Lectures
"Hirzebruch formula: generalizations and applications"

given by Alexander S. Mishchenko
at the Chern Institute of Mathematics, Nankai University,
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Some elementary and evident examples

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$$\textbf{sign}\ X = 2^{2k}\langle L(X), [X]\rangle\,.$$

Here

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is the signature of non degenerated quadratic form in the cohomology groups $H^{2k}(X, \mathbf{C})$,

defined by $\cup$–product:

$$([\omega_1], [\omega_2]) \overset{def}{=} \langle [\omega_1] \cup [\omega_2], [X]\rangle =$$

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The class

$$L(X) = \prod_j \frac{t_j/2}{\text{th } (t_j/2)}$$

is the Hirzebruch characteristic class defined by formal generators t_j such that

$$\sigma_k(t_1, \ldots, t_n) = c_k\,(cTX) \in H^{2k}(X; \mathbf{Z}),$$

where σ_k is elementary symmetric polynomial. Here c_k is the k-th Chern characteristic class of the complex vector bundle.

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There are different ways to generalize the Hirzebruch formula mainly for non simply connected manifolds.

1. Finite dimensional unitary representations

Let X be a closed orientable non simply connected manifold and let $\pi = \pi_1(X)$,

$$f_X : X \longrightarrow B\pi$$

be the canonical mapping defined up to homotopy which induces the isomorphism of fundamental groups

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Consider a finite dimensional representation

$$\rho : \pi \longrightarrow U(N).$$

Using the representation ρ one can construct several things:

1)The flat vector bundle ξ^ρ over $B\pi$, induced by the representation ρ.

2)The flat vector bundle ξ^ρ_X over X induced by the representation ρ, $\xi^\rho_X = f^*_X \xi^\rho$.

3)The cohomology groups $H^{2k}(X, \rho)$ with the local system of coefficients induced by the representation ρ.

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The $\cup-$product induces non degenerated quadratic form in the group $H^{2k}(X,\rho)$:

$$H^{2k}(X,\xi_X^\rho)\times H^{2k}(X,\xi_X^\rho)\xrightarrow{\cup}H^{4k}(X,\xi_X^\rho\otimes\xi_X^\rho)\xrightarrow{\langle\cdot,\cdot\rangle}H^{4k}(X,\mathbf{C})\approx\mathbf{C}.$$

The signature of this form we shall denote by

$$\textbf{sign}\ _\rho X=\textbf{sign}\ \left(H^{2k}(X,\rho),\cup\right).$$

It is easy to check that

$$\textbf{sign}\ _\rho X=2^{2k}\left\langle L(X)\mathrm{ch}\xi_X^\rho,[X]\right\rangle.$$

Since ξ^ρ is flat bundle one has $\mathrm{ch}\xi^\rho=\dim\xi^\rho=N$. Hence both left side and right side of the formula (2) coincide with that of (1) up to an integer factor N.

This means that at least the signature $\textbf{sign}\ _\rho X$ does depend only on the dimension of the finite dimensional unitary representation ρ. More of that this generalization might be useful for further generalizations. Namely one can construct at least righthand side of the formula (2) for more general representations of fundamental group π.

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2. Finite dimensional unitary representations with respect to pseudo Hermitian structure of type (p, q)

Consider a representation

$$\rho : \pi \longrightarrow U(p, q)$$

into the matrix group $U(p, q)$ which preserves an indefinite form of type (p, q).

Then again one can construct the operation of type $\cup$ which generates a non degenerated quadratic form on middle cohomologies $H^{2k}(X; \rho)$.

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On the other hand the flat vector bundle ξ^{ρ}_{X} can be split into the direct sum

$$\xi^{\rho}_{X} = \xi^{\rho}_{+} \oplus \xi^{\rho}_{-},$$

such that on each summand the form is definite (positively and negatively).

Then the Hirzebruch formula has the following form

$$\textbf{sign}\,_{\rho}X = 2^{2k}\left\langle L(X)\text{ch}(\xi^{\rho}_{+} - \xi^{\rho}_{-}), [X]\right\rangle,$$

Here the Chern character of the bundles $\xi^{\rho}_{\pm}$ may be non trivial (Lusztig, 1972).

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Finite dimensional unitary representations with respect to a skew Hermitian structure

Taking a skew Hermitian form φ on $\mathbf{C}^N$ and the matrix group $\mathbf{Sp}(N)$ which preserves this form one can consider a representation

$$\rho : \pi \longrightarrow \mathbf{Sp}(N)$$

and flat vector (complex) bundle ξ^ρ_X.

