Geometric structures on

Fano Manifolds

of Picard number 1

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X Fano Miyaoka-Mori, i.e. $K_X^{-1} > 0$

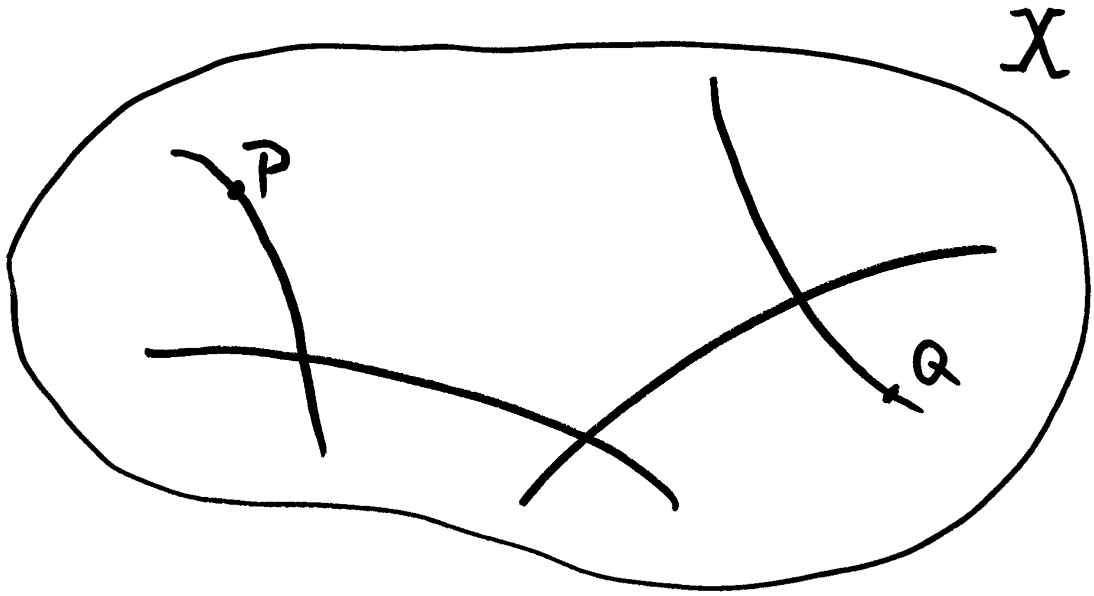
By Miyaoka-Mori,

X is uniruled, i.e.

"filled up by rational curves"

By Kollar-Miyaoka-Mori

X is rationally connected



Differential-geometric criterion:

X Fano $\Leftrightarrow \exists\, g$ Kähler, Ric $(X, g) > 0$

Grothendieck Splitting Theorem (1956)

$V \mapsto \mathbb{P}^1$ holomorphic vector bundle. Then

$$V \cong \mathcal{O}(a_1) \oplus \cdots \oplus \mathcal{O}(a_r) \ ,$$

where $a_1 \leq \cdots \leq a_r$ are unique.

Formulation in terms of matrices

Let $f : \mathbb{C} - \{0\} \mapsto GL(n, \mathbb{C})$ be holomorphic. Then there exist

$$g_1 : \mathbb{C} \to GL(n, \mathbb{C}) \ , \quad g_2 : \mathbb{P}^1 - \{0\} \to GL(n, \mathbb{C})$$

such that

$$g_1 f g_2^{-1}(z) = \begin{bmatrix} z^{a_1} & & \\ & \ddots & \\ & & z^{a_r} \end{bmatrix}$$

Hilbert (1905), Plemelj (1908), Birkhoff (1913), Hasse (1895)

<u>Deformation of Rational Curves</u>

X complex mfld, $f : \mathbb{P}^1 \to X$, $f(\mathbb{P}^1) = C$

$\{C_t\}$ hol. family of $\mathbb{P}^1$, defined by

$f_t : \mathbb{P}^1 \to X$, $f_0 = f$, $C_0 = C$.

Write $F(z,t) = f_t(z)$

$$\frac{\partial F}{\partial t}|_{t=0} = s \in \Gamma(\mathbb{P}^1, f^* T_X) \ .$$

Any section $s \in \Gamma(\mathbb{P}^1, f^* T_X)$ is a candidate for <u>infinitesimal deformation</u>.

Use power series to construct

$F(z,t) = f_t(z)$

<u>Obstruction</u> to construction given by $H^1(\mathbb{P}^1, f^* T_X)$

$$H^1(\mathbb{P}^1, f^* T_X) = \sum_{i=1}^{r} H^1(\mathbb{P}^1, \mathcal{O}(a_i))$$

$$H^1(\mathbb{P}^1, \mathcal{O}(a)) = 0 \quad \forall a \geq -1 \ .$$

Example of hol. vector bundles on $\mathbb{P}^1$

(A) $\mathbb{P}^1 \subset \mathbb{P}^2$; $V = T_{\mathbb{P}^2}|_{\mathbb{P}^1}$

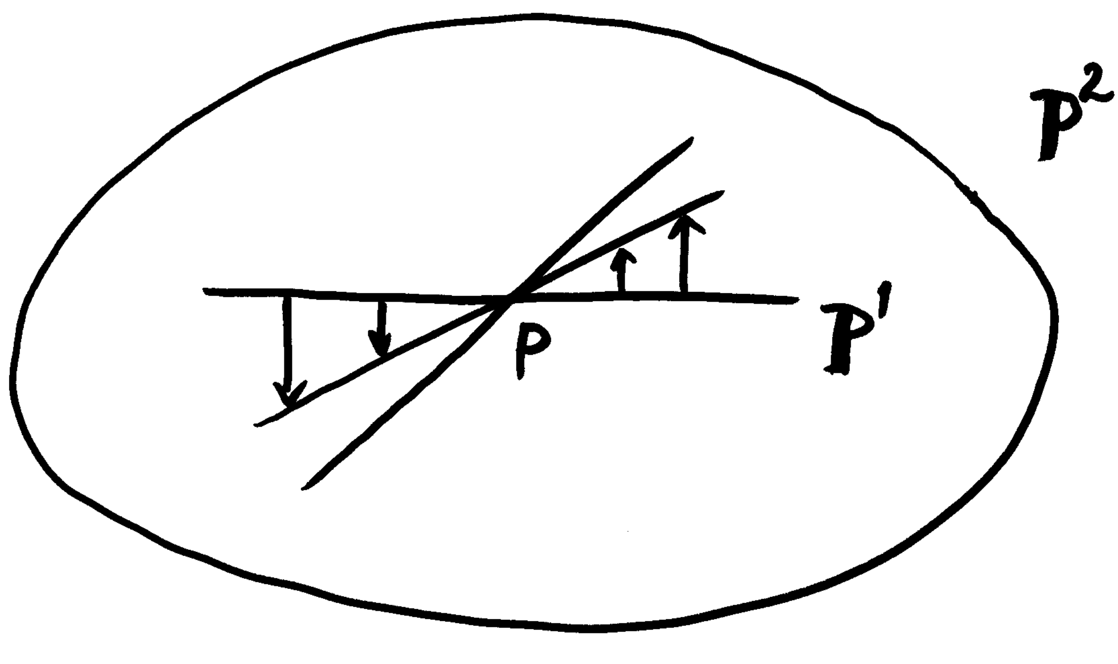
$$V/T_{\mathbb{P}^1} = N_{\mathbb{P}^1|\mathbb{P}^2},\ N = \text{normal bundle.}$$

