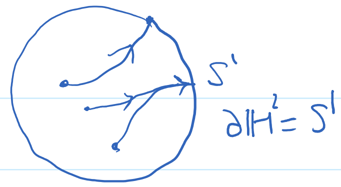
双曲群边界

2021年7月12日 9:51

Hyperbolic group (f.g. f.p) ⇝ Gromov boundary

$G \longmapsto \partial G \triangleq \partial\, Cay(G,S)$



$\boxed{\text{left multiplication } G \curvearrowright Cay(G,S)}$ ⇝ $G \curvearrowright \partial G$ by quasi-conformal $\boxed{\text{homeos}}$

hg —— hgs $\qquad h \mapsto \partial h : \partial G \to \partial G$

Fact: 1) If G is finite then $\partial G = \phi$ } elementary

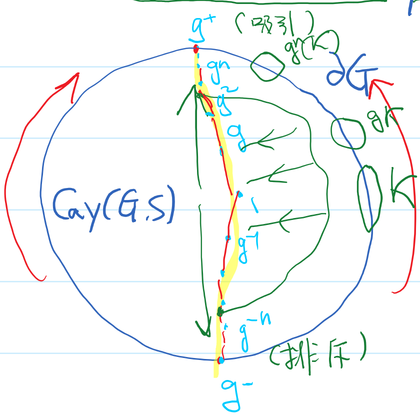
G is vir. cyclic then $\partial G = \{+,-\}$ } $\mathbb{Z}$ . . . $\mathbb{R}$

Otherwise

- ∂G = perfect compact metrizable space of ∞-cardinality. (no isolated pt.)
- $G \curvearrowright \partial G$ is minimal: any orbit is dense in ∂G.

2) Classification of elements in G: (No parabolic elements)

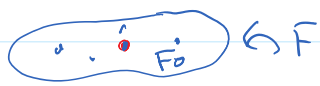
- finite order elements: $\boxed{\text{Elliptic}}$ $\boxed{\text{elements}}$
- infinite order elements: Hyperbolic elements (Loxodromic ⋯)
  - $n \in \mathbb{Z} \longmapsto g^n \in G$ is quasi-isometric embedding (quasi-geodesic)
  - g has exactly two fixed points on $G \cup \partial G$ with south-north dynamics:



$\forall K \subseteq \partial G \setminus \{g^-\}$

$g^n(K) \longrightarrow g^+$ as $n \to +\infty$

Thm: There are only finitely many conjugacy classes of finite subgps in a hyp gp.



Thm. (Tits alternative)

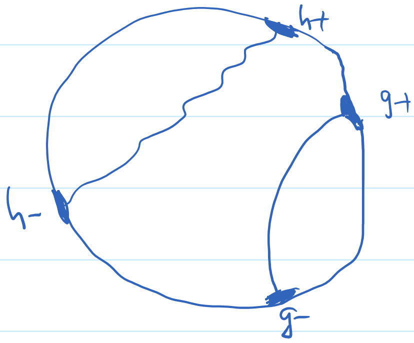
A infinite subgp of a $\boxed{\text{hyp gp}}$ is either

Thm. (Tits alternative)

Any infinite subgp of a $\boxed{\text{hyp. gp}}$ is either virtually cyclic or contains F_2.

Cor: No $\mathbb{Z}^2$ in hyp. group.

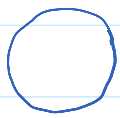
Proof idea: Play Ping-Pong between independent hyp elements



Thm: If a hyp gp G has $\partial G \cong$ Cantor set then G is virtually free.

Thm. (Tukia 88, Casson-Jungreis 94, Gabai 92)

If a hyp gp G has $\partial G \cong S^1$ then G is virtually closed surface group.



Rmk: This is the final step in the resolution of Seifert fiber space Conjecture. (+ Scott + Mess + ...)