If $\dim X = 4k + 2$ then in the middle dimension one has non degenerated Hermitian form in the group $H^{2k+1}(X, \rho)$ generated by the $\cup$–product:

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3. Continuous family of finite dimensional representations

Let us consider a continuous family of representations

$$\rho_t : \pi \longrightarrow U(N), \quad t \in T,$$

This family generates the family of quadratic forms with constant signature but non trivial vector bundles of positive and negative subspaces

$$\textbf{sign}\ {}_{\rho_t}X = \textbf{sign}\ \left(H^{2k}(X, \rho_t), \cup\right) \in K(T).$$

To construct the family of quadratic forms as a continuous family we need to include the family $\left(H^{2k}(X, \rho_t), \cup\right)$ into larger space with constant dimension unlike homologies $H^{2k}(X, \rho_t)$.

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Given combinatorial structure on X let $C_k = C_k(X)$ denotes the group of k–dimensional chains of X with coefficients in $\mathbf{C}^N$.

Then the representations ρ_t define boundary homomorphisms d_k and the Poincare duality homomorphisms D_k which are continuous with respect to $t \in T$:

$$\begin{array}{ccccccc}
C_0 & \xleftarrow{d_1} & C_1 & \xleftarrow{d_2} & \cdots & \xleftarrow{d_n} & C_n \\
\uparrow D_0 & & \uparrow D_1 & & & & \uparrow D_n \\
C_n^* & \xleftarrow{d_n^*} & C_{n-1}^* & \xleftarrow{d_{n-1}^*} & \cdots & \xleftarrow{d_1^*} & C_0^*
\end{array}$$

The properties are

$$d_{k-1} d_k = 0,$$

$$d_k D_k + (-1)^{k+1} D_{k-1} d^*_{n-k+1} = 0,$$

$$D_k = (-1)^{k(n-k)} D^*_{n-k}.$$

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D induces isomorphism in homology groups.

Put

$$F_k = i^{k(k-1)} D_k.$$

Then similar diagram

$$\begin{array}{ccccccc}
C_0 & \xleftarrow{d_1} & C_1 & \xleftarrow{d_2} & \cdots & \xleftarrow{d_n} & C_n \\
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has more natural properties

$$d_k F_k + F_{k-1} d_{n-k+1}^* = 0,$$
$$F_k = F_{n-k}^*.$$

Consider the cone of F, that is acyclic complex with respect to total graduation and sum of differentials d and F:

$$0 \longleftarrow A_0 \xleftarrow{H_1} A_1 \xleftarrow{H_2} \dots \xleftarrow{H_{2l}} A_{2l} \xleftarrow{H_{2l+1}} A_{2l+1} \xleftarrow{H_{2l+2}} \dots \xleftarrow{H_{4l}} A_{4l} \xleftarrow{H_{4l+1}} A_{4l+1} \longleftarrow 0$$

where

$$A_k = C_k \oplus C^*_{n-k+1},$$

$$H_k = \begin{pmatrix} d_k & F_{k-1} \\ 0 & d^*_{n-k+2} \end{pmatrix}$$

Put

$$A = \bigoplus_{k=0}^{n+1} A_k = A_{ev} \oplus A_{odd},$$

where

$$A_{ev} = \bigoplus_{k=0}^{2l} A_{2k}, \quad A_{odd} = \bigoplus_{k=0}^{2l} A_{2k+1}.$$

$$A_{ev} \approx A_{odd} \approx \bigoplus_{k=0}^{n} C_k,$$

$$d + d^* + F : A_{ev} \longrightarrow A_{odd}$$

is isomorphism.

Put

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$$d + d^* + F : A_{ev} \longrightarrow A_{odd}$$

is isomorphism.

Take in account that $d = d_t$, $F = F_t$ one has

Theorem 1

$$\textbf{sign}\ (A_t) = \textbf{sign}\ (X, \rho_t).$$

where $A_t = d_t + d_t^* + F_t$.

New notion of signature for continuous family

$$A_t : V_t \longrightarrow V_t, \quad V_t \approx V.$$

There is a splitting

$$V = V_t^+ \oplus V_t^-,$$

such that A_t is positive on V_t^+ and negative on V_t^-.

