Mathematical Aspects of String Duality

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Dedicated to the Memory of S.-S. Chern

Twenty years ago I started my mathematical career here at Nankai.

String Duality is to identify different theories in string theory. Such identifications have produced many surprisingly beautiful mathematical formulas.

The mathematical proofs of many of such formulas depend on *Localization Techniques* on various finite dimensional moduli spaces. More precisely integrals of Chern classes on moduli spaces:

Combined with various mathematics: Chern-Simons knot invariants, Kac-Moody algebras' representations, Calabi-Yau, geometry and topology of moduli space of stable maps, moduli of stable bundles....

A simple technique we use: *Functorial Localization* transfers computations on complicated spaces to simple spaces: Connects computations of mathematicians and physicists.

Atiyah-Bott Localization Formula:

For $\omega \in H_T^*(X)$ an equivariant cohomology class, we have

$$\omega = \sum_E i_{E*}\left(\frac{i_E^*\omega}{e_T(E/X)}\right).$$

where E runs over all connected components of T fixed points set, i_E denotes the inlcusion.

Equivariant cohomology

$$H_T^*(X) = H^*(X \times_T ET)$$

where ET is the universal bundle of T.

Example: $ES^1 = S^\infty$. If S^1 acts on $\mathbf{P}^n$ by

$$\lambda \cdot [Z_0, \ldots, Z_n] = [\lambda^{w_0} Z_0, \ldots, \lambda^{w_n} Z_n],$$

then we have

$$H_{S^1}(\mathbf{P}^n; \mathbb{Q}) \cong \mathbb{Q}[H, \alpha]/\langle (H - w_0\alpha) \cdots (H - w_n\alpha)\rangle$$

Functorial Localization Formula (LLY):

$f:\ X \to Y$ equivariant map. $F \subset Y$ a fixed component, $E \subset f^{-1}(F)$ fixed components in $f^{-1}(F)$. Let $f_0 = f|_E$, then for $\omega \in H_T^*(X)$ an equivariant cohomology class, we have identity on F:

$$f_{0*}[\frac{i_E^*\omega}{e_T(E/X)}] = \frac{i_F^*(f_*\omega)}{e_T(F/Y)}.$$

This formula is used to push computations on complicated moduli space to simple moduli space. In most cases,

Complicated moduli: moduli of stable maps.

Simple moduli: projective spaces.

I will discuss the following topics:

(1) The mirror principle; (2) Hori-Vafa formula: The discussion will be very brief. (Joint with Lian, Yau).

(3) A new localization proof of the ELSV formula; (4) A proof of the Witten conjecture through localization. (Joint with Kim; with Lin Chen, Yi Li).

(5) The proof of the Mariño-Vafa formula; (6) Work in progress: an effective method to derive recursion formulas from localization. (With M. Liu, Zhou; Kim)

(7) Mathematical theory of topological vertex; (8) Applications of topological vertex. (Joint with J. Li, M. Liu, Zhou; Peng).

(9) Recent work joint with Hao Xu: explicit formula for n-point functions.

(1). Mirror Principle:

Mathematical moduli: $M_d^g(X)$ = stable map moduli of degree $(1, d)$ into $\mathbf{P}^1 \times X = \{(f, C) : f : C \to \mathbf{P}^1 \times X\}$ with C a genus g (nodal) curve, modulo obvious equivalence.

Physical moduli: W_d for toric X. (Witten, Aspinwall-Morrison):

Example: $\mathbf{P}^n$ with homogeneous coordinate $[z_0, \cdots, z_n]$. Then

$$W_d = [f_0(w_0, w_1), \cdots, f_n(w_0, w_1)]$$

$f_j(w_0, w_1)$: homogeneous polynomials of degree d. A large projective space.

There exists an explicit equivariant collapsing map(LLY+Li; Givental for $g = 0$):

$$\varphi : \ M_d^g(\mathbf{P}^n) \longrightarrow W_d.$$

$M_d^g(X)$, embedded into $M_d^g(\mathbf{P}^n)$, is very "singular" and complicated. But W_d smooth and simple. The embedding induces a map of $M_d^g(X)$ to W_d.

Functorial localization formula pushes all computations on the stable map moduli to the simple moduli W_d.

Mirror formulas are to compute certain Chern numbers on the moduli spaces of stable maps in terms of hypergeometric type functions from mirror manifolds. In fact we can say:

Mirror formulas = Comparison of computations on the two moduli spaces M_d^g and W_d!

Hypergeometric functions arise naturally on W_d through localization formula.

(2). Proof of the Hori-Vafa Formula

Let $\mathcal{M}_{0,1}(d,X)$ be the genus 0 moduli space of stable maps of degree d into X with one marked point, $ev: \mathcal{M}_{0,1}(d,X) \to X$ be evaluation map, and c the first Chern class of the tangent line at the marked point. One needs to compute the generating series

$$HG[1]^X(t) = e^{-tH/\alpha} \sum_{d=0}^{\infty} ev_*[\frac{1}{\alpha(\alpha-c)}]\, e^{dt}.$$

Example: $X = \mathbf{P}^n$, then we have $\varphi_*(1) = 1$, trivially follow from functorial localization:

$$ev_*[\frac{1}{\alpha(\alpha-c)}] = \frac{1}{\prod_{m=1}^{d}(x-m\alpha)^{n+1}}$$

in $\mathbf{P}^n$, where the denominators of both sides are equivariant Euler classes of normal bundles of fixed points. Here x denotes the hyperplane class. We easily get the hypergeometric series.