$\exists$ hol. vector fields of $\mathbb{P}^2$, along $\mathbb{P}^1$, corresponding to inf. deformation of lines in $\mathbb{P}^2$. Using s, we have, $s(P) = 0$

$$\begin{aligned} V &\cong T_{\mathbb{P}^1} \oplus N_{\mathbb{P}^1|\mathbb{P}^2} \\ &\cong \mathcal{O}(2) \oplus \mathcal{O}(1)\ . \end{aligned}$$

In general,

$$T_{\mathbb{P}^n}|_{\mathbb{P}^1} \cong \mathcal{O}(2) \oplus [\mathcal{O}(1)]^{n-1}\ .$$



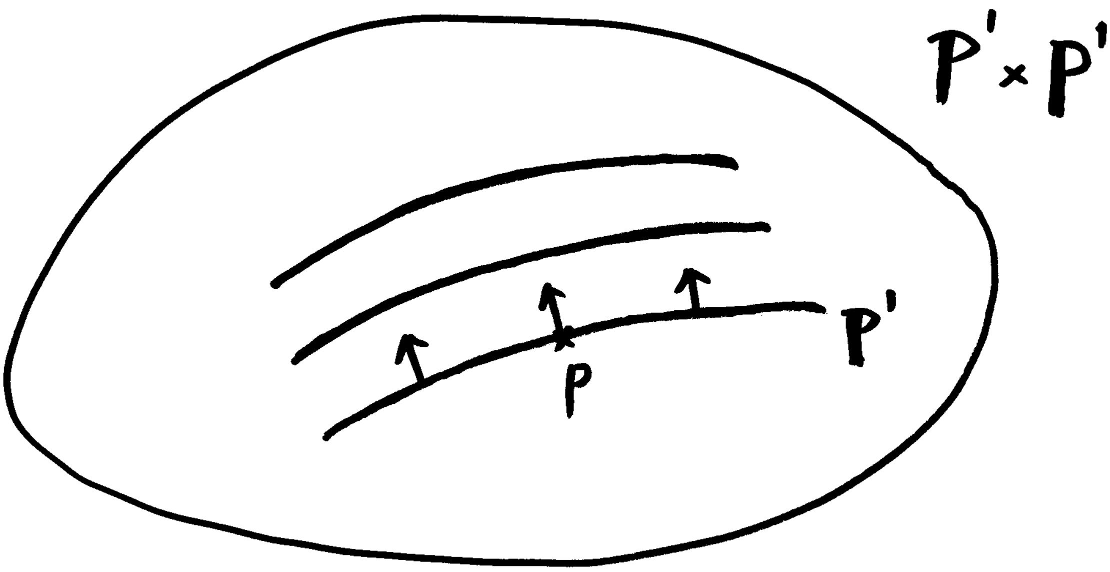
(B) $\mathbb{P}^1 \subset \mathbb{P}^1 \times \mathbb{P}^1$, $\quad z \to (z, 0)$

$$T_{\mathbb{P}^1 \times \mathbb{P}^1}\big|_{\mathbb{P}^1} \cong \mathcal{O}(2) \oplus \mathcal{O} \ .$$

(C) $Q^n \subset \mathbb{P}^{n+1}$ hyperquadric, defined by $z_0^2 + \cdots + z_{n+1}^2 = 0$

$$T_{Q^n}\big|_{\mathbb{P}^1} \cong \mathcal{O}(2) \oplus [\mathcal{O}(1)]^{n-2} \oplus \mathcal{O} \ .$$

Trivial factor: $Q^2 \subset Q^n$; $Q^2 \cong \mathbb{P}^1 \times \mathbb{P}^1$.



$s =$ nowhere zero section

X Fano, $\quad L > 0, \quad \delta_L = \deg.$

<u>minimal rational curve</u> C attains

$$\min\{\delta_L(C) : T_X|_C \geq 0\} \ .$$

Deformation Theory of Rational Curves

$\Longrightarrow$ For a very general point $P \in X$,

$$T_X|_C \geq 0 \quad \forall C \text{ rat. }, \quad P \in C \ .$$

<u>Consequence</u>

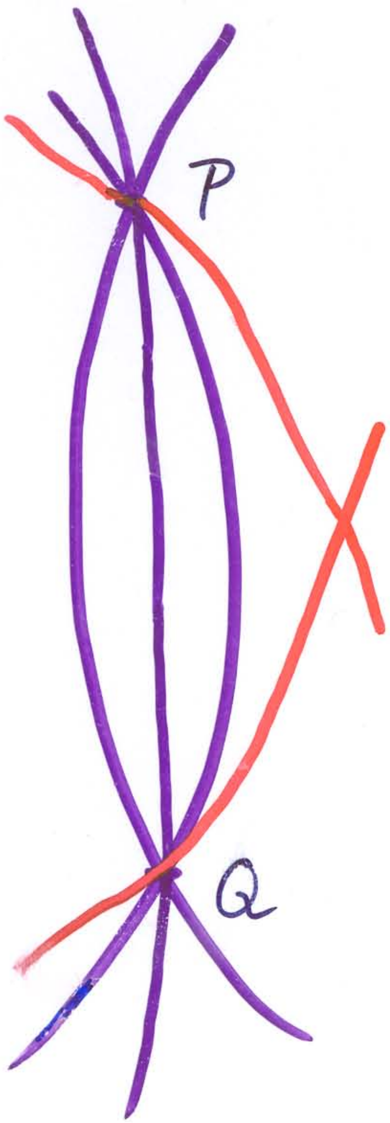
$\mathcal{K}$ = choice of irr. comp. of mrc

