HARMONIC SECTIONS OF RIEMANNIAN VECTOR BUNDLES

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The work is dedicated to Professors J. Eells and J.H. Sampson, mindful of their paper:

“Harmonic Mappings of Riemannian Manifolds”,

American J. Math, 1964

Introduction

(M, g) $\cdots$ smooth n-dimensional Riemannian manifold,

$(\mathcal{E}, \langle\, , \rangle, \nabla) \to M$ $\cdots$ smooth Riemannian vector bundle,

σ $\cdots$ smooth section of $\mathcal{E}$,

$$X.|\sigma|^2 = 2\langle \nabla_X \sigma, \sigma \rangle$$

Question: ***Which sections $\boldsymbol{\sigma}$ are "best"?***

1. **$\boldsymbol{\sigma}$ parallel ($\boldsymbol{\nabla\sigma = 0}$).**

Drawbacks ("Reduction of holonomy").

- $|\sigma| = \text{const.}$ $\quad (\Rightarrow\ \chi(\mathcal{E}) = 0)$
- **de Rham Decomposition** $(\mathcal{E} = TM)$

2. $\boldsymbol{\sigma}$ Hodge-de Rham harmonic ($\boldsymbol{\Delta\sigma = 0}$).

Drawbacks. For M compact:

- **Hodge's Theorem.**

  $\{\textit{H-dR harmonic } \sigma\} \cong H^1(M, \mathbb{R})$

  If $\beta_1(M) = 0$ then: $\quad \Delta\sigma = 0 \Rightarrow \sigma = 0.$

- **Bochner's Vanishing Theorem.**

  If $\mathrm{Ric}(M) > 0$ then: $\quad \Delta\sigma = 0 \Rightarrow \sigma = 0.$

  If $\mathrm{Ric}(M) \geqslant 0$ then: $\quad \Delta\sigma = 0 \Rightarrow \nabla\sigma = 0.$

3. $\boldsymbol{\sigma}$ harmonic section of $\mathcal{E}$.

$$E^v(\sigma) = \frac{1}{2}\int_M |\nabla\sigma|^2 \operatorname{vol}(g)$$

Defn. σ *harmonic section* of $\mathcal{E}$ if: $\frac{d}{dt}\Big|_{t=0} E^v(\sigma_t) = 0, \ \forall\sigma_t$

Euler-Lagrange eqns: $\boxed{\nabla^*\nabla\sigma = 0}$ $(\nabla^*\nabla = -\operatorname{Tr}\nabla^2)$

Drawbacks.

- If M compact and $\nabla^*\nabla\sigma = 0$ then:

$$0 = \int_M \langle\nabla^*\nabla\sigma, \sigma\rangle \operatorname{vol}(g) = \int_M \langle\nabla\sigma, \nabla\sigma\rangle \operatorname{vol}(g), \quad \text{so } \nabla\sigma = 0.$$

4. $|\boldsymbol{\sigma}| = \mathbf{k}$, $\boldsymbol{\sigma}$ harmonic section of $\mathbf{S}\boldsymbol{\mathcal{E}}(\mathbf{k})$.

$$S\mathcal{E}(k) = \{u \in \mathcal{E} : |u| = k\}, \quad \text{sphere bundle.}$$

Successes.

- E-L equations: $\boxed{\nabla^*\nabla\sigma = \frac{1}{k^2}\,|\nabla\sigma|^2\,\sigma}$ (nonlinear).
- Many interesting solns. (eg. Hopf vector fields on S^{2p+1}).
- Stability theory. (However...)

Drawbacks.

- Limited to bundles with $\chi(\mathcal{E}) = 0$.

Desiderata.

A variational theory on $\mathcal{C}^\infty(\mathcal{E})$ which:

- Works for all bundles $\mathcal{E} \to M$, including $\chi(\mathcal{E}) \neq 0$.
- Applies to all sections σ of $\mathcal{E}$.
- Includes all solutions of **4.** as critical points.