For $X = \mathrm{Gr}(k,n)$ and flag manifolds, Hori-Vafa conjectured a formula for $HG[1]^X(t)$ by which we can compute this series in terms of those of projective spaces:

Hori-Vafa Formula for Grassmannian:

$$HG[1]^{\mathrm{Gr}(k,n)}(t) = e^{(k-1)\pi\sqrt{-1}\sigma/\alpha}\frac{1}{\prod_{i<j}(x_i-x_j)}\cdot$$

$$\prod_{i<j}(\alpha\frac{\partial}{\partial x_i}-\alpha\frac{\partial}{\partial x_j})|_{t_i=t+(k-1)\pi\sqrt{-1}}HG[1]^{\mathbf{P}}(t_1,\cdots,t_k)$$

where $\mathbf{P} = \mathbf{P}^{n-1}\times\cdots\times\mathbf{P}^{n-1}$ is product of k copies of the projective spaces. Here σ is the generator of the divisor classes on $\mathrm{Gr}(k,n)$ and x_i the hyperplane class of the i-th copy $\mathbf{P}^{n-1}$:

$$HG[1]^{\mathbf{P}}(t_1,\cdots,t_k) = \prod_{i=1}^k HG[1]^{\mathbf{P^{n-1}}}(t_i).$$

We use another smooth moduli, the Grothendieck quot-scheme Q_d of quotient sheaves on $\mathbf{P}^1$

to play the role of the simple moduli, and apply the functorial localization formula, and a general set-up.

Plücker embedding $\tau: \mathsf{Gr}(k,n) \to \mathbf{P}^N$ induces embedding of the stable map moduli of $\mathsf{Gr}(k,n)$ into the corresponding stable map moduli of $\mathbf{P}^N$. Composite with the collapsing map gives us a map

$$\varphi: \quad M_d \to W_d$$

into the simple moduli space W_d of $\mathbf{P}^N$.

On the other hand the Plücker embedding also induces a map:

$$\psi: \quad Q_d \to W_d.$$

The above two maps have the same image in W_d: $\mathrm{Im}\,\psi = \mathrm{Im}\,\varphi$, and all the maps are equivariant with respect to the induced circle action from $\mathbf{P}^1$.

Functorial localization, applied to both φ and ψ, transfers the computations on the stable map moduli spaces to smooth moduli spaces, the quot-scheme Q_d.

Hori-Vafa formula = explicit computations of localization in the quot-scheme Q_d.

This can be explicitly done with combinatorial computations by analyzing fixed points in Q_d.

In the following discussions, we will study Hodge integrals (i.e. intersection numbers of λ classes and ψ classes) on the Deligne-Mumford moduli space of stable curves $\overline{\mathcal{M}}_{g,h}$.

A point in $\overline{\mathcal{M}}_{g,h}$ consists of $(C, x_1, \ldots, x_h)$, a (nodal) curve and h smooth points on C.

The Hodge bundle $\mathbb{E}$ is a rank g vector bundle over $\overline{\mathcal{M}}_{g,h}$ whose fiber over $[(C, x_1, \ldots, x_h)]$ is $H^0(C, \omega_C)$. The λ classes are Chern Classes:

$$\lambda_i = c_i(\mathbb{E}) \in H^{2i}(\overline{\mathcal{M}}_{g,h}; \mathbb{Q}).$$

The cotangent line $T^*_{x_i}C$ of C at the i-th marked point x_i gives a line bundle $\mathbb{L}_i$ over $\overline{\mathcal{M}}_{g,h}$. The ψ classes are also Chern classes:

$$\psi_i = c_1(\mathbb{L}_i) \in H^2(\overline{\mathcal{M}}_{g,h}; \mathbb{Q}).$$

(3). A New Proof of ELSV Formula

Given a partition $\mu = (\mu_1 \geq \cdots \geq \mu_{l(\mu)} \geq 0)$ of length $l(\mu)$, write $|\mu| = \sum_j \mu_j$.

$\overline{\mathcal{M}}_g(\mathbf{P}^1, \mu)$: moduli space of relative stable maps from a genus g (nodal) curve to $\mathbf{P}^1$ with fixed ramification type μ at ∞. (Jun Li)

$H_{g,\mu}$: the Hurwitz number of almost simple covers of $\mathbf{P}^1$ of ramification type μ at ∞ by connected genus g Riemann surfaces.