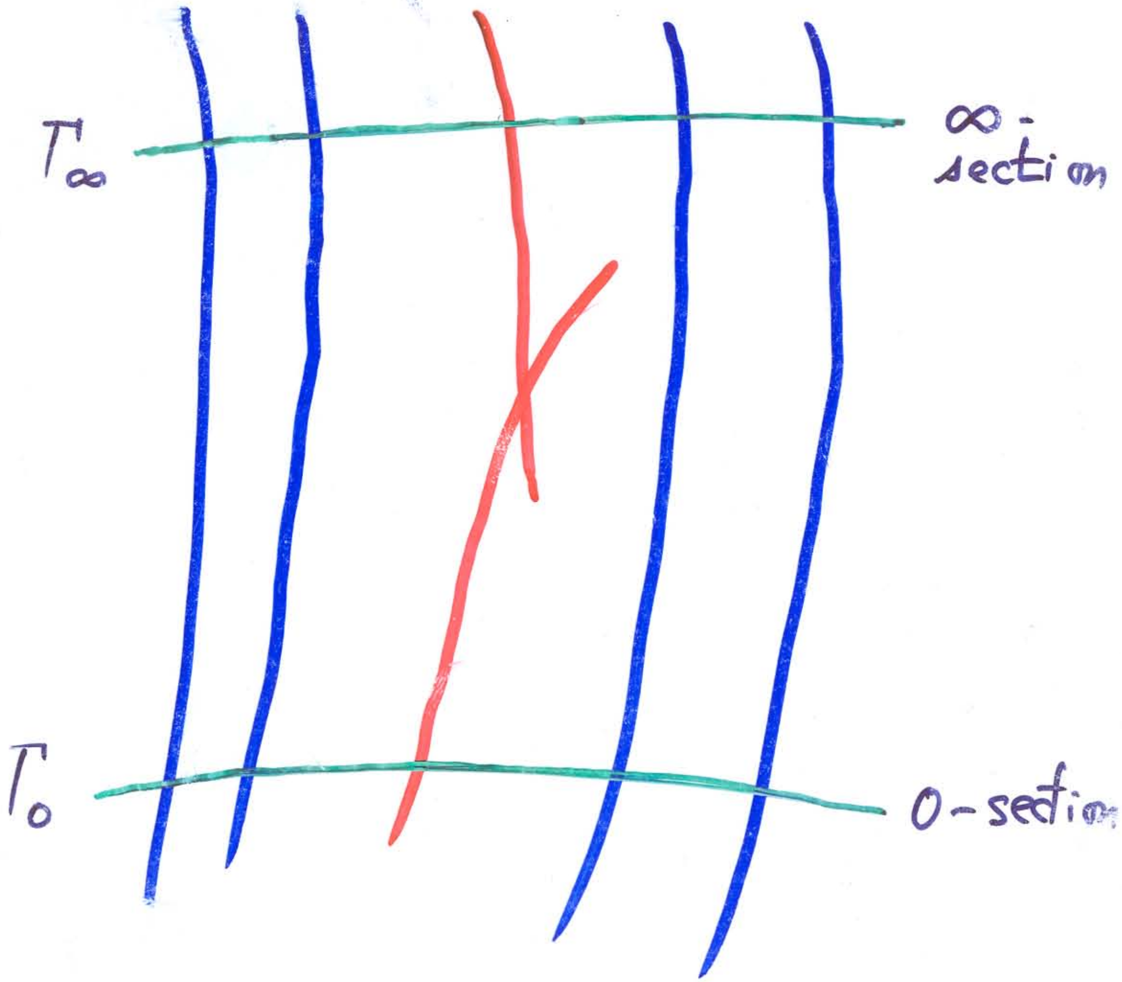
For P generic, $[C] \in \mathcal{K}$ generic

$f : \mathbb{P}^1 \to X$, $\quad C = f(\mathbb{P}^1)$. Then,

$$f^* T_X \cong \mathcal{O}(2) \oplus [\mathcal{O}(1)]^p \oplus \mathcal{O}^q \ .$$

Mori's "Breaking-up Lemma"





Family of curves fixing 2 points $P, Q \in X$ must break up. Otherwise $\Gamma_\infty \cdot \Gamma_\infty = -\Gamma_0 \cdot \Gamma_0$

Varieties of Minimal Rational Tangents

X uniruled,

$\mathcal{K}$ = component of Chow space of minimal rational curves

$\mu : \mathcal{U} \to X$; $\rho : \mathcal{U} \to \mathcal{K}$ universal family

$x \in X$ generic; $\mathcal{U}_x$ smooth

The tangent map $\tau : \mathcal{U}_x \to \mathbb{P}T_x(X)$ *is given by*

$$\tau([C]) = [T_x(C)] \ ;$$

for C *smooth at* $x \in X$.

τ is rational, generically finite,

a priori **undefined for** C *singular* **at** x.

We call the strict transform

$$\tau(\mathcal{U}_x) = \mathcal{C}_x \subset \mathbb{P}T_x(X)$$

variety of minimal rational tangents.

For C standard, $T_x(C) = \mathbb{C}\alpha$

$$T|_C \cong \mathcal{O}(2) \oplus \mathcal{O}(1)^p \oplus \mathcal{O}^q$$

$$P_\alpha := [\mathcal{O}(2) \oplus \mathcal{O}(1)^p]_x \ , \ \text{positive part} \ .$$

Then,

$$T_\alpha(\tilde{C}_x) = P_\alpha \ ;$$

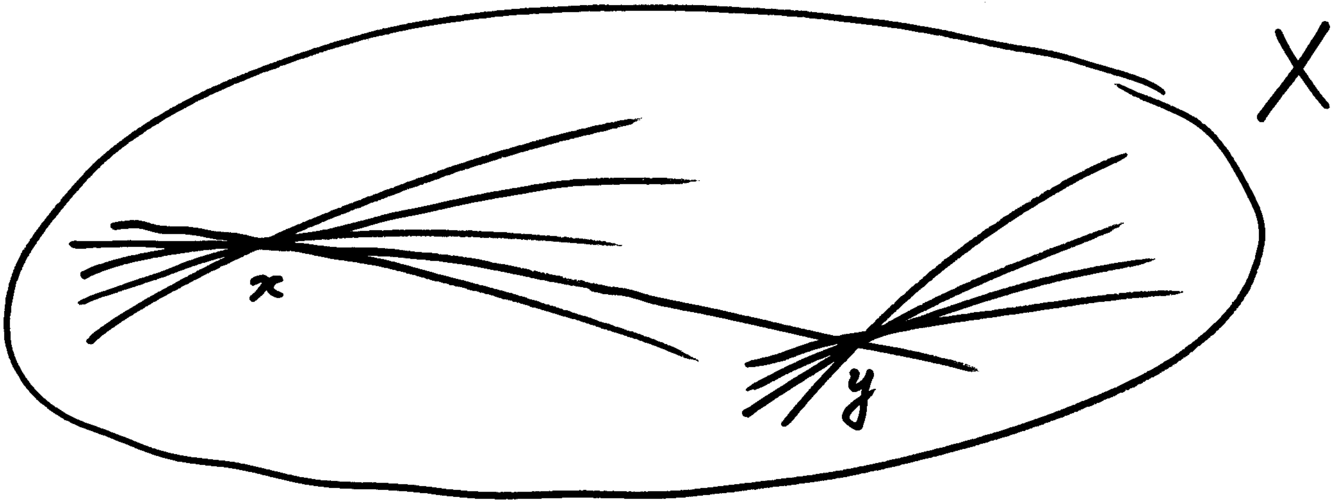
$$T_{[\alpha]}(C_x) = P_\alpha \ \mathrm{mod}\ \mathbb{C}\alpha \ .$$