Cannon Conjecture:

If a hyp gp G has $\partial G = S^2$ then G is virtually a uniform lattice in $\mathrm{Isom}(\mathbb{H}^3)$

i.e. G is virtually Kleinian cocompact group

$$G \doteq \pi_1(\text{compact. hyp 3-mfd})$$

Bonk-Kleiner 200? Hausdorff dim

Markovich: 2013. Special cube complex. Wise

收敛群作用

2021年7月27日 14:02

(Martin – Greenberg 1985)

Definition 12.1. Assume that a (discrete) group G acts by homeomorphisms on a compact metrizable space M. Assume that $|M| \geq 3$. Then G is called *a convergence group on* M if the induced action of G on the space of distinct triples

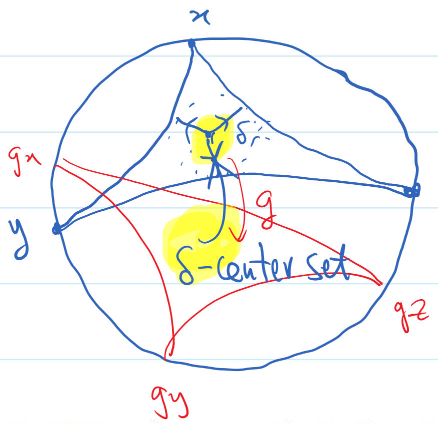
$$G \curvearrowright \Theta^3(M) := \{(x, y, z) \in M^3 : x \neq y \neq z \neq x\}$$ distinct triple space

is proper: the following set

$$\{g \in G : gK \cap K \neq \emptyset\}$$

is finite for any compact set K in $\Theta^3(M)$.

If $G \curvearrowright \Theta^3 M$ cocompact then it is called $\boxed{\text{uniform}}$ convergence group action.



$M = \partial X \hookleftarrow G \overset{\text{proper}}{\curvearrowright} X$ (hyp)

Fact: Any ideal triangle has δ-thin property.

Corollary 12.2. *Assume that G acts as a convergence group on M. Then every element of infinite order in G has* $\boxed{\textit{at most two}}$ *fixed points in M.* $K = \{x, y, z\} \curvearrowleft g^n$

Thm: Suppose a gp. G acts properly on a proper length hyperbolic space X. Then

- $G \curvearrowright \partial X$ is a convergence gp action.

If $G \curvearrowright X$ is cocompact, ($\Rightarrow$ G hyp group)
then
- $G \curvearrowright \partial X$ is a (uniform) convergence group action.

Cor: A hyp gp acts as a uniform convergence group action.

Lem: $G \curvearrowright M$ convergence gp action $\overset{\Leftarrow \text{EASY}}{\Longleftrightarrow}$ $\forall \{g_n\} \subseteq G$ $\#\{g_n\} = \infty$
$\exists \{g_{n_i}\}$ & $\exists\, a, b \in M$ s.t.

Lem:

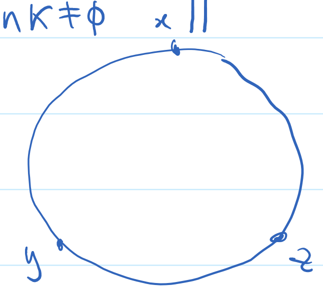
$\exists \{g_{n_i}\}$ & $\exists\, \underline{a,\ b} \in M$ s.t.

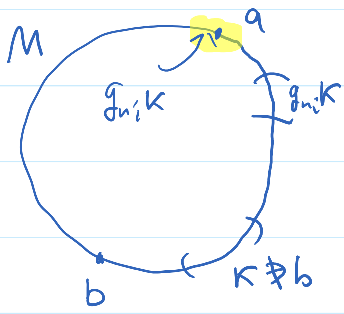
$K = \{(x,y,z)\}$

$g_n K \cap K \neq \emptyset$

$\Big\Vert$ $g_{n_i}\big|_{M \setminus b} \longrightarrow a$ loc. unif.