Theorem: *The ELSV formula:*

$$H_{g,\mu} = (2g - 2 + |\mu| + l(\mu))! I_{g,\mu}$$

where

$$I_{g,\mu} = \frac{1}{|\operatorname{Aut}(\mu)|} \prod_{i=1}^{l(\mu)} \frac{\mu_i^{\mu_i}}{\mu_i!} \int_{\overline{\mathcal{M}}_{g,l(\mu)}} \frac{\Lambda_g^\vee(1)}{\prod_{i=1}^{l(\mu)} (1 - \mu_i \psi_i)}.$$

Functorial localization formula, applied to the branching morphism:

$$\mathrm{Br}: \quad \overline{\mathcal{M}}_g(\mathbf{P}^1, \mu) \to \mathbf{P}^r,$$

where r denotes the dimension of $\overline{\mathcal{M}}_g(\mathbf{P}^1, \mu)$. Label the isolated fixed points $\{p_0, \cdots, p_r\}$ of $\mathbf{P}^r$ and denote by

$I^0_{g,\mu}$: the fixed points contribution in $\mathrm{Br}^{-1}(p_0)$.

$I^1_{g,\mu}$: the fixed points contribution in $\mathrm{Br}^{-1}(p_1)$.

Since $\mathrm{Br}_*(1)$ is a constant, we get

$$I^0_{g,\mu} = I^1_{g,\mu}$$

where $I^0_{g,\mu} = I_{g,\mu}$ and

$$I^1_{g,\mu} = \sum_{\nu \in J(\mu)} I_1(\nu) I_{g,\nu} + \sum_{\nu \in C(\mu)} I_2(\nu) I_{g-1,\nu} +$$

$$\sum_{g_1+g_2=g} \sum_{\nu^1 \cup \nu^2 \in C(\mu)} I_3(\nu^1, \nu^2) I_{g_1,\nu_1} I_{g_2,\nu_2},$$

where $I_1,\ I_2,\ I_3$ are some explicit combinatorial coefficients. This recursion formula is equivalent to that the generating function

$$R(\lambda,p)=\sum_{g\geq 0}\sum_{|\mu|\geq 1} I_{g,\mu}\lambda^{2g-2+|\mu|+l(\mu)}p_\mu,$$

where $p=(p_1,p_2,\ldots,p_n,\ldots)$ denotes formal variables and $p_\mu=p_{\mu_1}\cdots p_{\mu_{l(\mu)}}$, satisfies the *cut-and-join* equation:

$$\begin{aligned}\frac{\partial R}{\partial\lambda} &= \frac{1}{2}\sum_{i,j=1}^{\infty}\Bigg((i+j)p_ip_j\frac{\partial R}{\partial p_{i+j}}\\ &\qquad +ijp_{i+j}\left(\frac{\partial R}{\partial p_i}\frac{\partial R}{\partial p_j}+\frac{\partial^2 R}{\partial p_i\partial p_j}\right)\Bigg)\end{aligned}$$

The cut-and-join equation of same kind for the generating series P of $H_{g,\mu}$ is simple and explicit combinatorial computation. This equation is equivalent to systems of linear ODE.

R and P have the same initial value p_1 at $\lambda=0$, therefore are equal, which gives the ELSV formula.

(4). Localization Proof of the Witten Conjecture

The Witten conjecture for moduli spaces states that the generating series F of the integrals of the ψ classes for all genera and any number of marked points satisfies the KdV equations and the Virasoro constraint. For example the Virasoro constraint states that F satisfies

$$L_n \cdot F = 0, \ n \geq -1$$

where L_n denote certain Virasoro operators.

Witten conjecture was first proved by Kontsevich using combinatorial model of the moduli space and matrix model, with later approaches by Okounkov-Pandhripande using ELSV formula and combinatorics, by Mirzakhani using Weil-Petersson volumes on moduli spaces of bordered Riemann surfaces.

I will present a much simpler proof by using functorial localization and asymptotics, jointly with Y.-S. Kim. The method has more applications in deriving more general recursion formulas in the subject.

The basic idea is to directly prove the following recursion formula which, as derived in physics by Dijkgraaf, Verlinde and Verlinde using quantum field theory, implies the Virasoro and the KdV equation for the generating series F of the integrals of the ψ classes:

Theorem: (DVV Conjecture) *We have the identity:*

$$\langle\tilde\sigma_n \prod_{k\in S}\tilde\sigma_k\rangle_g = \sum_{k\in S}(2k+1)\langle\tilde\sigma_{n+k-1}\prod_{l\neq k}\tilde\sigma_l\rangle_g +$$

$$\frac{1}{2}\sum_{a+b=n-2}\langle\tilde\sigma_a\tilde\sigma_b\prod_{l\neq a,b}\tilde\sigma_l\rangle_{g-1} +$$

$$\frac{1}{2}\sum_{\substack{S=X\cup Y,\\ a+b=n-2,\\ g_1+g_2=g}}\langle\tilde\sigma_a\prod_{k\in X}\tilde\sigma_k\rangle_{g_1}\langle\tilde\sigma_b\prod_{l\in Y}\tilde\sigma_l\rangle_{g_2}.$$

Here $\tilde\sigma_n = (2n+1)!!\psi^n$ and

$$\langle\prod_{j=1}^n\tilde\sigma_{k_j}\rangle_g = \int_{\overline{\mathcal{M}}_{g,n}}\prod_{j=1}^n\tilde\sigma_{k_j}.$$

The notation $S = \{k_1,\cdots,k_n\} = X\cup Y$.