In other words,

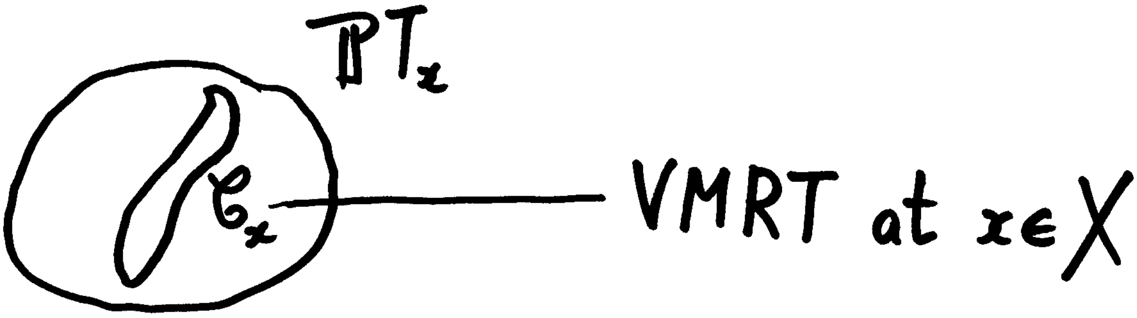
$$\dim(\mathcal{C}_x) = p \ ,$$

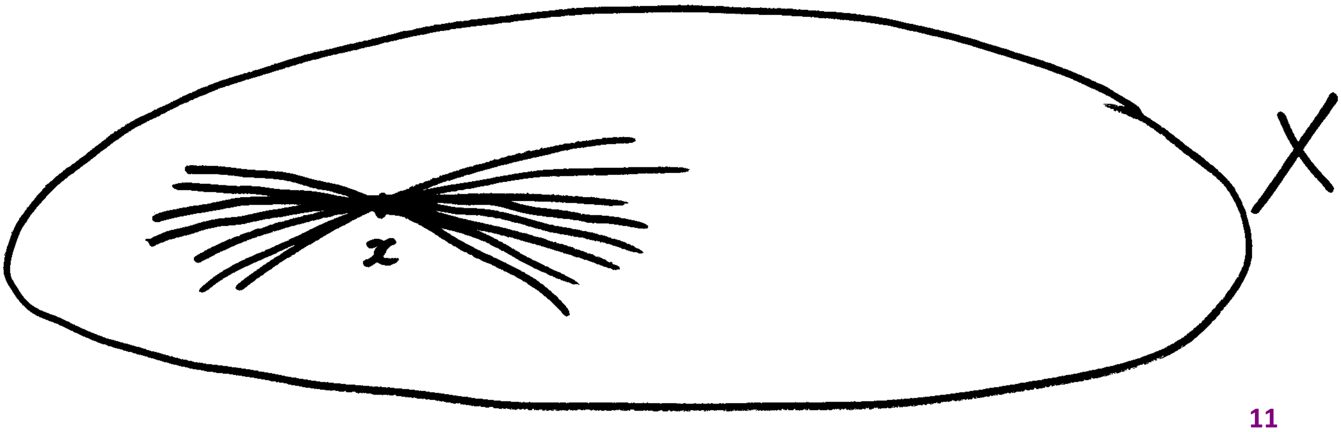
and $\mathcal{C}_x$ is infinitesimally determined by splitting types.

Minimal Rational Curves



Variety of Minimal Rational Tangents (VMRT)





Characterization of $\mathbb{P}^n$ (Cho-Miyaoka-Shepherd-Barron 2002)

X irr. normal variety, $\dim(X) = n$.

Suppose there exists a minimal component $\mathcal{K}$ on X such that

$$\mathcal{C}(\mathcal{K}) = \mathbb{P}T_X \ .$$

Then, there exists

$$\nu : \mathbb{P}^n \to X$$

étale over $X - \mathrm{Sing}(X)$ such that

$$\text{members of } \mathcal{K} = \text{images of lines in } \mathbb{P}^n.$$

In particular

$$X \text{ smooth} \Rightarrow X \cong \mathbb{P}^n \ .$$

Theorem (Kebekus 2002, JAG).

The tangent map

$$\tau_x : \mathcal{U}_x \to \mathbb{P}T_x(X)$$

is a morphism at a generic point $x \in X$.

Theorem (Hwang-Mok 2004, AJM).

The tangent map

$$\tau_x : \mathcal{U}_x \to \mathcal{C}_x \subset \mathbb{P}T_x(X)$$

is a birational morphism at a generic point $x \in X$.

Examples of VMRTs

Fermat hypersurface $1 \leq d \leq n-1$

$$X = \{Z_0^d + Z_1^d + \cdots + Z_n^d = 0\}$$

$x = [z_0, z_1, \ldots, z_n] \in X$.

FIND all $(w_0, w_r, \ldots, w_n)$ such that $\forall\ t \in \mathbb{C}$.

$$[z_0 + tw_0, z_1 + tw_1, \ldots, z_n + tw_n] \in X$$

$$(z_0 + tw_0)^d + \cdots + (z_n + tw_n)^d = 0$$

$$0 = (z_0^d + \cdots + z_n^d)$$

$$+t(z_0^{d-1} w_0 + \cdots + z_n^{d-1} w_n) \cdot d$$

$$+t^2(z_0^{d-2} w_0^2 + \cdots + z_n^{d-2} w_n^2) \cdot \frac{d(d-1)}{2}$$

$$+ \cdots + t^d(w_0^d + \cdots + w_n^d)\ .$$

When $(z_0, z_1, \ldots, z_n)$ is fixed, we get $d+1$ equations, and

$\mathcal{C}_x =$ complete intersection of $d-1$ hypersurfaces of degree $2, 3, \ldots, d$ in $\mathbb{P}T_x(X) \cong \mathbb{P}^{n-1}$

If $d \leq n-1$, $\dim(\mathcal{C}_x) = (n+1) - (d+1) - 1 = n - d - 1 \geq 0$.