Then

$$\xi^+ = \coprod V_t^+, \quad \xi^- \coprod V_t^-$$

are subbundles. By definition

$$\textbf{sign}\ (A_t) = [\xi^+] - [\xi_-] \in K(T).$$

Thus

$$\textbf{sign}_{\rho_t}(X) = \textbf{sign}\,(d_t + d_t^* + F_t) \in K(T).$$

Generalization of the Hirzebruch formula:

$$K(T) \ni \textbf{sign}_{\rho_t} X = 2^{2k} \left\langle L(X)\mathrm{ch}_X \left(\xi^{\rho_t}_{X\times T}\right), [X]\right\rangle \in K(T) \otimes Q,$$

4. Algebraic setting

The most general picture for the Hirzebruch formula for oriented smooth manifolds can be represented as follows.

Let $\Omega_*(B\pi)$ denote the bordism group of pairs (M, f_M). Recall that $\Omega_*(B\pi)$ is a module over the ring $\Omega_* = \Omega_*(\text{ pt })$.

One can construct a homomorphism

$$\textbf{sign} : \Omega_*(B\pi) \longrightarrow L_*(\mathbf{C}\pi)$$

which for every manifold (M, f_M) assigns the element **sign** $(M) \in L_*(\mathbf{C}\pi)$, where $L_*(\mathbf{C}\pi)$ is the Wall group for the group ring $\mathbf{C}\pi$.

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Algebraic construction of symmetric signature

Similar combinatorial diagram:

$$\begin{array}{ccccccc}
C_0 & \xleftarrow{d_1} & C_1 & \xleftarrow{d_2} & \cdots & \xleftarrow{d_n} & C_n \\
\uparrow D_0 & & \uparrow D_1 & & & & \uparrow D_n \\
C^n & \xleftarrow{d_n^*} & C^{n-1} & \xleftarrow{d_{n-1}^*} & \cdots & \xleftarrow{d_1^*} & C^0
\end{array}$$

where $C_k \stackrel{def}{=} C_k(\widetilde{X})$, $C^k \stackrel{def}{=} C_0^k(\widetilde{X}) \approx Hom_{C\pi}(C_k, C\pi)$.

The homomorphism **sign** satisfies the following conditions:

(a) **sign** (M) does depend only on the homotopy equivalence class of the manifold M.

(b) if N is a simply connected manifold and $\tau(N)$ is its signature then

$$\textbf{sign}\,(M \times N) = \textbf{sign}\,(M)\tau(N) \in L_*(\mathbf{C}\pi).$$

We shall be interested only in the groups after tensor multiplication with the field of rational numbers $\mathbf{Q}$, in other words in the homomorphism

$$\textbf{sign} : \Omega_*(B\pi) \otimes \mathbf{Q} \longrightarrow L_*(\mathbf{C}\pi) \otimes \mathbf{Q}.$$

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$$\textbf{sign}\ : \Omega_*(B\pi)\otimes\mathbf{Q}\longrightarrow L_*(\mathbf{C}\pi)\otimes\mathbf{Q}.$$

Since $\Omega_*(B\pi)\otimes\mathbf{Q}\approx H_*(B\pi;\mathbf{Q})\otimes\Omega_*$ the homomorphism **sign** can be considered as a homomorphism

$$\textbf{sign} : H_*(B\pi;\mathbf{Q})\longrightarrow L_*(\mathbf{C}\pi)\otimes\mathbf{Q}.$$

Therefore the homomorphism **sign** represents the cohomology class

$$\sigma\in H^*(B\pi;L_*(\mathbf{C}\pi)\otimes\mathbf{Q}) = \textbf{Hom}\,(H^*(B\pi;\mathbf{Q}),L_*(\mathbf{C}\pi)\otimes\mathbf{Q}).$$

The key idea is that for any manifold (M,f_M) the signature can be presented by a version of the general Hirzebruch formula

$$\textbf{sign}\,(M,f_M) = \langle L(M)f_M^*(\sigma),\,[M]\rangle\in L_*(\mathbf{C}\pi)\otimes\mathbf{Q}. \qquad (4)$$

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The key idea is that for any manifold (M,f_M) the signature can be presented by a version of the general Hirzebruch formula

$$\textbf{sign}\ (M,f_M)=\langle L(M)f_M^*(\sigma),\ [M]\rangle\in L_*(\mathbf{C}\pi)\otimes\mathbf{Q}. \qquad (6)$$

Higher signature

Let $= x \in H^*(B\pi; \mathbf{Q})$ be arbitrary cohomology class. Then the number

$$\textbf{sign}\,_x(M, f_M) = \langle L(M) f_M^*(x), [M] \rangle \in \mathbf{Q}$$

is called the higher signature due to S.P.Novikov. In the case of an additive functional $\alpha : L_*(\mathbf{C}\pi)\otimes\mathbf{Q} \longrightarrow \mathbf{Q}$ the higher signature $\textbf{sign}\,_x(M, f_M)$, where $x = \alpha(\bar{\sigma}) \in H^*(B\pi; \mathbf{Q})$ arises from the Hirzebruch formula (6) above.