To prove the above recursion relation, recall that the functorial localization applied to the map

$$\mathrm{Br}:\ \overline{\mathcal{M}}_g(\mathbf{P}^1,\mu)\to\mathbf{P}^r,$$

and by comparing the contributions in $\mathrm{Br}^{-1}(p_0)$ and $\mathrm{Br}^{-1}(p_1)$, we easily get the cut-and-join equation for one Hodge integral:

$$I_{g,\mu} = \sum_{\nu \in J(\mu)} I_1(\nu) I_{g,\nu} + \sum_{\nu \in C(\mu)} I_2(\nu) I_{g-1,\nu} +$$

$$\sum_{g_1+g_2=g} \sum_{\nu^1 \cup \nu^2 \in C(\mu)} I_3(\nu^1,\nu^2) I_{g_1,\nu^1} I_{g_2,\nu^2}$$

where $I(\nu)$, $I(\nu^1,\nu^2)$ are some explicit combinatorial coefficients. Here recall

$$I_{g,\mu} = \frac{1}{|\mathrm{Aut}\ \mu|} \prod_{i=1}^{n} \frac{\mu_i^{\mu_i}}{\mu_i!} \int_{\overline{\mathcal{M}}_{g,n}} \frac{\Lambda_g^\vee(1)}{\prod(1-\mu_i\psi_i)}.$$

We then do asymptotic expansion. Write $\mu_i = N x_i$ and let N go to infinity and expand in N and x_i, and take the coefficient of $N^{m+\frac{1}{2}}$ with $m = 3g-3+\frac{n}{2}$.

We obtained the identity:

$$\sum_{i=1}^{n}\Big[\frac{(2k_i+1)!!}{2^{k_i+1}k_i!}x_i^{k_i}\prod_{j\neq i}\frac{x_j^{k_j-\frac{1}{2}}}{\sqrt{2\pi}}\int_{\overline{\mathcal{M}}_{g,n}}\prod\psi_j^{k_j}-$$

$$\sum_{j\neq i}\frac{(x_i+x_j)^{k_i+k_j-\frac{1}{2}}}{\sqrt{2\pi}}\prod_{l\neq i,j}\frac{x_l^{k_l-\frac{1}{2}}}{\sqrt{2\pi}}\int_{\overline{\mathcal{M}}_{g,n-1}}\psi^{k_i+k_j-1}\prod\psi_l^{k_l}$$

$$-\frac{1}{2}\sum_{k+l=k_i-2}\frac{(2k+1)!!(2l+1)!!}{2^{k_i}\,k_i!}x_i^{k_i}\prod_{j\neq i}\frac{x_j^{k_j-\frac{1}{2}}}{\sqrt{2\pi}}$$

$$\Big[\int_{\overline{\mathcal{M}}_{g-1,n+1}}\psi_1^k\psi_2^l\prod\psi_j^{k_j}+$$

$$\sum_{\substack{g_1+g_2=g,\\ \nu_1\cup\nu_2=\nu}}\int_{\overline{\mathcal{M}}_{g_1,n_1}}\psi_1^k\prod\psi_j^{k_j}\int_{\overline{\mathcal{M}}_{g_2,n_2}}\psi_1^l\prod\psi_j^{k_j}\Big]\Big]=0$$

Performing Laplace transforms on the x_i's, we get the recursion formula which implies both the KdV equations and the Virasoro constraints.

4'. Another Simple Proof of the Witten Conjecture.

Direct expansions by using symmetric polynomials and the cut-and-join equation.

Chen-Li-Liu: Localization, Hurwitz Numbers and the Witten Conjecture, math.AG/0609263.

$$\Phi(\lambda, p) = \sum_{\mu} \sum_{g \geq 0} I_{g,\mu} \frac{\lambda^{2g-2+|\mu|+l(\mu)}}{(2g-2+|\mu|+l(\mu))!} p_\mu.$$

We have the following version of cut-and-join equation

$$\frac{\partial \Phi}{\partial \lambda} = \frac{1}{2} \sum_{i.j \geq 1} [ijp_{i+j} \frac{\partial^2 \Phi}{\partial p_i \partial p_j} + ijp_{i+j} \frac{\partial \Phi}{\partial p_i} \frac{\partial \Phi}{\partial p_j} +$$

$$(i+j) p_i p_j \frac{\partial \Phi}{\partial p_{i+j}}].$$

Introduce

$$\Phi_{g,n}(z, p) = \sum_{d \geq 1} \sum_{\mu \vdash d, l(\alpha)=n} \frac{I_{g,\mu}}{(2g-2+|\mu|+l(\mu))!} p_\mu z^d,$$

where p_μ formal variables. By simple calculation, we can rewrite it in the following form

$$\Phi_{g,n}(z;p) = \frac{1}{n!} \sum_{b_1,\cdots,b_n \geq 0, 0 \leq k \leq g} (-1)^k \langle \tau_{b_1} \cdots \tau_{b_n} \lambda_k \rangle_g$$

$$\cdot \prod_{i=1}^{n} \phi_{b_i}(z;p),$$

where

$$\phi_i(z;p) = \sum_{m \geq 0} \frac{m^{m+i}}{m!} p_m z^m, \quad i \geq 0.$$

Then apply the symmetrization operators and direct expansions in symmetric polynomials to get the recursions in the ψ-classes.