Examples of VMRT

X	(generic) VMRT $\mathcal{C}_x$
$\mathbb{P}^n$	$\mathbb{P}^{n-1}$
Q^n	Q^{n-2}
cubic in $\mathbb{P}^{n+1}$	codim $2 \subset \mathbb{P}^{n-1}$ $=$ quadric $\cap$ cubic, deg. 6
$X_3^3 \subset \mathbb{P}^4$	6 points
$X_3^4 \subset \mathbb{P}^5$	deg. 6 curve of genus 4
$X_3^5 \subset \mathbb{P}^6$	$K3$ – surfaces
$X_d^n \subset \mathbb{P}^{n+1}$, $d < n$	complete intersection $\subset \mathbb{P}^n$ of degrees $1, 2, \ldots, d$

In these examples,

$$\{\text{mrc}\} = \{\text{lines in } \mathbb{P}^n \text{ contained in } X\} \ .$$

Type	G	K	$G/K = S$	$\mathcal{C}_o$	Embedding
I	$SU(p+q)$	$S(U(p) \times U(q))$	$G(p,q)$	$\mathbb{P}^{p-1} \times \mathbb{P}^{q-1}$	Segre
II	$SO(2n)$	$U(n)$	$G^{II}(n,n)$	$G(2, n-2)$	Plücker
III	$Sp(n)$	$U(n)$	$G^{III}(n,n)$	$\mathbb{P}^{n-1}$	Veronese
IV	$SO(n+2)$	$SO(n) \times SO(2)$	Q^n	Q^{n-2}	by $\mathcal{O}(1)$
V	E_6	$\mathrm{Spin}(10) \times U(1)$	$\mathbb{P}^2(\mathbb{O}) \otimes_{\mathbb{R}} \mathbb{C}$	$G^{II}(5,5)$	by $\mathcal{O}(1)$
VI	E_7	$E_6 \times U(1)$	exceptional	$\mathbb{P}^2(\mathbb{O}) \otimes_{\mathbb{R}} \mathbb{C}$	Severi

Scope

- Algebraic Geometry
  - Mori theory
  - Hilbert schemes
  - projective geometry
- Differential Geometry
  - distributions
  - G-structures
- Several Complex Variables
  - Hartogs phenomenon
  - analytic continuation
- Lie Theory
  - Hermitian symmetric spaces
  - rational homog. spaces G/P

Examples of G-structures

Riemannian Geometry

A Riemannian metric $\Sigma g_{ij} dx^i \otimes dx^j$ gives a reduction of the structure group from $GL(n, \mathbb{R})$ to $O(n, \mathbb{R})$; $G = O(n, \mathbb{R})$.

Holomorphic Metrics

X complex manifold,

$$\sum g_{ij} dz^i \otimes dz^j$$

hol. symmetric 2-tensor,

$$\det(g_{ij}) \neq 0 \ ;$$

g a *holomorphic metric*;
Hol. G-structure with $G = O(n; \mathbb{C})$.

Theorem (Hwang-Mok, Crelle 1997)

V model vector space $\cong \mathbb{C}^n$,

G reductive complex Lie group,

$G \subsetneq GL(V)$ irreducible faithful representation,

M Fano manifold with holomorphic G-structure.

Then, the G-structure is flat

$$M \cong S \ ,$$

where S = irr. HSS, compact type of rank ≥ 2.

Lazarsfeld's Problem

Theorem (Hwang-Mok, Invent. 1999).
(2nd proof: Asian J. Math. 2004)

$Y = G/P$ rational homogeneous

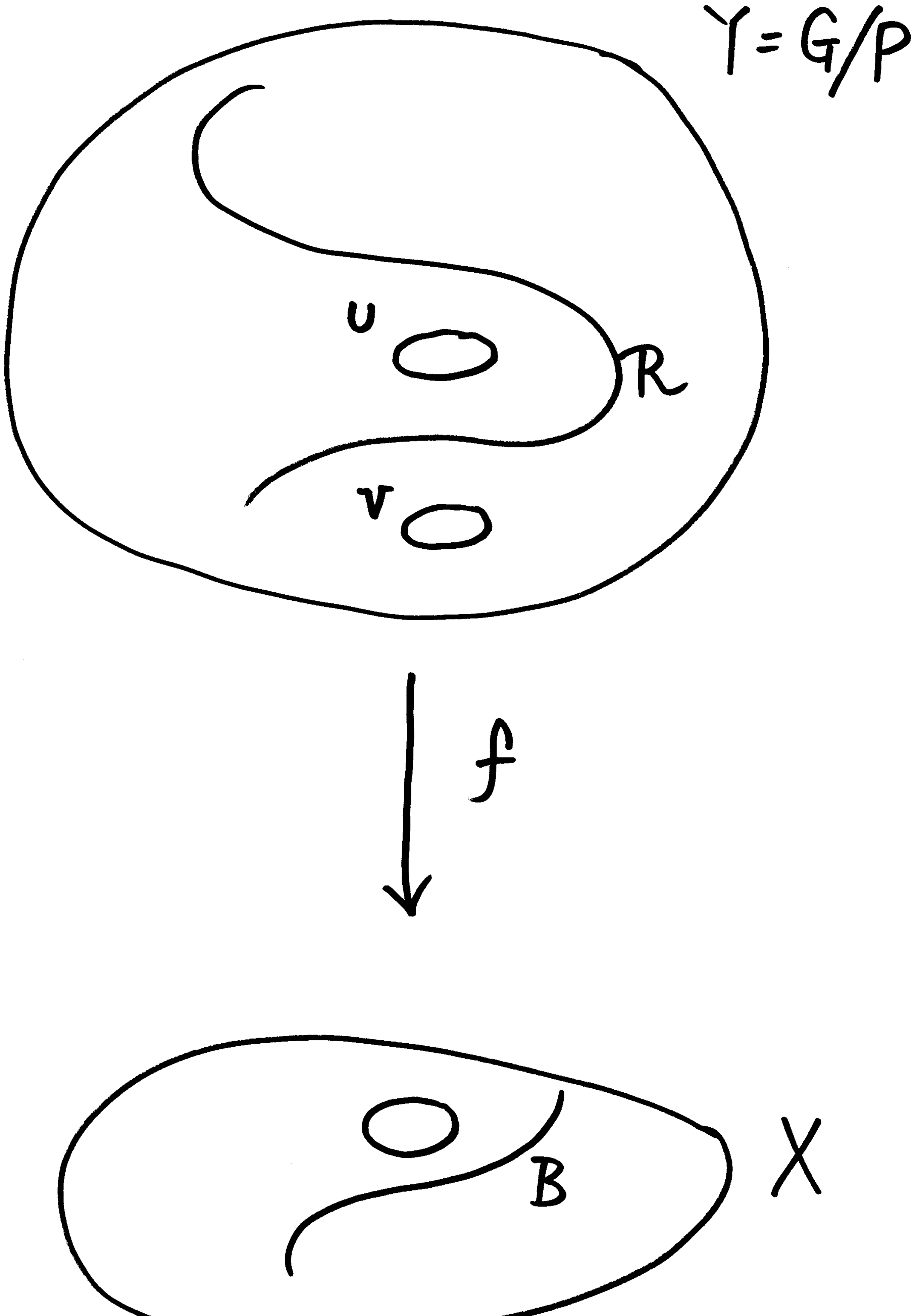
P maximal parabolic, i.e. $b_2(Y) = 1$

X projective manifold

$f : Y \to X$ finite holomorphic map

Then,

EITHER
(a) $X \cong \mathbb{P}^n$; OR
(b) $f : Y \xrightarrow{\cong} X$ is a biholomorphism.


