Hyperbolic group \( \longrightarrow \) Gromov boundary

\[ S \subset G \]

G acts by quasi-affine homeomorphisms on \( \mathbb{G} \cap \mathbb{G} \) by left multiplication.

Fact: If \( G \) is finite, then \( \partial G = \emptyset \)

Otherwise,

- \( \partial G \) is perfect, connected, metrizable space of \( \infty \)-cardinality.
- \( \partial G \) is minimal: any orbit is dense in \( \partial G \)

2) Classification of elements in \( G \): (No parabolic elements)

- Finite order elements: Elliptic elements
- Inifinite order elements: Hyperbolic elements (loxodromic)
  - \( n \in \mathbb{Z} \rightarrow g^n \in G \) is quasi-isometric embedding (quasi-geodesic)
  - \( g \) has exactly two fixed points on \( G \cup \partial G \)

Thm: There are only finitely many conjugacy classes of finite subgroups in a hyperbolic group.

Thm. (Tits alternative)

- A finite subgroup of a hyperbolic group is either...
Thm. (Tits alternative)

Any infinite subgp of a hyp. gp is either virtually cyclic or contains $\mathbb{Z}_2$.

Cor. No $\mathbb{Z}^2$ in hyp. group.

Proof idea: Play Ring-Pong between independent hyp. elements

\[ \text{Diagram} \]

Thm. If a hyp. gp $G$ has $\partial G \cong \mathbb{S}^1$, then $G$ is virtually free.

Thm. (Tuckia 89, Casson-Jungreis 94, Gabai 92)

If a hyp. gp $G$ has $\partial G \cong S^1$, then $G$ is virtually closed surface group.

Remk: This is the final step in the resolution of Seifert fiber space conjecture. (T Scott + Mess+...)

 Cannon Conjecture:

If a hyp. gp $G$ has $\partial G = \mathbb{S}^2$,
then $G$ is virtually a uniform lattice in Isom($\mathbb{H}^3$)
 i.e. $G$ is virtually Kleinian group

$G^* \cong \pi_1(\text{compact, hyp. 3-mfd})$

Bonk- Kleiner 2007, Hausdorff dim
Markovich: 2013, Special cube complex. Wise
Definition 12.1. Assume that a (discrete) group $G$ acts by homeomorphisms on a compact metrizable space $M$. Assume that $|M| \geq 3$. Then $G$ is called a convergence group on $M$ if the induced action of $G$ on the space of distinct triples

$$\Theta^3(M) := \{(x, y, z) \in M^3 : x \neq y \neq z \neq x\}$$

distinct triple space

is proper: the following set

$$\{g \in G : gK \cap K \neq \emptyset\}$$

is finite for any compact set $K$ in $\Theta^3(M)$.

If $G \not\leq \Theta^3(M)$ cocompact then it is called uniform convergence group action. Proper

$M = \partial \mathcal{X} \prec \prec G \not\leq \mathcal{X}$ (hyp)

Fact: Any ideal triangle has $\delta$-thin property.

Corollary 12.2. Assume that $G$ acts as a convergence group on $M$. Then every element of infinite order in $G$ has at most two fixed points in $M$.

Thm: Suppose a gp $G$ acts properly on a proper length hyperbolic space $\mathcal{X}$. Then

- $G \not\leq \mathcal{X}$ is a convergence gp action.

If $G \not\leq \mathcal{X}$ is cocompact, (\Rightarrow $G$ hyp group)

then

- $G \not\leq \mathcal{X}$ is a uniform convergence group action.

Cor: A hyp gp acts as a uniform convergence group action.

Lem: $G \not\leq M$ convergence gp action $\iff \forall \{g_n\} \subseteq G \# \{g_n\} = \infty$

$\exists g_0, \exists a, b \in M$ s.t.
If a group $G$ admits a uniform convergence group action on a compact metric space $M$, then $G$ is a hyp gp. and $\partial G \equiv \text{limit set of } G \text{ in } M$.

In particular, if $G \not\subseteq M$ is minimal, then $\partial G \equiv M$. 

Thm (Bowditch, 90's)

\[ k = \frac{e^{-M}}{M/nK \neq \phi} \]

\[ g_k \mid M \setminus b \rightarrow a \text{ wc, uniq.} \]

\[ \forall K \subset M \setminus b \text{ compact} \]

\[ g_k(\bar{k}) \rightarrow a \]
Free products:

\[ F_2 = \mathbb{Z} \times \mathbb{Z} \]

\[ \mathbb{Z}^2 \times \mathbb{Z}^2 \]

Free product of Two groups \( H \) and \( K \): \( G \triangleleft H \ast K \)

Precisely, a group \( G \) is called a free product of \( H \) and \( K \) if there exist a pair of homomorphisms \( \iota_H : H \to G \) and \( \iota_K : K \to G \) such that they are universal in the following diagram:

\[
\begin{array}{ccc}
H & \xrightarrow{\iota_H} & G \\
\downarrow{\phi_H} & & \downarrow{\Phi} \\
\Gamma & \xleftarrow{\iota_K} & K
\end{array}
\]