Cut-and-Join operator also has root in representation theory of infinite dimensional Lie algebras.

(5). Proof of the Mariño-Vafa Formula

Introduce the total Chern classes of the Hodge bundle on the moduli space of curves:

$$\Lambda_g^\vee(u) = u^g - \lambda_1 u^{g-1} + \cdots + (-1)^g \lambda_g.$$

Mariño-Vafa formula: the generating series over g of triple Hodge integrals

$$\int_{\overline{\mathcal{M}}_{g,h}} \frac{\Lambda_g^\vee(1)\Lambda_g^\vee(\tau)\Lambda_g^\vee(-\tau-1)}{\prod_{i=1}^h (1-\mu_i\psi_i)},$$

can be expressed by close formulas of **finite** expression in terms of representations of symmetric groups, or Chern-Simons knot invariants. Here τ is a parameter.

Mariño-Vafa conjectured the formula from large N duality between Chern-Simons and string theory following Witten, Gopakumar-Vafa, Ooguri-Vafa.

The Mariño-Vafa Formula:

Geometric side: For every partition $\mu = (\mu_1 \geq \cdots \geq \mu_{l(\mu)} \geq 0)$, define triple Hodge integral:

$$G_{g,\mu}(\tau) = A(\tau) \cdot \int_{\overline{\mathcal{M}}_{g,l(\mu)}} \frac{\Lambda_g^\vee(1)\Lambda_g^\vee(-\tau-1)\Lambda_g^\vee(\tau)}{\prod_{i=1}^{l(\mu)}(1-\mu_i\psi_i)},$$

with

$$A(\tau) = -\frac{\sqrt{-1}^{|\mu|+l(\mu)}}{|\operatorname{Aut}(\mu)|}[\tau(\tau+1)]^{l(\mu)-1} \prod_{i=1}^{l(\mu)} \frac{\prod_{a=1}^{\mu_i-1}(\mu_i\tau+a)}{(\mu_i-1)!}.$$

Introduce generating series

$$G_\mu(\lambda;\tau) = \sum_{g\geq 0} \lambda^{2g-2+l(\mu)} G_{g,\mu}(\tau).$$

Special case when $g = 0$:

$$\int_{\overline{\mathcal{M}}_{0,l(\mu)}} \frac{\Lambda_0^\vee(1)\Lambda_0^\vee(-\tau-1)\Lambda_0^\vee(\tau)}{\prod_{i=1}^{l(\mu)}(1-\mu_i\psi_i)}$$

$$= \int_{\overline{\mathcal{M}}_{0,l(\mu)}} \frac{1}{\prod_{i=1}^{l(\mu)}(1-\mu_i\psi_i)} = |\mu|^{l(\mu)-3}$$

for $l(\mu) \geq 3$, and we use this expression to extend the definition to the case $l(\mu) < 3$.

Introduce formal variables $p = (p_1, p_2, \dots, p_n, \dots)$, and define

$$p_\mu = p_{\mu_1} \cdots p_{\mu_{l(\mu)}}$$

for any partition μ.

Generating series for all genera and all possible marked points:

$$G(\lambda; \tau; p) = \sum_{|\mu| \geq 1} G_\mu(\lambda; \tau) p_\mu.$$

Representation side: χ_μ: the character of the irreducible representation of symmetric group $S_{|\mu|}$ indexed by μ with $|\mu| = \sum_j \mu_j$,

$C(\mu)$: the conjugacy class of $S_{|\mu|}$ indexed by μ.

Introduce:

$$\mathcal{W}_\mu(q) = q^{\kappa_\mu/4} \prod_{1\le i<j\le \ell(\mu)} \frac{[\mu_i - \mu_j + j - i]}{[j-i]}$$

$$\prod_{i=1}^{\ell(\mu)} \frac{1}{\prod_{v=1}^{\mu_i}[v - i + \ell(\mu)]}$$

where

$$\kappa_\mu = |\mu| + \sum_i (\mu_i^2 - 2i\mu_i), \quad [m] = q^{m/2} - q^{-m/2}$$

and$q = e^{\sqrt{-1}\lambda}$. The expression $\mathcal{W}_\mu(q)$ has an interpretation in terms of *quantum dimension* in Chern-Simons knot theory.