This gives a description of the family of all homotopy-invariant higher signatures.

5. Functional version of the Hirzebruch formula

Infinite dimensional representations

Let $C^*[\pi]$ be a C^*- group algebra of the group π.

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Let $C^*[\pi]$ be a C^*- group algebra of the group π.

Any unitary representation of the group π can be uniquely extended to a representation $\bar{\rho}$ of the algebra $C^*[\pi]$. Put $A = \mathbf{Im}\,\bar{\rho}$, $\bar{\rho} : C^*[\pi] \longrightarrow A$.

By ξ^ρ we denote the vector bundle over $B\pi$ with the fiber A, whose transition functions are induced by the action of the group π on the algebra A by the representation ρ. The vector bundle ξ^ρ generates the element of the $K-$ group

$$\xi^\rho \in K_A(B\pi).$$

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$$\xi^\rho \in K_A(B\pi).$$

Now we can write the right side part of the formula (2):

$$? = 2^{2k}\left\langle L(X)\mathrm{ch}_A \xi_X^\rho, [X]\right\rangle \in K_A(\mathrm{pt})\otimes Q.$$

The left side part of the formula can be calculated as a symmetric signature of the manifold X by replacing of rings, induced by the representation ρ, so we receive so called generalized Hirzebruch formula for arbitrary C^*–algebra A:

$$K_A(\mathrm{pt}) \ni \mathbf{sign}\,_\rho(X) = 2^{2k}\left\langle L(X)\mathrm{ch}_A \xi_X^\rho, [X]\right\rangle \in K_A(\mathrm{pt})\otimes Q.$$

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6. Smooth version of the Hirzebruch formula

The left side part of the Hirzebruch formula (2) is described in the terms of the combinatorial structure of the manifold X. There is smooth version of this expression as well. Namely, consider the de Rham complex of differential forms on the manifold X with values in the flat vector bundle ξ^ρ:

$$0 \longrightarrow \Omega_0(X, \xi^\rho) \xrightarrow{d} \Omega_1(X, \xi^\rho) \xrightarrow{d} \cdots \xrightarrow{d} \Omega_{4k}(X, \xi^\rho) \longrightarrow 0.$$

It is well known that the cohomology groups of the de Rham complex are isomorphic to the cohomology groups $H^*(X, \xi^\rho)$.

Then $\cup-$ product is induced by external product of differential forms, so the Hermitian form which defines the Poincare duality can be determine by

$$\langle \omega_1, \omega_2 \rangle = \int_X \omega_1 \wedge \omega_2.$$

On the other hand using a Riemannian metric on the manifold X, (ω_1, ω_2), the Poincare duality can be determine with a bounded operator $*$:

$$(\omega_1, \omega_2) = \int\limits_X \omega_1 \wedge *\omega_2,$$

where

$$* : \Omega_k(X) \longrightarrow \Omega_{n-k}(X).$$

Put

$$\alpha = i^{k(k+1)} * .$$

Then

$$\alpha d \alpha = -d^*;\ \alpha^2 = 1.$$

Let

$$\Omega^+(X) = \textbf{Ker}\ (\alpha - 1);\ \Omega^-(\alpha + 1).$$

It is evident that

$$(d+d^*)\left(\Omega^+(X)\right)\subset\Omega^-(X).$$

Consider the elliptic operator

$$D=(d+d^*):\Omega^+(X)\longrightarrow\Omega^-(X).$$

Then we have

$$\textbf{index}\ D=\textbf{sign}\ (X).$$

Using the Atiyah–Singer index formula for elliptic operator we have

$$\textbf{index}\ (D\otimes\xi)=2^{2k}\left\langle L(X)\mathrm{ch}\xi,[X]\right\rangle.$$

for arbitrary vector bundle ξ over the manifold X.