Basic Idea.

Eliminate all constraints by perturburbing the background geometry (of $\mathcal{E}$).

The Harmonic Section Variational Problem

$\sigma: M \to \mathcal{E}$, section. $\mathcal{V} \subset T\mathcal{E}$, vertical subbundle.

- For each $x \in M$ form: $d^v\sigma(x): T_x M \to \mathcal{V}_{\sigma(x)} \subset T_{\sigma(x)}\mathcal{E}$
- Compute: $|d^v\sigma(x)|^2 = \sum_i |d^v\sigma(x)(E_i)|^2$ where:

  $\{E_i\}$ is a $g(x)$-orthonormal basis of $T_x M$,

  $|\,.\,|^2$ on $T_{\sigma(x)}\mathcal{E}$ is induced by the *Sasaki metric* h on $\mathcal{E}$.
- Then: $$\boxed{E^v(\sigma) = \frac{1}{2}\int_M |\nabla\sigma|^2 \,\mathrm{vol}(g) = \frac{1}{2}\int_M |d^v\sigma|^2 \,\mathrm{vol}(g)}$$

Metrics

Suppose $V \in \mathcal{V}_u$, $u \in \mathcal{E}$.

Then $V = v'(0)$ for a unique str. line $v(t) = u + tv$, $v \in \mathcal{E}_{\pi(u)}$

Sasaki metric. Define: $|V|^2 = |v|^2$.

Cheeger-Gromoll metric. $|V|^2 = \dfrac{1}{1+|u|^2}\left(|v|^2 \,+\, \langle u, v\rangle^2\right)$

LBW metric. For any $(m, r) \in \mathbb{R}^2$ define:

$$|V|^2_{m,r} \;=\; \frac{1}{(1+|u|^2)^m}\left(|v|^2 \,+\, r\langle u, v\rangle^2\right)$$

Obtain a "metric" $h_{m,r}$ on $\mathcal{E}$.

Note. $h_{0,0} =$ Sasaki, $h_{1,1} =$ Cheeger-Gromoll.

Note. If $r < 0$ then $h_{m,r}$ is Riemannian only on a tubular nhd. of the zero section:

$$B\mathcal{E}(1/\sqrt{-r}) = \{u \in \mathcal{E} : |u|^2 < -1/r\}$$

Definition. Say σ is r-*Riemannian* if $r|\sigma(x)|^2 \geqslant -1, \forall x$.

Note. If $r > 0$ then every σ is r-Riemannian.

Note. If $r_1 < r_2$ then r_1-Riemannian $\Rightarrow r_2$-Riemannian.

Replacing Sasaki with LBW yields:

$$\boxed{E^v(\sigma) = E^v_{m,r}(\sigma) = \frac{1}{2}\int_M w^m(\sigma)\big(|\nabla\sigma|^2 + r|X(\sigma)|^2\big)\mathrm{vol}(g)}$$

where $\quad w(\sigma) = \dfrac{1}{1+|\sigma|^2} \quad$ and $\quad X(\sigma) = \frac{1}{2}\nabla|\sigma|^2.$

Remark. If $r < 0$ and $|\sigma|^2 \leqslant -1/r$, deduce *Kato's Inequality:*

$$|\nabla\sigma|^2 + r\,|X(\sigma)|^2 \geqslant 0, \quad \text{with equality iff } \nabla\sigma = 0.$$

Definition. A harmonic section of $\mathcal{E}$ w.r.t. $h_{m,r}$ is (m,r)-*harmonic.*

Note. Not necessarily r-Riemannian.

Euler-Lagrange Equations

Remark. If $|\sigma| = k$ then: $\quad E^v_{m,r}(\sigma) \;=\; \dfrac{1}{(1+k^2)^m}\, E^v_{0,0}(\sigma)$

σ is an (m,r)-harmonic section of $S\mathcal{E}(k)$ if and only if

σ is a harmonic section of $S\mathcal{E}(k)$.