Define:

$$R(\lambda;\tau;p) = \sum_{n\ge 1} \frac{(-1)^{n-1}}{n} \sum_\mu [\sum_{\cup_{i=1}^n \mu^i = \mu}$$

$$\prod_{i=1}^n \sum_{|\nu^i|=|\mu^i|} \frac{\chi_{\nu^i}(C(\mu^i))}{z_{\mu^i}} e^{\sqrt{-1}(\tau+\frac{1}{2})\kappa_{\nu^i}\lambda/2} \mathcal{W}_{\nu^i}(\lambda)] p_\mu$$

where μ^i are sub-partitions of μ, $z_\mu = \prod_j \mu_j! j^{\mu_j}$ for a partition μ: standard for representations of symmetric groups.

Theorem: *Mariño-Vafa Conjecture is true:*

$$G(\lambda; \tau; p) = R(\lambda; \tau; p).$$

Remark: (1). Equivalent expression:

$$G(\lambda; \tau; p)^\bullet = \exp\left[G(\lambda; \tau; p)\right] = \sum_\mu G(\lambda; \tau)^\bullet p_\mu =$$

$$\sum_{|\mu|\geq 0} \sum_{|\nu|=|\mu|} \frac{\chi_\nu(C(\mu))}{z_\mu} e^{\sqrt{-1}(\tau+\frac{1}{2})\kappa_\nu \lambda/2} \mathcal{W}_\nu(\lambda) p_\mu$$

(2). Each $G_\mu(\lambda,\tau)$ is given by a **finite and closed** expression in terms of representations of symmetric groups:

$$G_\mu(\lambda,\tau) = \sum_{n\geq 1}\frac{(-1)^{n-1}}{n}\sum_{\cup_{i=1}^n \mu^i=\mu}\prod_{i=1}^n$$

$$\sum_{|\nu^i|=|\mu^i|}\frac{\chi_{\nu^i}(C(\mu^i))}{z_{\mu^i}}e^{\sqrt{-1}(\tau+\frac{1}{2})\kappa_{\nu^i}\lambda/2}\mathcal{W}_{\nu^i}(\lambda).$$

$G_\mu(\lambda,\tau)$ gives triple Hodge integrals for moduli spaces of curves of all genera with $l(\mu)$ marked points.

(3). Mariño-Vafa formula gives explicit values of many interesting Hodge integrals up to three Hodge classes:

• Taking limit $\tau \longrightarrow 0$ we get the λ_g conjecture (Faber-Pandhripande),

$$\int_{\overline{\mathcal{M}}_{g,n}}\psi_1^{k_1}\cdots\psi_n^{k_n}\lambda_g = \binom{2g+n-3}{k_1,\ldots,k_n}\frac{2^{2g-1}-1}{2^{2g-1}}\frac{|B_{2g}|}{(2g)!},$$

for $k_1+\cdots+k_n=2g-3+n$, and the following identity for Hodge integrals:

$$\int_{\overline{\mathcal{M}}_g} \lambda_{g-1}^3 = \int_{\overline{\mathcal{M}}_g} \lambda_{g-2}\lambda_{g-1}\lambda_g$$

$$= \frac{1}{2(2g-2)!}\frac{|B_{2g-2}|}{2g-2}\frac{|B_{2g}|}{2g},$$

B_{2g} are Bernoulli numbers. And other identities.

• Taking limit $\tau \longrightarrow \infty$, we get the ELSV formula.

• Yi Li recently has derived more Hodge integral identities from the MV formula.

The idea to prove the Mariño-Vafa formula is to prove that both G and R satisfy the Cut-and-Join equation:

Theorem : *Both R and G satisfy the following differential equation:*

$$\frac{\partial F}{\partial \tau} = \frac{1}{2}\sqrt{-1}\lambda \sum_{i,j=1}^{\infty}\Bigg((i+j)p_i p_j \frac{\partial F}{\partial p_{i+j}} + ijp_{i+j}\left(\frac{\partial F}{\partial p_i}\frac{\partial F}{\partial p_j} + \frac{\partial^2 F}{\partial p_i \partial p_j}\right)\Bigg)$$

This is equivalent to linear systems of ODE. They have the same initial value at $\tau = 0$:

The solution is unique!

$$G(\lambda;\tau;p) = R(\lambda;\tau;p).$$

Cut-and-Join operator, denoted by (CJ), in variables p_j on the right hand side gives a nice match of Combinatorics and Geometry from collecting the following operations:

Combinatorics: Cut and join of cycles:

Cut: a k-cycle is cut into an i-cycle and a j-cycle, denote the set by $C(\mu)$:

Join: an i-cycle and a j-cycle are joined to an $(i+j)$-cycle, denote the set by $J(\mu)$:

Geometry: How curves stably vary,

Cut: One curve split into two lower degree or lower genus curves.

Join: Two curves joined together to give a higher genus or higher degree curve.

The proof of cut-and-join equation for R is a direct computation in combinatorics.

The first proof of the cut-and-join equation for G used functorial localization formula.