If the bundle ξ is flat that is if there is a representation ρ such that $\xi = \xi^\rho$ then

$$\textbf{index } (D \otimes \xi) = \textbf{sign }_\rho(X)$$

and we again receive the Hirzebruch formula:

$$\textbf{sign }_\rho(X) = \textbf{index } (D \otimes \xi^\rho) = 2^{2k} \langle L(X)\mathrm{ch}\xi^\rho, [X] \rangle \,,$$

7. Short proof of topological invariance of rational Pontrygin classes (due to M.Gromov).

The theorem says:

Theorem 2 *Let* $f : M_1 \longrightarrow M_2$ *be a topological homeomorphism of smooth manifolds,* p_k *be the rational Pontryagin class,* $p_k(M_j) \in H^{4k}(M_j; \mathbf{Q})$ *Then*

$$p_k(M_1) = f^*(p_k(M_2)) \in H^{4k}(M_1; \mathbf{Q}). \tag{7}$$

Proof . We can substitute the Pontryagin classes for the Hirzebruch genus, $L(M) = \sum_{k=0}^{\infty} L_k(M)$, $L_k(M) \in H^{4k}(M_1; \mathbf{Q})$ since each Pontryagin class can be expressed as a polynomial of variables L_k:

$$L_k(M_1) = f^* \left(L_k(M_2)\right) \in H^{4k}(M_1; \mathbf{Q}). \tag{8}$$

Then the equations (8) one can substitute for the number relations

$$\langle L_k(M_1), y\rangle = \langle f^* \left(L_k(M_2)\right), y\rangle \in \mathbf{Q}, \quad \forall y \in H_{4k}(M_1; \mathbf{Z}) \tag{9}$$

or

$$\langle L_k(M_1), y\rangle = \langle L_k(M_2), f_*(y)\rangle \in \mathbf{Q}, \quad \forall y \in H_{4k}(M_1; \mathbf{Z}). \tag{10}$$

Assume that the homology class $y \in H_{4k}(M_1; \mathbf{Z})$ is realized by a closed orientable submanifold $Y \subset M_1$, $\dim Y = 4k$ with trivial normal bundle. Let $\varphi : Y \hookrightarrow M_1$ be the inclusion. Then

$$\begin{aligned} \langle L_k(M_1), y\rangle = \ & \langle L_k(TM_1), y\rangle = \langle L_k(\varphi^*(TM_1)), [Y]\rangle = \\ & = \langle L_k(TY \oplus \nu(Y)), [Y]\rangle = \langle L_k(TY), [Y]\rangle = \\ & = \textbf{sign}\ (Y). \end{aligned} \tag{11}$$

So we should check that

$$\textbf{sign}\ (Y) = \langle L_k(M_2), f_*\varphi_*([Y])\rangle \in \mathbf{Q} \tag{12}$$

or

$$\textbf{sign}\ (Y) = \langle L_k(M_2), [f(\varphi(Y))]\rangle \in \mathbf{Q}. \tag{13}$$

So for the proof of the theorem is sufficient to establish that manifolds Y and $Y' \sim f(\varphi(Y))$ have the same signatures. Here the proof by Novikov and Gromov are different. Novikov has proven that Y and $Y' \sim f(\varphi(Y))$ can be chosen homotopy equivalent.

Gromov suggested that the signatures of Y and Y' can be calculated as higher signatures of other manifolds, Z and Z' which are bordant with respect to fundamental groups.

Namely let $Y \subset U$ be a tubular neighborhood of Y which is diffeomorphic to a Cartesian product

$$Y \subset U = Y \times R^{n-4k}. \tag{14}$$

Consider a hypersurface $V \subset R^{n-4k}$ of codimension one with trivial normal bundle. One can take V as torus $T^{(}n-4k-1)$. It is known that all higher signatures are generated by representations of fundamental groups. Then the signature of Y coincides with a higher signature of $Z = Y \times V$.

More over, let

$$\varphi : Z \longrightarrow V$$

be a smooth map. Then $\textbf{sign}\,_v(Z)$ coincides with the signature of the inverse image of the regular point:

$$\textbf{sign}\,_v(Z) = \textbf{sign}\,(\varphi^{-1}(p_0)).$$

So, we have submanifolds $Y \subset Z \subset M_1$. Let $Z \subset U \subset U$ be a tubular neighbourhood of Z,

$$Z \subset U = Z \times R^1.$$

The homeomorphism f maps the open set $U \subset M_1$ onto an open set $W \subset M_2$, which can be splitted by a hypersurface $Z' \subset W$. What we need is to establish that

$$\textbf{sign}\ _v(Z) = \textbf{sign}\ _v(Z'). \tag{15}$$

The crucial here is that two compact (topological) submanifolds Z and $f^{-1}(Z')$ in the neighbourhood $Z \times R^1$ are topological bordant!

Hence the signatures (15) coincide.