Y = G/P
U
R
V
f
B
X

Lazarsfeld's Problem

Principle of Proof:

$$f : Y \to X \ ; \quad Y = G/P \ , \quad b_2(Y) = 1 \ .$$

Suppose $X \not\cong \mathbb{P}^n$; f not a biholomorphism. To derive a contradiction let

$$\boxed{\begin{array}{c} \varphi : U \xrightarrow{\cong} V \ ; \ U, V \subset Y \\ \textit{such that} \quad f \circ \varphi \equiv f. \end{array}}$$

$\mathcal{C} \subset \mathbb{P}T(X)$ varieties of mrt

$\mathcal{D} := f^*\mathcal{C} \subset \mathbb{P}T(Y)$

$$\varphi_* \mathcal{D}|_U = \mathcal{D}|_V \text{ tautologically.}$$

Prove that $\varphi = \Phi|_U$ for some $\Phi \in Aut(Y)$ to derive a contradiction!

Varieties of distinguished tangents

$\mathcal{N}$ = irr. comp. of Chow space of curves on X passing through $x \in X$

$\mathcal{N}' \subset \mathcal{N}$ subset smooth of curves smooth at x

$\mathcal{N}' = N^1 \cup \cdots \cup N^\ell$ decomposition in terms of geometric genus

$\tau : N^j \to \mathbb{P}T_x(X)$ tangent map

$N^j = M_1^j \cup \cdots \cup M_k^j$ τ-stratification

Definition.
An irreducible subvariety $\mathcal{D} \subset \mathbb{P}T_x(X)$ is called a variety of distinguished tangents (VMRT) if $\mathcal{D} = \overline{\tau(M_i^j)}$ for some choice of $\mathcal{N}$, N^j and M_i^j.

Varieties of distinguished tangents

Properties

(i) Given an irreducible smooth projective variety X and $x \in X$, there are only countably many varieties of distinguished tangent in $\mathbb{P}T_x(X)$.

(ii) Let $\mathcal{D} \subset \mathbb{P}T_x(X)$ be a variety of distinguished tangents associated to some choice of $\mathcal{N}$, N^j and M_i^j. Then for any tangent vector v to $\mathcal{D}$, we can find a family of curves $\{l_t, t \in \Delta\}$ belonging to $\mathcal{N}$ smooth at x so that the derivative of the tangent directions $\mathbb{P}T_y(l_t) \in \mathbb{P}T_x(X)$ at $t = 0$ is v.

(iii) Suppose a connected Lie group P acts on X fixing x. Then any variety of distinguished tangents in $\mathbb{P}T_x(X)$ is invariant under the isotropy action of P on $\mathbb{P}T_x(X)$.

Theorem. (Hwang-Mok, Invent. 2005)

G simple Lie group over $\mathbb{C}$, $\mathfrak{g}$ = Lie algebra

$P \subset G$ maximal parabolic subgroup

S = rational homogeneous of type $(G; \alpha)$

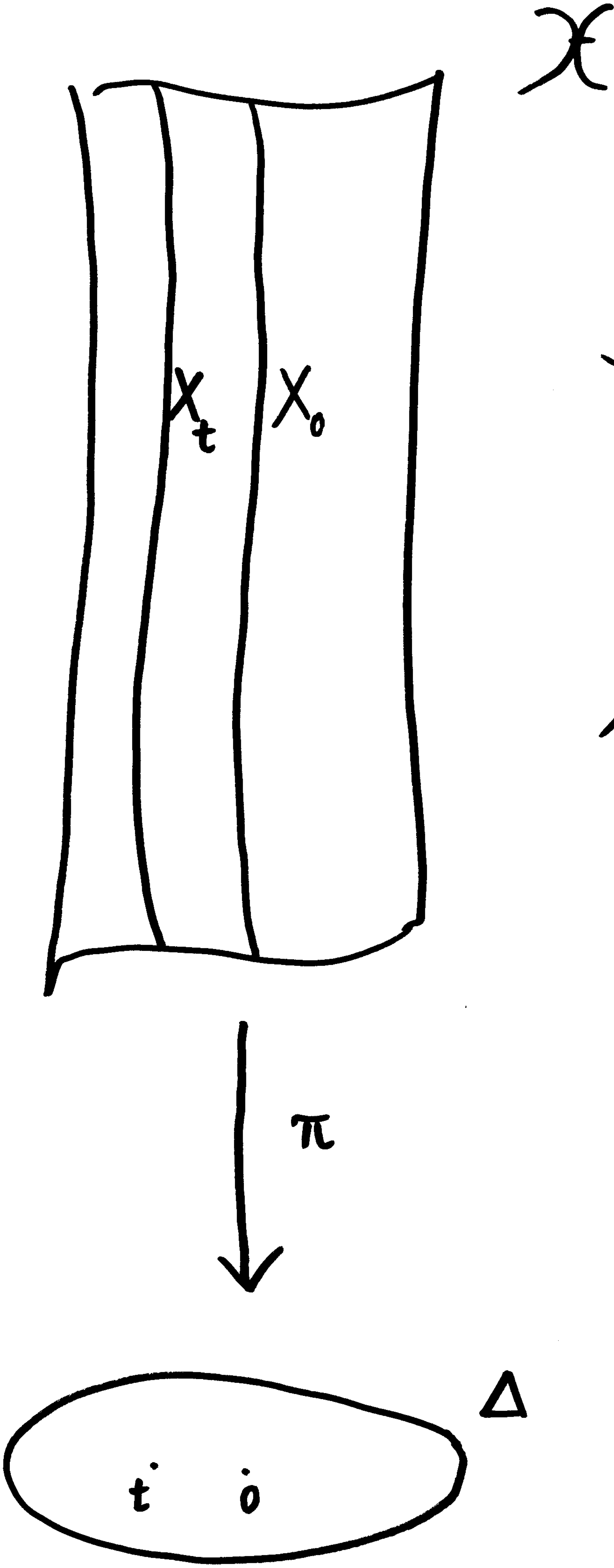
$\pi : \mathcal{X} \to \triangle = \{t \in \mathbb{C} : |t| < 1\}$ regular family such that

(i) $X_t := \pi^{-1}(t) \cong S$ for $t \neq 0$ and

(ii) $X_0 := \pi^{-1}(0)$ is Kähler.