Label the isolated fixed points $\{p_0, \cdots, p_r\}$ of $\mathbf{P}^r$:

$J^0_{g,\mu}$: the fixed points contribution in $\mathrm{Br}^{-1}(p_0)$

$J^1_{g,\mu}$: the fixed points contribution in $\mathrm{Br}^{-1}(p_1)$.
Then

$$J^0_{g,\mu}(\tau) = \sqrt{-1}^{|\mu|-l(\mu)} G_{g,\mu}(\tau),$$

$$J^1_{g,\mu}(\tau) = \sqrt{-1}^{|\mu|-l(\mu)-1}.$$

$$\left(\sum_{\nu \in J(\mu)} I_1(\nu) G_{g,\nu}(\tau) + \sum_{\nu \in C(\mu)} I_2(\nu) G_{g-1,\nu}(\tau) \right.$$

$$\left. + \sum_{g_1+g_2=g, \nu^1 \cup \nu^2 \in C(\mu)} I_3(\nu^1, \nu^2) G_{g_1,\nu^1}(\tau) G_{g_2,\nu^2}(\tau) \right).$$

We proved the following identity by analyzing equivariant cohomology classes on $\mathbf{P}^r$:

$$\frac{d}{d\tau} J^0_{g,\mu}(\tau) = -J^1_{g,\mu}(\tau).$$

which is the cut-and-join equation we need.

In fact we have more higher order cut-and-join equations:

$$\frac{(-1)^k}{k!}\frac{d^k}{d\tau^k}J^0_{g,\mu}(\tau) = J^k_{g,\mu}(\tau)$$

from comparing contributions from the first and the k-th fixed point on $\mathbf{P}^r$.

Remark: Cut-and-join equation is encoded in the geometry of the moduli spaces of stable maps: convolution formula of the form:

$$G^\bullet_\mu(\lambda,\tau) = \sum_{|\nu|=|\mu|} \Phi^\bullet_{\mu,\nu}(-\sqrt{-1}\tau\lambda) z_\nu K^\bullet_\nu(\lambda)$$

where $\Phi^\bullet_{\mu,\nu}$ is series of double Hurwitz numbers. This gives the explicit solution of the cut-and-join equation, with initial value $K^\bullet(\lambda)$, the integrals of Euler classes on moduli of relative stable maps.

Another approach by Okounkov-Pandhripande using ELSV formula and the λ_g conjecture.

(6). Work in Progress:

We note that *asymptotic cut-and-join equation* from the functorial localization formula and moduli space of relative stable maps always holds. This gives an effective way to derive recursive type formulas in Hodge integrals and in GW invariants.

Applications may include:

(1) The recursion formula related to the generalized Witten conjecture for W-algebra constraints.

(2) Recursion formula for GW invariants of general projective manifolds.

(3) Recursion formula for other Hodge integrals.

(7). Mathematical Theory of Topological Vertex

Mirror symmetry used periods and holomorphic anomaly to compute Gromov-Witten series, difficult for higher genera. Topological vertex theory, as developed by Aganagic-Klemm-Marino-Vafa from string duality and geometric engineering, gives complete answers for all genera and all degrees in the toric Calabi-Yau cases in terms of Chern-Simons knot invariants!

We developed the mathematical theory of topological vertex by using localization technique, to first prove a three partition analogue of the Mariño-Vafa formula. This formula gives closed formula for the generating series of the Hodge integrals involving three partitions in terms of Chern-Simons knot invariants of Hopf links.

The corresponding cut-and-join equation has the form:

$$\frac{\partial}{\partial \tau} F^\bullet = (CJ)^1 F^\bullet + \frac{1}{\tau^2}(CJ)^2 F^\bullet + \frac{1}{(\tau+1)^2}(CJ)^3 F^\bullet$$

where (CJ) denotes the cut-and-join operator with respect to the three groups of infinite numbers of variables associated to the three partitions.

We derived the convolution formulas both in combinatorics and in geometry. Then we proved the identity of initial values at $\tau = 1$.

We introduced the new notion of *formal toric Calabi-Yau manifolds* to work out the gluing of Calabi-Yau and the topological vertices. We then derived all of the basic properties of topological vertex, like the fundamental gluing formula.

By using gluing formula of the topological vertex, we can derive closed formulas for generating series of GW invariants, all genera and all degrees, open or closed, for all toric Calabi-Yau, in terms Chern-Simons invariants, by simply looking at

The moment map graph of the toric Calabi-Yau.

Each **vertex** of the moment map graph contributes a closed expression to the generating series of the GW invariants in terms of explicit combinatorial Chern-Simons knot invariants.

Let us look at an example to see the computational power of topological vertex.

Let $N_{g,d}$ denote the GW invariants of a toric Calabi-Yau, total space of canonical bundle on a toric surface S.

It is the Euler number of the obstruction bundle on the moduli space $\overline{\mathcal{M}}_g(S,d)$ of stable maps of degree $d \in H_2(S,\mathbb{Z})$ from genus g curve into the surface S:

$$N_{g,d} = \int_{[\overline{\mathcal{M}}_g(S,d)]^v} e(V_{g,d})$$

with $V_{g,d}$ a vector bundle induced by the canonical bundle K_S.