8. Noncommutative signature of topological manifolds.

There are two difficulties in writing the correct Hirzebruch formula for topological manifolds and families of representations of the fundamental group. In first, we must define the rational Pontryagin characteristic classes. Such a definition needs the Novikov theorem about topological invariance of the rational Pontryagin classes.

In second, classic construction of the signature must be modified for topological manifolds. It seems to be impossible to define the Poincare duality as a homomorphism of finite generated vector spaces because homology groups itself are defined using either singular chains or spectrum of open coverings of manifolds. In both cases one should deal with infinite generated vector spaces although homology here turns out to be finite generated spaces.

We want to develop techniques related with infinite generated vector spaces, obtained as a direct h or inverse H limit of spectrum of finite dimensional vector spaces. These spaces can be endowed with the natural (minimal) topologies. When in some special cases it is possible to define for linear and continuous mapping $f : H \oplus h \to h \oplus H$ the integer number, its signature. General results will be applied to the spectrum of chain and cochain groups related with system of open coverings of topological manifold. So for topological manifold X and given unitary representation σ of the fundamental group $\pi(X)$ the signature $Sign_\sigma X$ can be defined by this way.

Let $\mathcal{U} = \{U_\alpha\}$ be a finite covering of the compact topological manifold X. Let $N_{\mathcal{U}}$ be the nerve of the covering $\mathcal{U}$. The nerve $N_{\mathcal{U}}$ determines a finite simplicial polyhedron. Hence chain and cochain complexes with local system of coefficients defined by a finite dimensional representation σ of the fundamental group $\pi_1(X)$ can be determined.

Consider a refining sequence of covering $\mathcal{U}_n = \{U_\alpha^n\}$, $\mathcal{U}_{n+1} \succ \mathcal{U}_n$. This means that $U_\alpha^{n+1} \subset U_\beta^n$ for a proper $\beta = \beta(\alpha)$. Hence the function $\beta = \beta(\alpha)$ defines a simplicial mapping

$$\pi_n^{n+1} : N_{\mathcal{U}_{n+1}} \to N_{\mathcal{U}_n},$$

a mapping of simplicial chains and cochains complexes

$$(\pi_n^{n+1})_* : C_*\left(N_{\mathcal{U}_{n+1}}\right) \to C_*\left(N_{\mathcal{U}_n}\right),$$
$$(\pi_n^{n+1})^* : C^*\left(N_{\mathcal{U}_n}\right) \to C_*\left(N_{\mathcal{U}_{n+1}}\right).$$

and the homomorphism of the homology and cohomology groups

$$(\pi_n^{n+1})_* : H_*\left(N_{\mathcal{U}_{n+1}}\right) \to H_*\left(N_{\mathcal{U}_n}\right),$$
$$(\pi_n^{n+1})^* : H^*\left(N_{\mathcal{U}_n}\right) \to H_*\left(N_{\mathcal{U}_{n+1}}\right).$$

Using this information the spectral chains and cochains of X can be defined as

$$C^*(X) = \varinjlim C^*\left(N_{\mathcal{U}_n}\right),$$
$$C_*(X) = \varprojlim C_*\left(N_{\mathcal{U}_n}\right).$$

Spectral homologies and cohomologies of X can be constructed as

$$H^*(X) = \varinjlim H^*\left(N_{\mathcal{U}_n}\right),$$
$$H_*(X) = \varprojlim H_*\left(N_{\mathcal{U}_n}\right).$$

The natural question is about connection between these spectral homologies and cohomologies and homology groups of $C^*(X), C_*(X)$.

The simple argument based on the Mittag-Leffler condition shows that

$$H_*(X) = H\left(C_*(X)\right)$$
$$H^*(X) = H\left(C^*(X)\right)$$

Note that one can choose the refining sequence of covering in such way that each covering had multiplicity equal to $N+1$, $N = \dim(X)$. The the Poincare duality can be defined as the intersection operator with the cycle $D_n \in C_n\left(\tilde{N}_{\mathcal{U}_n}\right)$, where $(\pi_n^{n+1})_*(D_{n+1}) = D_n$. And the Poincare homomorphism D is induced by the intersection operator with the cycle $D_\infty = \lim\limits_{\longleftarrow}(D_n)$.