$\boxed{\forall K \subset M \setminus b \text{ compact} \quad g_{n_i}(K) \longrightarrow a}$





Thm (Bowditch, 90's)

If a group G admits a $\boxed{\text{uniform}}$ convergence group action on a compact metric space M

then G is a hyp gp. and $\underline{\partial G \cong \text{Limit set of } G \text{ in } M}$

in particular

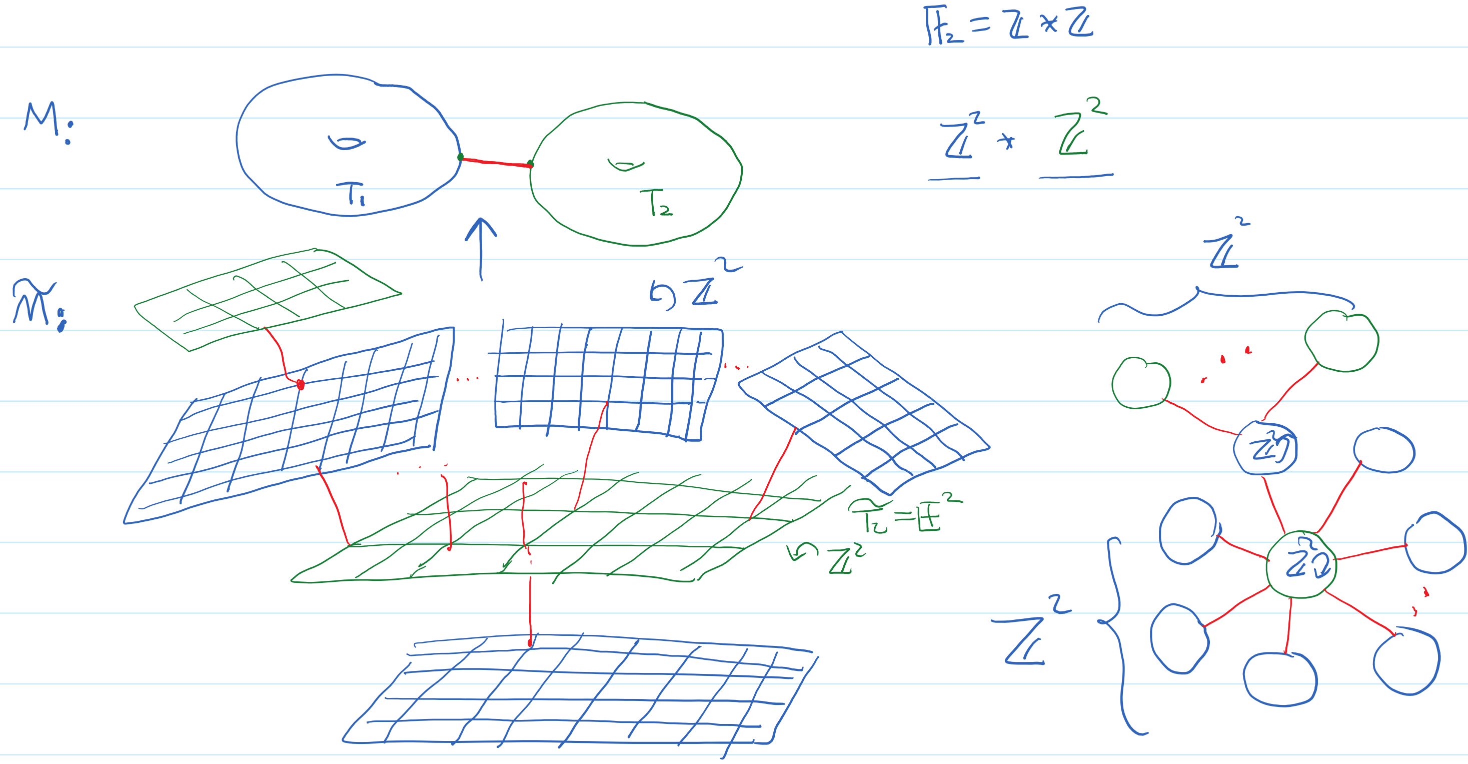
$\Big\Vert$ if $\underline{G \curvearrowright M \text{ is minimal}}$

$\Big\Vert$ then $\partial G \cong M$

融合积

2021年7月27日 17:55

Free products:



Free product of Two groups H and K: $\quad G \triangleq H * K$

Precisely, a group G is called a *free product* of H and K if there exist a pair of homomorphisms $\iota_H : H \to G$ and $\iota_K : K \to G$ such that they are universal in the following diagram:

$$\begin{tikzcd} H \arrow[r, "\iota_H"] \arrow[dr, "\phi_H"'] & G \arrow[d, dashed, "\exists \Phi"] & K \arrow[l, "\iota_K"'] \arrow[dl, "\phi_K"] \\ & \forall \Gamma & \end{tikzcd}$$

By the universal property, it is easy to see that $\iota_H : H \to G$ and $\iota_K : K \to G$ are both injective. Moreover, G is unique up to isomorphism, so G must be generated by H and K.

$$\mathbb{Z}^2 * \mathbb{Z}^2 = \langle a, b, c, d \mid ab = ba,\ cd = dc \rangle$$

Fact: • suppose $H = \langle S \mid \mathbb{R} \rangle \quad K = \langle T \mid \mathbb{Q} \rangle$

then $H * K = \langle S \amalg T \mid \mathbb{R}, \mathbb{Q} \rangle$

[Normal Form Thm] • Every element $g \in H * K$ can be written as a unique product of ALTERNATING words

$$1 \neq g = h_1 k_1 h_2 k_2 \cdots h_n k_n$$

where $\forall h_i, k_j \neq 1$

$F_2 = \mathbb{Z} * \mathbb{Z} = \langle a, b \rangle$ $a\,b\,a^2 b^2 a^3 b^{-1}$

Thm: $G = H * K$ acts on a tree T by graph isom, s.t.

- $T/G = $ (H) —— (K)

- The vertex stabilizer is conjugate either to $\underline{H}$ or $\underline{K}$.

- Edge stabilizer is trivial.

Proof: Let us construct the Bass-Serre Tree: « Tree »

$VT \triangleq \{gH, gK : g \in G\}$

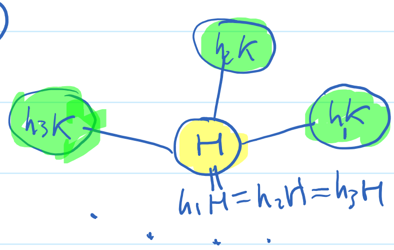
$ET \triangleq \{g : g \in G\}$

(H) - (gH) $\xrightarrow{\;g\;}$ (gK)

$\rightsquigarrow$ • T/G = (H) —— (K).

• T is a tree: (normal form thm)

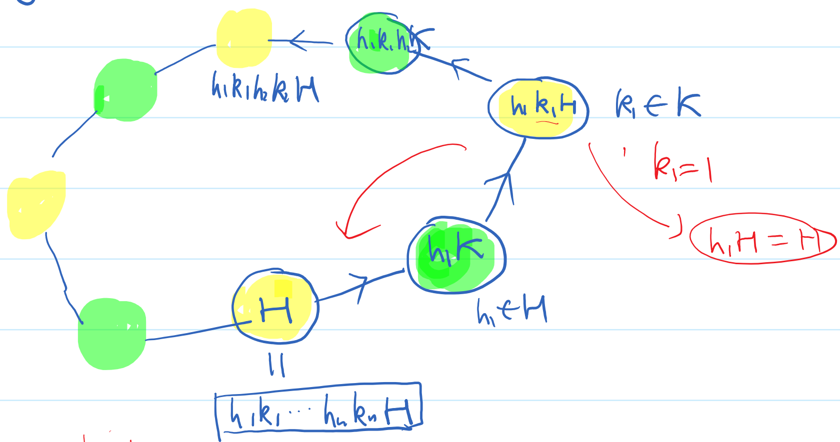
$\to$ Vertex (H) is adjacent to (hK)
where $\underline{h \in H}$



$\to$ vertex (K) is adjacent to (kH)
where $\boxed{k \in K.}$

$\to$ Any loop in T looks like:

→ Any loop in T looks like:



$\Rightarrow \quad h_1k_1k_2h_2k_2 \cdots h_nk_n = h \in H$

By uniqueness of normal form, we have

$\exists \; k_j = 1$

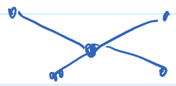
This gives a backtracking in the loop.