At point $(\Sigma; f) \in \overline{\mathcal{M}}_g(S,d)$, its fiber is $H^1(\Sigma, f^*K_S)$. Write

$$F_g(t) = \sum_d N_{g,d}\, e^{-d\cdot t}.$$

Example: Topological vertex formula of GW generating series in terms of Chern-Simons invariants. For the total space of canonical bundle $\mathcal{O}(-3)$ on $\mathbf{P}^2$:

$$\exp\big(\sum_{g=0}^{\infty}\lambda^{2g-2}F_g(t)\big)=\sum_{\nu_1,\nu_2,\nu_3}\mathcal{W}_{\nu_1,\nu_2}\mathcal{W}_{\nu_2,\nu_3}\mathcal{W}_{\nu_3,\nu_1}\cdot$$

$$(-1)^{|\nu_1|+|\nu_2|+|\nu_3|}q^{\frac{1}{2}\sum_{i=1}^{3}\kappa_{\nu_i}}\,e^{t(|\nu_1|+|\nu_2|+|\nu_3|)}.$$

Here $q=e^{\sqrt{-1}\lambda}$, and $\mathcal{W}_{\mu,\nu}$ are from the Chern-Simons knot invariants of Hopf link. Sum over three partitions $\nu_1,\ \nu_2,\ \nu_3$.

Three vertices of moment map graph of $\mathbf{P}^2$ $\leftrightarrow$ three $\mathcal{W}_{\mu,\nu}$'s, explicit in Schur functions.

For general (formal) toric Calabi-Yau, the expressions are just similar: **closed formulas**.

8. Applications of Topological Vertex.

Recall the interesting:

Gopakumar-Vafa conjecture: *There exists expression:*

$$\sum_{g=0}^{\infty} \lambda^{2g-2} F_g(t) = \sum_{k=1}^{\infty} \sum_{g,d} n_d^g \frac{1}{d} (2\, sin \frac{d\lambda}{2})^{2g-2} e^{-kd\cdot t},$$

such that n_d^g are integers, called instanton numbers.

By using the explicit knot invariant expressions from topological vertex in terms of the Schur functions, we have the following applications:

(1). First motivated by the Nekrasov's work, by comparing with Atiyah-Bott localization formulas on instanton moduli we have proved:

Theorem: *For conifold and the toric Calabi-Yau from the canonical line bundle of the*

Hirzebruch surfaces, we can identify the n_d^g *as equivariant indices of twisted Dirac operators on moduli spaces of anti-self-dual connections on* $\mathbf{C}^2$*.*

A complicated change of variables like mirror transformation is performed.

(2). The following theorem was first proved by Pan Peng:

Theorem: *The Gopakumar-Vafa conjecture is true for all (formal) local toric Calabi-Yau for all degree and all genera.*

(3). The proof of the flop invariance of the GW invariants of (toric) Calabi-Yau by Konishi and Minabe, with previous works of Li-Ruan and Chien-Hao Liu-Yau.

More applications expected from the computational power of the topological vertex.

We have seen close connection between knot invariants and Gromov-Witten invariants. There should be a more interesting and grand duality picture between Chern-Simons invariants for real three dimensional manifolds and Gromov-Witten invariants for complex three dimensional toric Calabi-Yau.

General correspondence between the geometry of real dimension 3 and complex dimension 3?!.

In any case *String Duality* has already inspired exciting duality and unification among various mathematical subjects.

9. Recent Results.

We call the following generating function

$$F(x_1,\ldots,x_n)=\sum_{g=0}^{\infty}\sum_{\sum d_j=3g-3+n}\langle\tau_{d_1}\cdots\tau_{d_n}\rangle_g\prod_{j=1}^{n}x_j^{d_j}$$

the n-point function.

Consider the following "normalized" n-point function

$$G(x_1,\ldots,x_n)=\exp\left(\frac{-\sum_{j=1}^{n}x_j^3}{24}\right)\cdot F(x_1,\ldots,x_n).$$

Theorem: For $n\geq 2$,

$$G(x_1,\ldots,x_n)=\sum_{r,s\geq 0}\frac{(2r+n-3)!!}{2^{2s}(2r+2s+n-1)!!}$$

$$P_r(x_1,\ldots,x_n)\cdot\Delta(x_1,\ldots,x_n)^s,$$

where P_r and Δ are homogeneous symmetric polynomials defined by

$$\Delta(x_1,\ldots,x_n)=\frac{(\sum_{j=1}^{n}x_j)^3-\sum_{j=1}^{n}x_j^3}{3},$$

$$P_r(x_1, \ldots, x_n) = \frac{1}{2 \cdot \sum_{j=1}^n x_j} \sum_{\underline{n} = I \coprod J}$$

$$(\sum_{i \in I} x_i)^2 \cdot (\sum_{i \in J} x_i)^2 \cdot G(x_I) \cdot G(x_J))_{3r+n-3}$$

where $G_g(x_I)$ denotes the degree $3g + |I| - 3$ homogeneous component of the normalized $|I|$-point function $G(x_{k_1}, \ldots, x_{k_{|I|}})$, where $k_j \in I$. If $I = \emptyset$, then $\sum_{i \in I} x_i = 0$.

This has many corollaries about the intersection numbers of the moduli spaces of Riemann surfaces.

This is a field full of interesting problems!

Thank You Very Much!