We have the following picture

$$\begin{array}{ccccccc} C_0 & \xleftarrow{d_1} & C_1 & \xleftarrow{d_1} & \ldots & \xleftarrow{d_n} & C_n \\ \uparrow D_0 & & \uparrow D_1 & & & & \uparrow D_n \\ C^n & \xrightarrow{d_n} & C^{n-1} & \xrightarrow{d_{n-1}} & \ldots & \xrightarrow{d_1} & C^0 \end{array}$$

where the homomorphism of Poincare duality induces an isomorphism in homologies and the following condition hold:

$$d_{k-1}d_k = 0$$
$$d_k D_k + D_{k-1}d^*_{n-k+1} = 0$$
$$D_k = (-1)^{k(n-k)/2} D^*_{n-k}$$

Now let us construct the cone of the operator D for the special case $n = 4l$

$$0 \longleftarrow C_0 \stackrel{A_1}{\longleftarrow} C_1 \oplus C^n \stackrel{A_2}{\longleftarrow} ...$$

$$... \stackrel{A_{2l}}{\longleftarrow} C_{2l} \oplus C^{2l+1} \stackrel{A_{2l+1}}{\longleftarrow} C_{2l+1} \oplus C^{2l} \stackrel{A_{2l+2}}{\longleftarrow} ...$$

$$... \stackrel{A_n}{\longleftarrow} C_n \oplus C^1 \stackrel{A_{n+1}}{\longleftarrow} C^0 \longleftarrow 0$$

It is the acyclic and self adjoint complex with the differential

$$A_k = \left\{ \begin{array}{cc} d_k & D_{k-1} \\ 0 & d^*_{n-k+2} \end{array} \right\}$$

For the reduction to the case of an individual operator we must in the long exact complex

$$\ldots \stackrel{A_{k-1}}{\longleftarrow} H_k \oplus h_k \stackrel{A_k}{\longleftarrow} H_{k+1} \oplus h_{k+1} \longleftarrow 0$$

to split the linear space $\mathsf{Im}\, A_k = \mathsf{Ker}\, A_{k-1}$ be the following way

$$H_k \oplus h_k = \mathsf{Im}\, A_k \oplus H_k^{'} \oplus h_k^{'},$$

where the spaces $H_k^{'}$ and $h_k^{'}$ can be represented as a direct and inverse limit of the spectrum of finite dimensional vector spaces.

The left side of the long complex will be splitted by duality. After every splitting the complex become shorter. So for the interesting case $2l$ this complex will be reduced to two dual spaces and linear self adjoint bijection A

$$0 \longrightarrow H \oplus j \xrightarrow{A} h \oplus J \longrightarrow 0,$$

Theorem 3 *For linear continuous self adjoint bijection*

$$A : H \oplus j \to (H \oplus j)^{*} \equiv h \oplus J$$

there exists a correctly defined number $Sign(A)$*, which does not depend on adding of direct hyperbolic summand and coincides with the signature for finite dimensional case.*

The procedure of splitting will be described here. In first, let us note that the image of A_k is equal to th kernel of A_{k-1}. It means that this space is a closed subspace of $H_k \oplus h_k$. There are two possibilities to the closed subspace of the inverse limit of the finite dimensional vector spaces. Its dimensional can be finite or continuum.

The operator A_k has the following form:

$$A_k = \left\{ \begin{array}{cc} A_{HH} & A_{hH} \\ A_{Hh} & A_{hh} \end{array} \right\}.$$

The left-bottom corner is finite dimensional, so without loss of generality we can think that $A_{Hh} = 0$. We can do this by "cutting and pasting" of some finite dimensional space.

So we can suppose that

$$A = A_k = \left\{ \begin{array}{cc} A_{HH} & A_{hH} \\ 0 & A_{hh} \end{array} \right\}.$$

The sequence is exact, so A_k is monomorphic: $\mathrm{Ker} = 0$. Consequently:

$$\mathrm{Ker}\, A_{HH} = 0,$$
$$\mathrm{Ker}\, A_{hH}|_{\mathrm{Ker}\, A_{hh}} = 0,$$
$$\mathrm{Im}\, A_{hH}|_{\mathrm{Ker}\, A_{hh}} \cap \mathrm{Im}\, A_{HH} = 0.$$

Let

$$H_1 = \operatorname{Im} A_{HH} \subset H$$

Then

$$A : H \oplus h \longrightarrow H_0 \oplus H_1 \oplus h.$$

Operator $A_{hH}|\operatorname{Ker} A_{hh}$ has closed and finite dimensional image. The possibility of dimension of $\operatorname{Im} A_{hH}|\operatorname{Ker} A_{hh}$ to be continuum is forbidden by a smaller (only countable) dimension of h. The splitting can be easily constructed now.