By the universal property, it is easy to see that \( \iota_H : H \to G \) and \( \iota_K : K \to G \) are both injective. Moreover, \( G \) is unique up to isomorphism, so \( G \) must be generated by \( H \) and \( K \).

\[ \mathbb{Z}^2 \times \mathbb{Z}^2 = \langle a, b, c, d \mid ab = ba, cd = dc \rangle \]

Fact: • Suppose \( H = \langle S | R \rangle \) \( K = \langle T | \Omega \rangle \)

then \( H \ast K = \langle SUT | IR, \Omega \rangle \)

[Normal Form Theorem] • Every element \( g \in H \ast K \) can be written as

a unique product of alternating words

\[ 1 \neq g = h_1 k_1 h_2 k_2 \cdots h_n k_n \]
where $\forall h_i, k_j \neq 1$

$F_2 = \mathbb{Z} \times \mathbb{Z} = \langle a, b | a^2 b a b^{-1} \rangle$

Thm: $G = H \times K$ acts on a tree $T$ by graph isom., s.t.

- $T/G = \overset{h}{H} \overset{k}{K}$

- The vertex stabilizer is conjugate either to $H$ or $K$.

- Edge stabilizer is trivial.

Proof: Let us construct the Bass–Serre Tree: \textbf{<< Tree >>}

$\mathcal{V}_T = \{ gH, gK : g \in G \}$

$\mathcal{E}_T = \{ g \cdot gH : g \in G \}$

\begin{center}
\begin{tikzpicture}
  \node (H) at (0,0) {$H$};
  \node (gH) at (1,0) {$gH$};
  \node (gK) at (1,1) {$gK$};
  \draw (H) edge (gH) edge (gK);
\end{tikzpicture}
\end{center}

\[ \sim \quad T/G = \overset{H}{H} \overset{K}{K} \]

- $T$ is a tree; (normal form thm)

  $\rightarrow$ vertex $H$ is adjacent to $HK$

  where $h \in H$

  $\rightarrow$ vertex $K$ is adjacent to $KH$

  where $k \in K$

$\rightarrow$ Any loop in $T$ looks like: \textbf{<< Loop }}
Any loop in $T$ loops twice.

$\Rightarrow$ Any loop in $T$ loops twice.

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$\Rightarrow$ Any loop in $T$ loops twice.

By uniqueness of normal form, we have

$\exists k_j = 1$

This gives a backtracking in the loop.

$\Rightarrow$ No Embedded Loop in $T$

Cor: $\forall F \neq H \times K \#F < \infty$

Then $F$ can be conjugated into $H$ or $K$.

Proof: To prove $F$ fix a vertex in $T$.

$T \Rightarrow x \cdot F(\text{or } F')$
Q: what is the universal covering of $M$?

"Cayley graph U filling holes" 

$\langle a, t^2 | tat^{-1} = a^2 \rangle$

$\cong \mathbb{Z}^2$

$t a t^{-1} = a^2$
HNN Extension: Let $G$ be a group, $H, K < G$ be two isomorphic subgroups.

A bigger group $\Gamma = G \ast_{H \triangleleft K} S.t.

H, K$ are conjugate in $\Gamma$.

If $G$ is given by $\langle S \mid R \rangle$, $H \triangleright K < G$ then

$G \ast_{H \triangleleft K} \cong \langle S \mid \cup_{\phi \in \Delta} t \theta t^{-1} = k = \phi(h) \rangle$

where $\phi : H \to K$ is given isomorphism.

Thm: Any element $g \in G \ast_{H \triangleleft K}$ can be written as

- $g = g_0 t^{c_1} g_1 t^{c_2} g_2 \cdots t^{c_n} g_n$
  where $c_i \in \{ \pm 1 \}$, $g_i \in G$

[Britton's Lemma]: If $g = 1$ then $\exists \ t h t^{-1} a r r o w \ k$ or $\exists \ t h k a r r o w \ h$ in the word.

Thm: $\Gamma = G \ast_{H \triangleleft K}$ acts on a tree $T$ s.t.

- $T/\Gamma = \bigcirc$
- The vertex stabilizer is conjugate to $G$
- The edge stabilizer is conjugate to $H$ or $K$. 
Ex. $\mathbb{Z} = HNN$ extension of $?$ over $?$

$\mathbb{Z} \uparrow \mathbb{R}$
Groups acting
Simplicial Trees
without inversions

Groups obtained by
taking free product
 amalgamations, HNNs

Bass-Serre tree
\[ \Gamma \]
\[ \Gamma = \prod_i \text{(Graph of groups)} \]

Graph of groups
\[ H \rightarrow K \]
\[ G \]

Groups acting on Trees
with finite stabilizer

Groups splitting over
finite groups

Stallings' Group Ends Thm
Assume \( \Gamma \) is not vir. \( \mathbb{Z} \)
\( \Gamma \) has \( \infty \)-many ends
Rmk. \* \#Ends of group $\Gamma = \{0, 1, 2, +\infty\}$

1. \#Ends $= 2 \iff \Gamma \cong \text{vir. } \mathbb{Z}$.

2. \#Ends is quasi-isometric invariant.