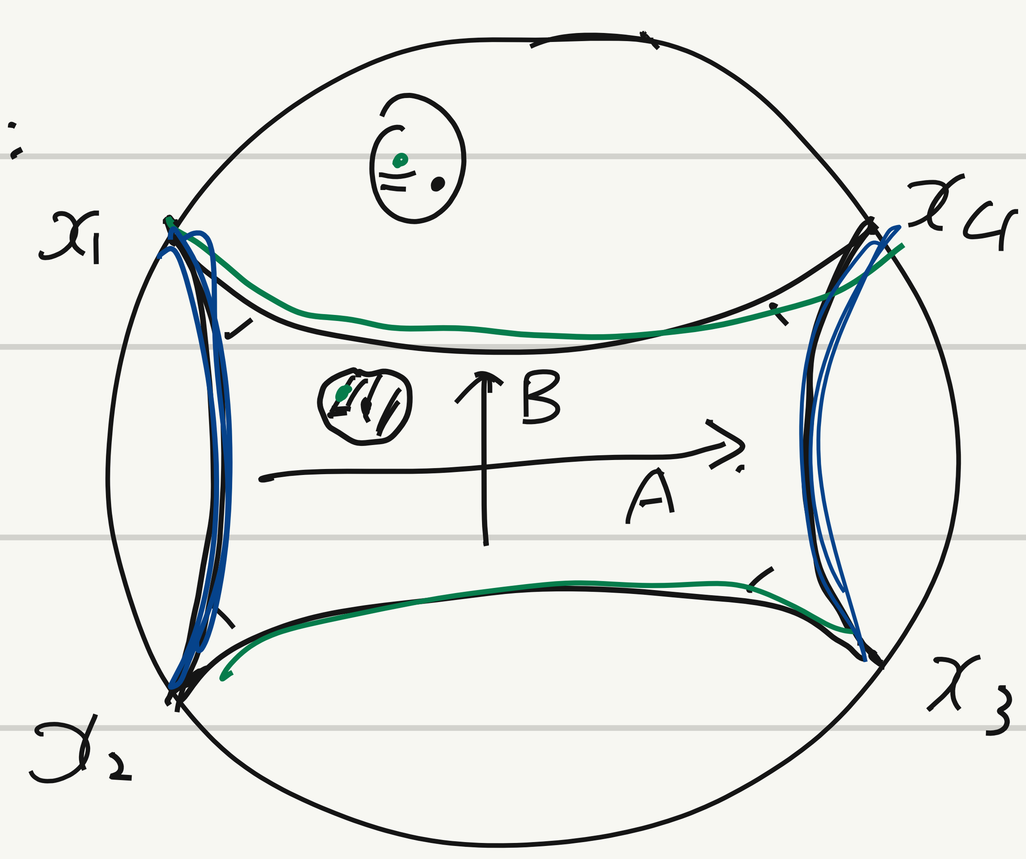
$$A_{\mathbb{H}}(D) = A_{\mathbb{H}}(S)$$
$$S = \mathbb{H}/\Gamma.$$

7. Hyperbolic surface. (oriented / finite type).



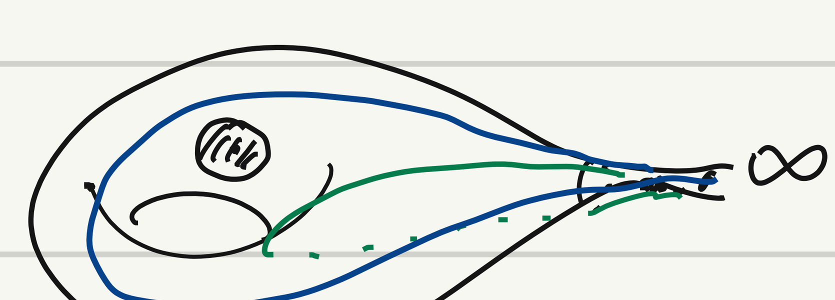
Def: An oriented hyp sf S is a quotient $\mathbb{H}/\Gamma$ where
Γ: finitely generated discrete group of $\mathrm{Isom}^+(\mathbb{H})$.
torsion free.
↑ no finite order elements

Ex:



$A(x_1) = x_4$
$A(x_2) = x_3$
$B(x_2, x_3) \mapsto (x_1, x_4)$

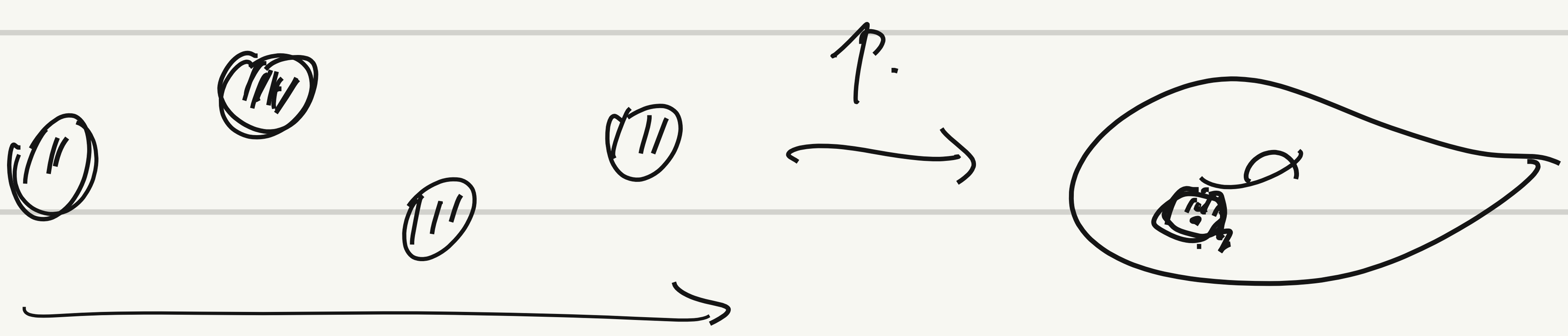
$S = \mathbb{H}/\Gamma$



$\boxed{p: \mathbb{H} \longrightarrow \mathbb{H}/\Gamma}$
$z \longmapsto [z]_\Gamma$
$A(z)$

$\forall z \in \mathbb{H},\ \exists R,$ if $\forall A,\ A(\bar{D}(z,R)) \cap \bar{D}(z,R) = \emptyset$

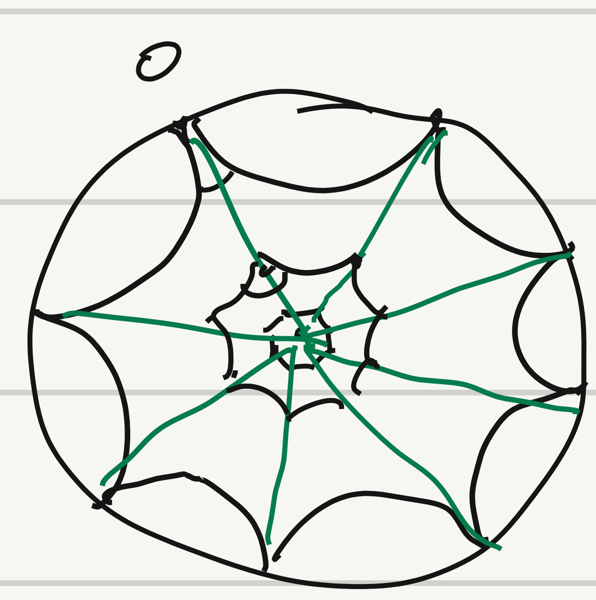
Prop: $\bar{D}(z,R)$ is isometric to $p(\bar{D}(z,R))$



Prop. p is a covering map.
$\forall q \in S.\ \exists R$ s.t.
$p^{-1}(D(q,R)) = \bigsqcup_{A \in \Gamma} D_A \quad \forall A.\ D_A \sim D(q,R)$

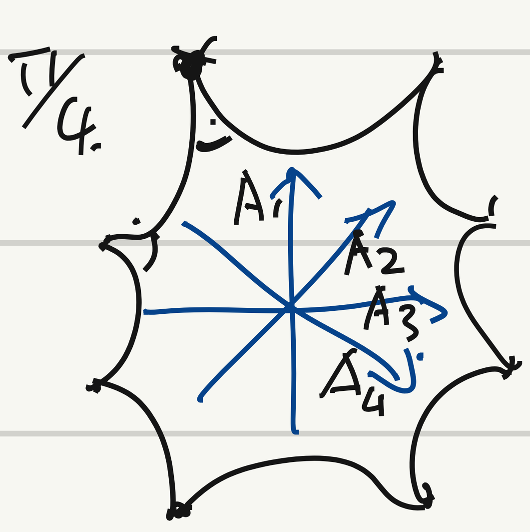
p is locally isometric.