Technical Theorem. *σ is an (m,r)-harmonic section of $\mathcal{E}$ iff:*

$$\boxed{T_m(\sigma) \;=\; \phi_{m,r}(\sigma)\,\sigma} \quad \textit{where:}$$

$$T_m(\sigma) \;=\; (1+2F)\nabla^*\nabla\sigma \;+\; 2m\,\nabla_{X(\sigma)}\sigma$$

$$\phi_{m,r}(\sigma) \;=\; m\,|\nabla\sigma|^2 \;-\; mr\,|X(\sigma)|^2 \;-\; r(1+2F)\Delta F$$

$$2F \;=\; |\sigma|^2$$

Theorem A. *Suppose* $|\sigma(x)| = k > 0$ *for all* $x \in M$.

(a) *If* $m \neq 1 + 1/k^2$ *then* σ *is an* (m, r)*-harmonic section of* $\mathcal{E}$ *if and only if* $\nabla\sigma = 0$;

(b) *If* $m = 1 + 1/k^2$ *then* σ *is an* (m, r)*-harmonic section of* $\mathcal{E}$ *if and only if* σ *is a harmonic section of* $S\mathcal{E}(k)$.

Example. For $m > 1$ define: $\quad \sigma = \dfrac{1}{\sqrt{m-1}}\,\xi,$

where ξ is the Hopf vector field on $M = S^{2p+1}$.

Since ξ is a harmonic section of $S\mathcal{E}(1)$, σ is harmonic section of $S\mathcal{E}(1/\sqrt{m-1})$, hence a (non-parallel) (m, r)-harm. section of $\mathcal{E}$.

Theorem B. *Suppose M is compact, $\chi(\mathcal{E}) \neq 0$, and $\sigma \neq 0$.*

For each $m \in \mathbb{R}$ there exists at most one $r \in \mathbb{R}$ such that σ is (m, r)-harmonic, and:

(a) *if $-4 \leqslant m \leqslant -1$ then $r < -1 - m$;*

(b) *if $-1 \leqslant m \leqslant 1$ then $r < 0$;*

(c) *if $1 < m \leqslant 2$ and $\|\sigma\|_\infty \leqslant 1/\sqrt{m-1}$ then $r < 0$;*

(d) *if $2 \leqslant m$ and $\|\sigma\|_\infty \leqslant 1/\sqrt{m-1}$ then $r < 1 - m/2$.*

Remark. Restrictions when $m < -4$?

Remark. $\|\sigma\|_\infty$ indicates non-linearity of (m, r)-harmonic section equations.

Definition. Define $\rho\colon [-4,\infty) \to \mathbb{R}$ as follows:

$$\rho(m) = \begin{cases} -1-m, & m \in [-4,-1] \\ 0, & m \in [-1,2] \\ (2-m)/2, & m \in [2,\infty) \end{cases}$$

Definition. σ is *strictly r-Riemannian* if σ is r-Riemannian and:

$$r\,|\sigma(x)|^2 > -1, \quad \text{for some } x \in M.$$

Theorem B+. *Suppose M compact.*

Suppose $m \geqslant -4$, $r \geqslant \rho(m)$ and σ is strictly $(1-r)$-Riemannian.

Then σ is (m,r)-harmonic if and only if $\nabla\sigma = 0$.

$$\mathcal{F}_- = \{(m,r) : m < 0,\ r \leqslant 2m\}, \qquad \mathcal{F}_0 = \{(m,r) : 0 \leqslant m \leqslant 1\},$$

$$\mathcal{F}_1 = \{(m,r) : m > 1,\ r < 1 - m\}, \qquad \mathcal{F} = \mathcal{F}_- \cup \mathcal{F}_0 \cup \mathcal{F}_1$$

Theorem C. *Suppose that:*

- $(m,r) \in \mathcal{F}$;
- σ *is* r*-Riemannian;*
- $|\sigma|^2 \colon M \to \mathbb{R}$ *is a harmonic function.*

Then σ *is an* (m,r)*-harmonic section if and only if* $\nabla\sigma = 0$.