$\rightsquigarrow$ $\boxed{\text{No}}$ Embedded Loop in T ▨

Cor: $\forall F < H * K \quad \# F < \infty$

Then F can be conjugated into H or K.

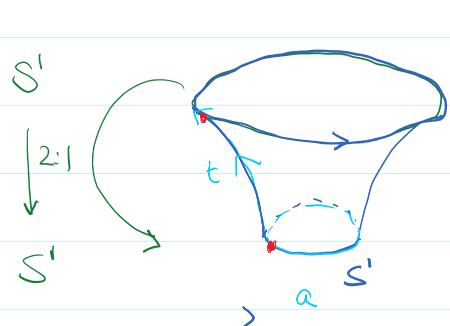
proof: To prove F fix a vertex in T.

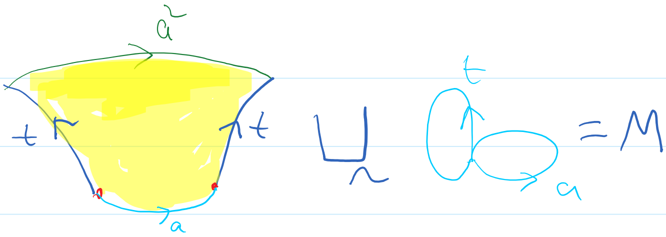
$T \ni x$  $Fx \curvearrowleft F$

$\lrcorner$

HNN扩张

2021年7月27日 18:09



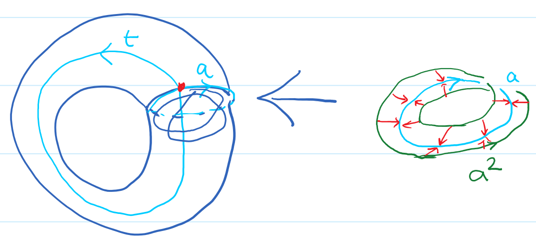


Annulus ⇝ $\pi_1 M \triangleq \langle a, t \mid tat^{-1} = a^2 \rangle$

$= \langle a \rangle *_{\langle a \rangle \sim \langle a^2 \rangle}$

Baumslag-Solitar groups.

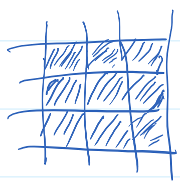
M:



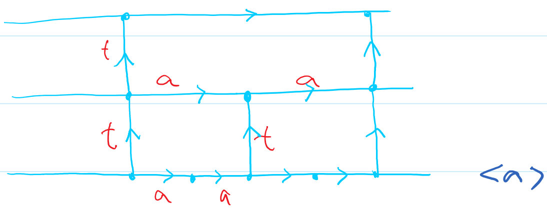
Q: what is the universal covering of M?

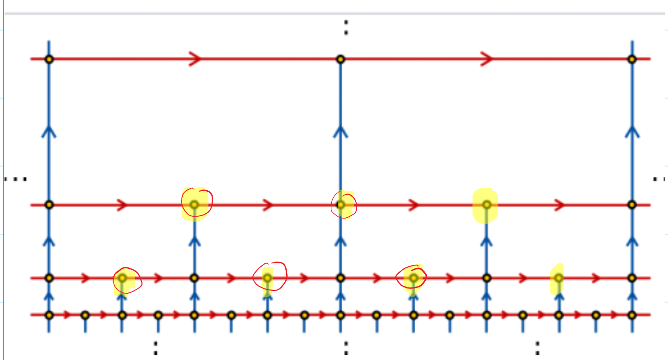
"Cayley graph ⊔ filling holes"

$\mathbb{Z}^2$

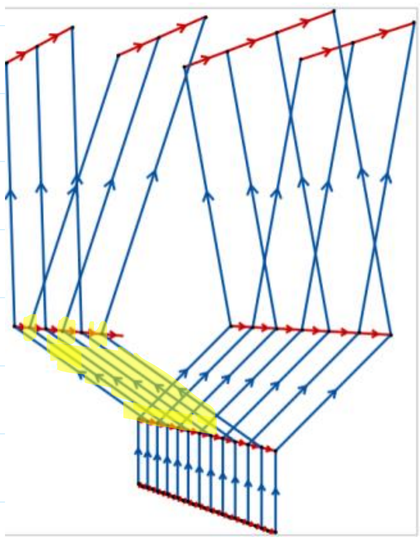


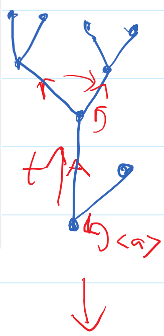
$tat^{-1} = a^2$





Sheet.