Then,

$$X_0 \cong S \ .$$



$S = G/P$

G simple

P maximal parabolic

$X_t \cong S, \ \forall t \neq 0$

Q. $X_0 \overset{?}{\cong} S$

Deformation rigidity in the Kähler case

Scheme

(1) S Hermitian symmetric
[Hwang-Mok, Invent. Math 1998]

(2) S of type (G, α), α a long simple root
[Hwang, Crelle 1997] for the contact case
[Hwang-Mok, Ann. ENS 2002] in general

(3) S of type (F_4, α_1)
[Hwang-Mok, Springer-Verlag 2004]

(4) S of type (C_n, α_k), $1 < k < n$; or (F_4, α_2)
[Hwang-Mok, Invent. Math 2005]

Deformation rigidity in the Kähler case

Methods

(1) Distribution spanned by VMRT
Integrability

(2) Differential systems generated by distributions spanned by VMRT

(3) Methods of (2)

(4) Holomorphic vector fields on uniruled projective manifolds.
Uses also conditions on integrability of (1).

Distributions Spanned by MRT

X uniruled,

$\mathcal{K}$: component of Chow space of minimal rational curves

$\mathcal{C}_x$: variety of mrt;

$\mathcal{C}_x \subset \mathbb{P}T_x(X)$; $\tilde{\mathcal{C}}_x \subset T_x(X)$;

$W_x = \mathrm{Span}(\tilde{\mathcal{C}}_x) \subset T_x(X)$.

Assume $\underline{W \neq T(X)}$.

Q. Is W integrable?

Pic$(X) = 1 \Rightarrow W$ *not integrable*
Projective-geometric properties of $\mathcal{C}_x$
$\Rightarrow W$ *integrable*

For $\mathcal{C}$ on X_0, $W = T(X_0)$, i.e. $\mathcal{C}_x$ lin. nondeg.

Integrability of Distributions

Proposition.

$\Omega \subset \mathbb{C}^n$, $W \subset T_\Omega$ hol. distribution. Then, W is integrable iff

(∗) Given $x \in \Omega$, $\exists$ hol. vector fields α_j, β_j def. on a nbd of x s.t.

(i) $[\alpha_j, \beta_j](x) \in W_x$.

(ii) $\mathrm{Span}\{\alpha_j \wedge \beta_j\} = \Lambda^2 W_x$.

Verification of Integrability

$C \subset X_0$ be a smooth standard mrc.

$$T_{X_0}|_C \cong \mathcal{O}(2) \oplus [\mathcal{O}(1)]^p \oplus \mathcal{O}^q \ .$$

For $x \in C$; $T_x(C) \cong \mathbb{C}\alpha_x$. Define

$$P_{\alpha_x} = (\mathcal{O}(2) \oplus [\mathcal{O}(1)]^p)_x \ .$$

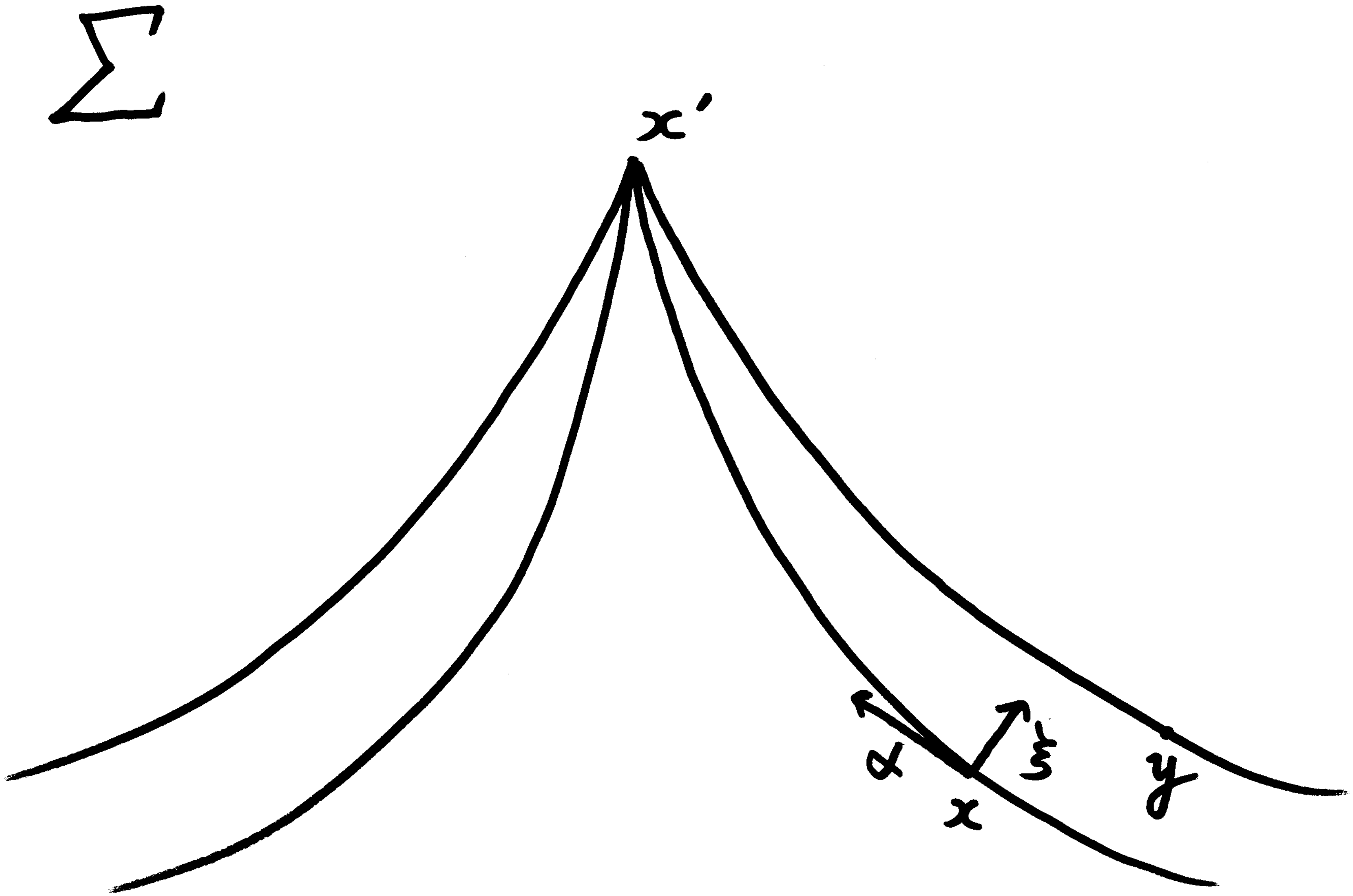
Proposition

$C \subset X_0$ standard mrc; $x \in C$. $\xi_x \in P_{\alpha_x}$ s.t. (α_x, ξ_x) linearly independent. Then, there exists a loc. smooth complex-analytic surface Σ at x such that

(i) $T_x(\Sigma) = \mathbb{C}\alpha_x + \mathbb{C}\xi_x$;

(ii) at every $y \in \Sigma$ near x;

$$T_y(\Sigma) \subset W_y \ .$$



$$T_x(\Sigma) = \mathbb{C}\alpha + \mathbb{C}\xi$$

Proposition.