Remark. If $|\sigma| = k$ and σ is r-Riemannian, then $r \geqslant -1/k^2$.

Thus if $m = 1 + 1/k^2$ then $r \geqslant 1 - m$ (ie. $(m,r) \notin \mathcal{F}$).

New Examples

Remark. From Theorems B and C:

If $0 \leqslant m \leqslant 1$ and σ an (m, m)-harmonic section of $\mathcal{E}$ (with $|\sigma|^2$ harmonic if M non-compact) then $\nabla \sigma = 0$.

Moral. Cheeger-Gromoll no better than Sasaki!

$M = S^n \subset \mathbb{R}^{n+1}$

Let ξ be a *standard gradient field:* $\xi = \nabla \lambda$,

where $\lambda \colon S^n \to \mathbb{R}$ is the restriction of a unit vector of $(\mathbb{R}^{n+1})^*$.

Define $\sigma = k\,\xi$, $k \in \mathbb{R}$.

Theorem.

σ is a non-trivial (m,r)-harmonic section of TM iff $n \geqslant 3$ and:

$$m = n + 1, \qquad r = 2 - n, \qquad k^2 = -1/r.$$

Remark. $\|\sigma\|_\infty = |k| = 1/\sqrt{-r}$.

- σ is r-Riemannian, but only just!
- $r \leqslant 1 - m/2$, with equality only when $n = 3$. However:

$$\|\sigma\|_\infty \;=\; 1/\sqrt{n-2} \;=\; 1/\sqrt{m-3} \;>\; 1/\sqrt{m-1}$$

So consistent with Theorem B.

Remark. Non-invariance under scaling!

Idea. Try to find new examples by rescaling old ones!

Theorem. *Suppose $\sigma = f\xi$ where:*

- *ξ is the Hopf vector field on $M = S^{2p+1}$;*
- *$f\colon M \to \mathbb{R}$ is any smooth function.*

Then σ is a non-trivial (m,r)-harmonic section of TM iff:

$$m > 1 \quad \textit{and} \quad f = \pm 1/\sqrt{m-1}\,.$$

Remark. These are the (m,r)-harmonic sections of Theorem A.

Theorem. *Let $n = 2q - 1$, and let $\lambda\colon S^n \to \mathbb{R}$ be the restriction of the following harmonic quadratic form on $\mathbb{R}^{n+1}$:*

$$k\left(x_1^2 + \cdots + x_q^2 - x_{q+1}^2 - \cdots - x_{2q}^2\right), \quad k \in \mathbb{R}.$$

Vector field $\sigma = \frac{1}{2}\nabla\lambda$ is an (m, r)-harmonic section of TS^n iff:

- $n \geqslant 5$;
- $2m = n + 3 \quad$ (*ie.* $m = q + 1$);
- k^2 *is determined by:*

$$4(m-3)k^2 = 4 + 2m - m^2 + \sqrt{m-2}\,\sqrt{m^3 - 2m^2 - 8}$$

- r *is determined by:* $\quad 0 = m(m-3) + 2r(m + k^2)$

Low dimensions.

$$n = 5, \quad m = 4, \quad r = \frac{1}{\sqrt{3}} - 1, \quad k^2 = \sqrt{3} - 1$$

$$n = 7, \quad m = 5, \quad r = \frac{\sqrt{201} - 24}{16}, \quad k^2 = \frac{\sqrt{201} - 11}{8}$$

$$n = 9, \quad m = 6, \quad r = \frac{\sqrt{34} - 13}{5}, \quad k^2 = \frac{\sqrt{34} - 5}{3}$$