Graham Higman, Bernhard Neumann, and Hanna Neumann

Graham Higman, Bernhard Neumann, and Hanna Neumann

$= T_3/BS(1,2)$

HNN Extension: Let G be a group, $H, K < G$ be two isomorphic subgroups.

$\Downarrow$

A bigger group $\Gamma \triangleq G *_{H\sim K}$ s.t. H, K are conjugate in Γ.

If G is given by $\langle S \mid R \rangle$ $\phi: H \xrightarrow{\text{iso}} K < G$

then

$$G *_{H \sim K} \triangleq \langle S \sqcup \{t\} \mid \underbrace{tht^{-1} = k = \phi(h)}_{\forall h \in H} \rangle$$

wher $\phi: H \to K$ is given isomorphism.

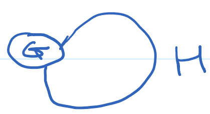
Thm: Any element $g \in G *_{H\sim K}$ can be written as

- $g = g_0 \boxed{t^{\varepsilon_1} g_1 t^{\varepsilon_2}} g_2 \cdots t^{\varepsilon_n} g_n$

where $\varepsilon_i \in \{\pm 1\}$, $g_i \in G$

[Britton's Lemma] • If $g = 1$ then $\exists$ $\underset{\substack{\| \\ k}}{\underline{tht^{-1}}}$ or $\exists$ $\underset{\substack{\| \\ h}}{\underline{t^{-1}kt}}$ in the word.

Thm: $\Gamma = G *_{H\sim K}$ acts on a tree T s.t.

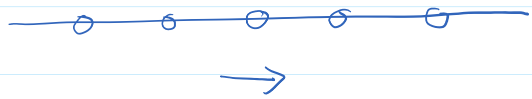
- $T/\Gamma =$ 

- the vertex stabilizer is conjugate to $\boxed{G}$

- The edge stabilizer is conjugate to H or K.

Ex: $\mathbb{Z}$ = HNN extension of ? over ?

$\mathbb{Z} \curvearrowright \mathbb{R}$



群在树上的作用

2021年7月28日　9:14

Groups acting Simplicial Trees without inversions $\longleftrightarrow$ Groups obtained by taking free product almagamations, HNN's

Bass-Serre tree

T

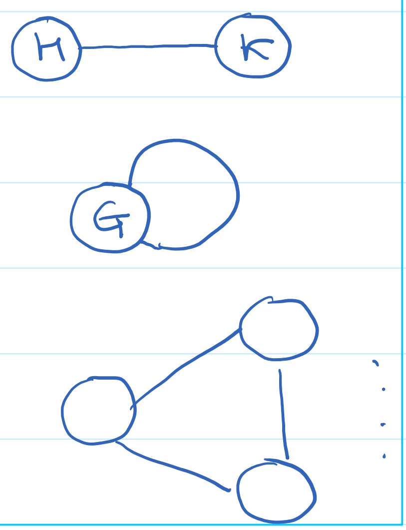
$\circlearrowright$

$\Gamma = \pi_1(\text{Graph of gps})$

$\longleftarrow$

$\xrightarrow{T/\Gamma}$

Graph of groups



Groups acting on Trees with finite stabilizer $\longleftrightarrow$ Groups spliting over finite groups.

$\updownarrow$

Stalling's Group Ends Thm

Assume Γ is NOT vir. $\mathbb{Z}$

Γ has ∞-many ends

Rmk: • # Ends of group $\Gamma = \{0, 1, 2, +\infty\}$

• # Ends $= 2 \iff \Gamma \cong$ vir. $\mathbb{Z}$.

• # Ends is Quasi-isometric invariant.