Ex: Bolza surface.

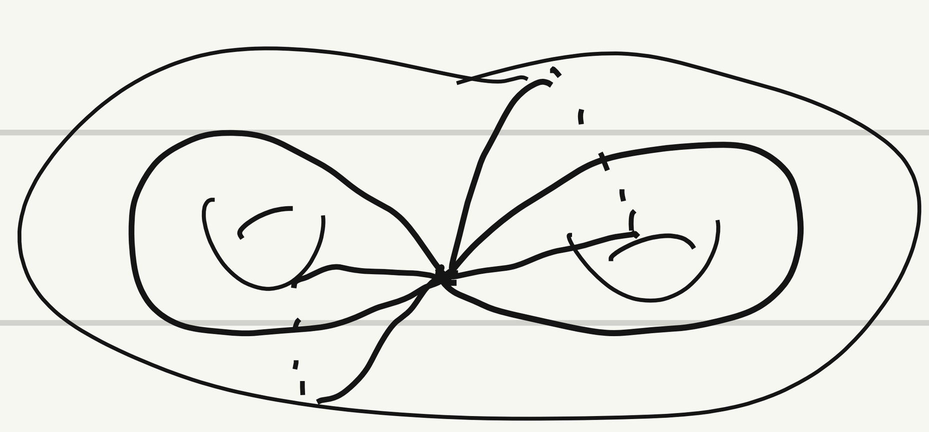


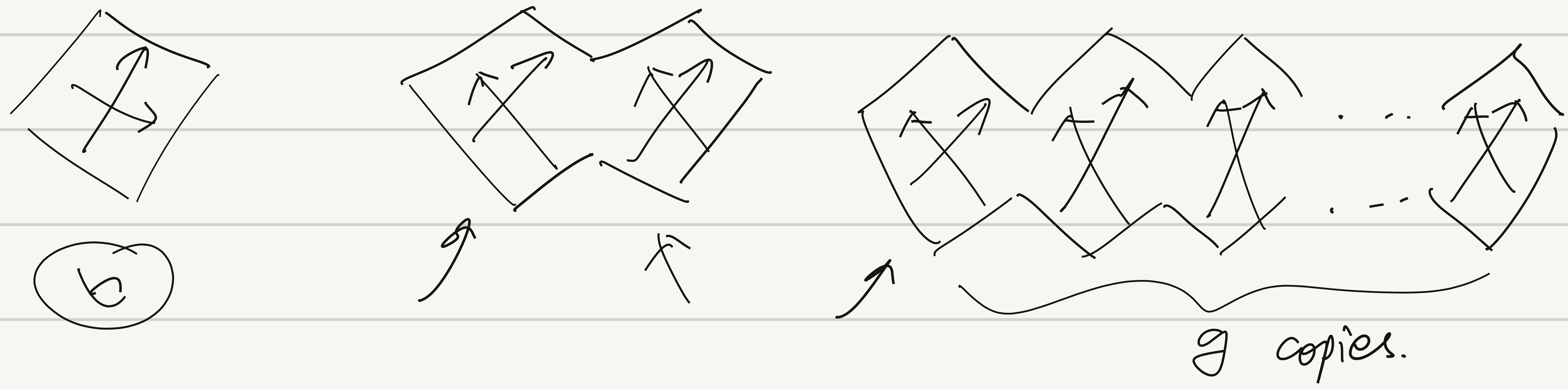
$\frac{3}{4}\pi \times 8 = 6\pi.$
$0 \times 8 = 0$

regular octogon.