9. The notion of almost flat vector bundle. Combinatorial local Hirzebruch formula.

Naive point of view is that all transition functions $\varphi_{\alpha\beta}(x)$, $x \in U_{\alpha\beta} = U_\alpha \cap U_\beta$ for vector bundle ξ are almost constant. Then one can construct a so called almost algebraic Poincare complex of formal dimension n which consists of chains and cochains with values on fibers of the bundle ξ:

$$\begin{array}{ccccccc}
C_0(\xi) & \xleftarrow{d_1} & C_1(\xi) & \xleftarrow{d_2} & \cdots & \xleftarrow{d_n} & C_n(\xi) \\
\uparrow D & & \uparrow D & & & & \uparrow D \\
C^n(\xi) & \xleftarrow{d_n^*} & C^{n-1}(\xi) & \xleftarrow{d_{n-1}^*} & \cdots & \xleftarrow{d_1^*} & C^0(\xi)
\end{array}$$

such that

$$\|d^2\| \leq \varepsilon, \quad \|Dd^* \pm dD\| \leq \varepsilon$$

$$\|D\| \leq \textbf{const}\ , \quad D^* = \pm D.$$

If ε is sufficiently small and number of neighbors for each cell is bounded then the operator

$$d + d^* + D : C_*(\xi) \longrightarrow C_*(\xi)$$

is invertible and the version of the Hirzebruch formula (Local combinatorial Hirzebruch formula) holds

$$\textbf{sign}\, C_*(X, \xi) = 2^{2k} \langle L(X)\text{ch}\xi, [X] \rangle \, .$$

If moreover the size of all cells is sufficiently large then we come to the notion of almost flat bundle for which the signature **sign** $C_*(X, \xi)$ is homotopy invariant.

Almost flat bundles from the point of view of C*-algebras

(Jointly with N.Teleman)

Connes, Gromov and Moscovici [1] showed that for any almost flat bundle α over the manifold M, the index of the signature operator with values in α is a homotopy equivalence invariant of M. From here it follows that a certain integer multiple n of the bundle α comes from the classifying space $B\pi_1(M)$. The geometric arguments allow to show that the bundle α itself, and not necessarily a certain multiple of it, comes from an arbitrarily large compact subspace $Y \subset B\pi_1(M)$ trough the classifying mapping.

For this we modify the notion of almost flat structure on bundles over smooth manifolds and extend this notion to bundles over arbitrary CW-spaces using quasi-connections by N.Teleman ([3]).

Using a natural construction by B. Hanke and T. Schick ([2]), one can present a simple description of such bundles as a bundle over a C^*–algebra and clarify the homotopy equivalence invariance of corresponding higher signatures.

More of that it is possible to construct so called classifying space for almost flat bundles.

Twisted K-theory due to M.Atiyah and G.Segal

(Jointly with A.Irmatov)

In the paper [4] M. Atiyah and G. Segal have considered families of Fredholm operators parametrized by points of a compact space K which are continuous in a topology weaker than the uniform topology, i.e. the norm topology in the space of bounded operators $B(H)$ in a Banach space H.

Therefore, it is interesting to ascertain whether the conditions, characterized families of Fredholm operators, from the paper [4] precisely describe the families of Fredholm operators which forms a Fredholm operator over the C^*–algebra $\mathcal{A} = C(K)$ of all continuous functions on K.

It is not supposed by the authors of the paper [4] that an operator over algebra $\mathcal{A}$ admits the adjoint one or in their terms continuity of the adjoint family.

Here we aim to clarify the question of description of the class of Fredholm operators which in general case do not admit the adjoint operator. For the first time, operators which play the role of Fredholm operators and may not have the adjoint ones were considered in the paper [5]. Since the main class of operators considered in the paper [5] is the class of pseudodifferential operators, for any element of which the adjoint operator automatically is a bounded one, then existence of the adjoint operator was not the actual question for the main goals of this paper.

However, in their paper [4] authors have considered operators, which may not have the adjoint one, in the form of families of operators continuous in the compact-open topology the adjoint families of which, in general case, may not be a continuous one. We can show that the class of Fredholm operators over arbitrary C^*−algebra, which may not admit the adjoint ones, can be extended in a such way that the class of compact operators used in the definition of the class of Fredholm operators contains compact operators both with and without existence the adjoint ones.

In the case when the C^*−algebra is a commutative algebra of continuous functions on a compact space appropriate topologies in the classic spaces of Fredholm and compact operators in the Hilbert space can be constructed which fully describe the sets of Fredholm and compact operators over the C^*−algebra without assumption of existence bounded adjoint operators over the algebra.

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