$\Gamma = \langle A_1\ A_2\ A_3\ A_3 \mid [A_1, A_3][A_2, A_4] \rangle$

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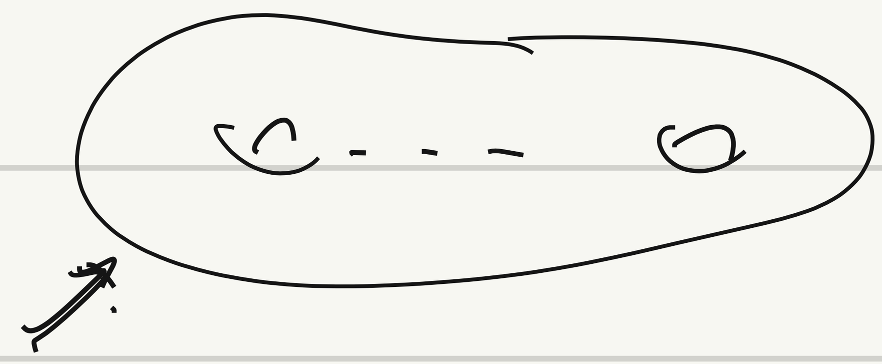

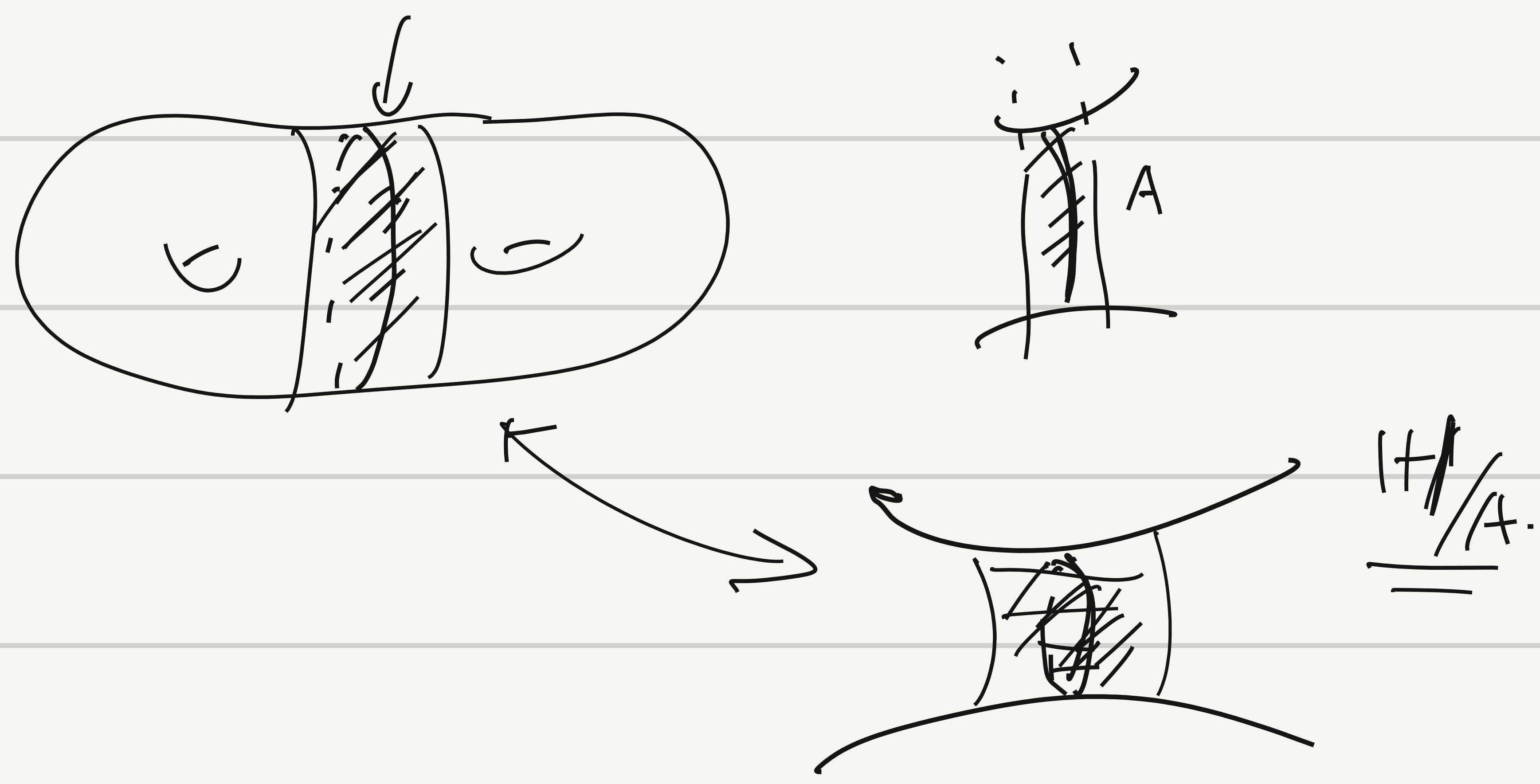
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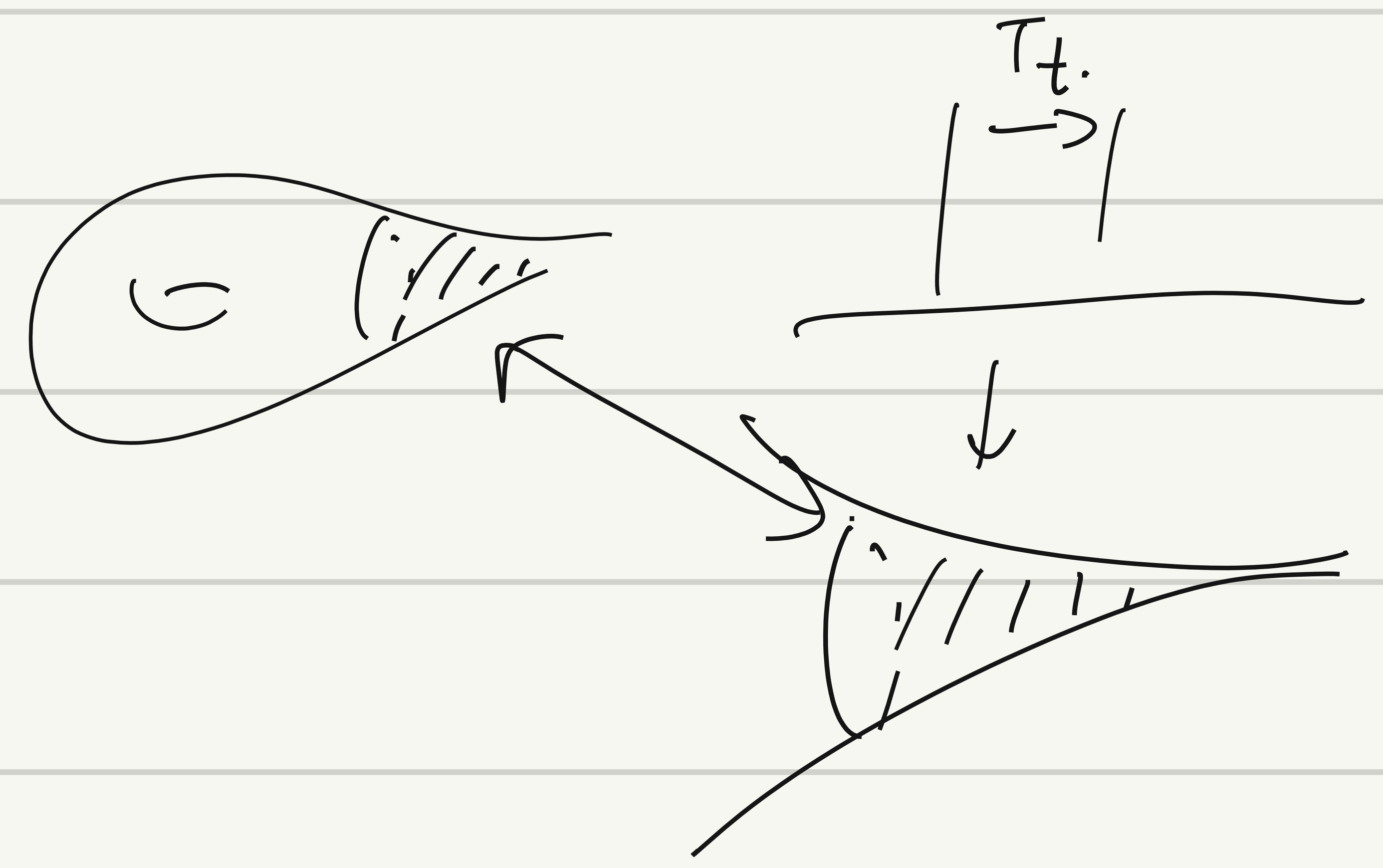
H/A.


T_t.