$\mathcal{C}_x \subset \mathbb{P}W_x$ VMRT at generic x

$\mathcal{T}_x \subset \mathbb{P}(\wedge^2 W_x)$ variety of tangents.

Then,

$$\mathcal{T}_x \subset \mathbb{P}(\wedge^2 W_x) \text{ lin. nondeg.}$$

$$\Rightarrow W \text{ integrable.}$$

Proposition. $\mathcal{T}_x \subset \mathbb{P}(\wedge^2 W_x)$ is linearly non-degenerate if

$$\dim \mathcal{C}_x \geq \operatorname{codim} \mathcal{C}_x \text{ in } \mathbb{P}W_x \ ,$$

$$\mathcal{C}_x \subset \mathbb{P}W_x \text{ is smooth .}$$

Differential system

$$0 \neq D_1 \subsetneqq D_2 \subsetneqq \cdots \subsetneqq D_m \subset T_U$$

filtration of X by hol. distributions.

Weak derived system (X, D)

$$D^1 = D \text{ , meromorphic distribution}$$
$$D^k = D^{k-1} + [D, D^{k-1}].$$

- On a Fano manifold X, $b_2(X) = 1$, $D^m = T_X$ for some m.

Symbol algebra of a weak derived system:

$$\mathfrak{s}(X, D) := D^1 \oplus D^2/D^1 \oplus \cdots \oplus D^m/D^{m-1}$$

- On a rational homogeneous space $S = G/P$, $b_2(S) = 1$, with $D =$ min. nontrivial G-inv. hol. distribution,

$$\mathfrak{n}^+ := \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_m \cong \mathfrak{s}(S, D).$$

Serre relations

$\mathfrak{g}$ simplie Lie algebra over $\mathbb{C}$

$\Sigma = \{\alpha_1, \ldots, \alpha_\ell\}$ system of simple roots

$n(i,j)$ = entries of Cartan matrix

Then, $\mathfrak{g}$ is the universal Lie algebra generated by $\{x_i, y_i, h_i : 1 \leq i \leq \ell\}$ subject to the identities

- $[h_i, h_j] = 0$
- $[x_i, y_i] = h_i$, $[x_i, y_j] = 0$ if $i \neq j$
- $[h_i, x_j] = n(i,j)x_j$, $[h_i, y_j] = -n(i,j)y_j$
- $ad(x_i)^{-n(i,j)+1}(x_j) = 0$ if $i \neq j$
- $ad(y_i)^{-n(i,j)+1}(y_j) = 0$ if $i \neq j$

Objective

For the regular family $\pi : \mathfrak{X} \to \triangle$ consider $D \subset T_{X_0}$ spanned by VMRTs. Show that $\mathfrak{s}(X_0, D) \cong \mathfrak{n}^+ = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_m$ for the model $S = G/P$.

Serre relations for $\mathfrak{n}^+$

Write $\mathfrak{n}^+ \subset \mathfrak{g}$ subalgebra generated by $\{x_1, x_2, \ldots, x_\ell\}$. Then, $\mathfrak{n}^+$ is the universal Lie algebra generated by $\{x_1, \ldots, x_\ell\}$ subject to

$$ad(x_i)^{-n(i,j)+1}(x_j) = 0.$$

Note that

- When α_i is a long simple root,

$$n(i,j) = \frac{2(\alpha_i, \alpha_j)}{\|\alpha_i\|^2} = 0 \text{ or } -1 .$$

For us the crucial relations are

$$[x_i, [x_i, x_j]] = 0 \text{ if } n(i,j) \neq 0.$$

Concluding argument:

$\mathfrak{s}(X_0, D)$ is a quotient of the universal Lie algebra $\mathbf{U}$ gen. by $\{x_1, \ldots, x_\ell\}$ subject to

$$ad(x_j)^{-n(i,j)+1}(x_i) = 0.$$

By Serre relations,

$$\mathbf{U} \cong \mathfrak{n}^+ \ , \quad \mathfrak{s}(X_0, D) \cong \mathfrak{n}^+/J.$$

If $J \neq 0$, the weak derived system (X, D) would terminate at D^m, $\dim D^m < n$, giving an *integrable* distribution $W = D^m$ containing VMRTs, which contradicts with $b_2(X_0) = 1$.

□

Conjecture 1

X Fano, $b_2(X) = 1$

$x \in X$ generic point

$Z \in \Gamma(X, T_X)$.

Then,

$$\mathrm{ord}_x(Z) \geq 3 \Rightarrow Z \equiv 0 \ .$$

Conjecture 2

X Fano, $b_2(X) = 1$, $\dim_{\mathbb{C}} X = n$

$\Rightarrow \dim_{\mathbb{C}}(\mathrm{Aut}(X)) \leq n^2 + 2n$;

$= n^2 + 2n \Leftrightarrow X \cong \mathbb{P}^n$.

Remark:

(1) For $\Sigma_k = \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(k))$, the k-th Hirzebruch surface,

$$\dim(\mathrm{Aut}(\Sigma_k)) > \dim \Gamma(\mathbb{P}^1, \mathcal{O}(k)) = k + 1.$$

Bounds fail in general for projective uniruled projective manifolds.

(2) If $\exists \mathcal{K}$ on X such that $\dim \mathcal{C}_x = 0$, Hwang shows that there are no hol. v.f. vanishing at a generic point $x \in X$. In that case, $\dim(\mathrm{Aut}(X)) \leq n$.

(3)

$$\dim\{Z \in \Gamma(X, T_X) : \mathrm{ord}_x(Z) \leq 2\}$$
$$\leq \frac{n(n+1)(n+2)}{2} \cong \frac{n^3}{2} .$$

Theorem 1 (Hwang-Mok)

X projective uniruled manifold

$\mathcal{K}$ = minimal rational component

$x \in X$ generic point

$\mathcal{C}_x \subset \mathbb{P}T_x(X)$, VMRT at x, $\dim \mathcal{C}_x = p > 0$

Assume $\mathcal{C}_x \subset \mathbb{P}T_x(X)$
nonsingular, irreducible,
linearly non-degenerate.

Then,

$$\boxed{Z \in \Gamma(X, T_X)\ ,\ \mathrm{ord}_x(Z) \geq 3 \Rightarrow Z \equiv 0\ .}$$

Theorem 2

Assume $\mathcal{C}_x \subset \mathbb{P}T_x(X)$, $\dim X = n$

nonsingular, irreducible,

linearly non-degenerate,

linearly normal.

Then,

$$\dim(\mathrm{Aut}(X)) \leq n^2 + 2n$$
$$= n^2 + 2n \Leftrightarrow X \cong \mathbb{P}^n$$

Corollary

X Fano, $b_2(X) = 1$, $\dim X = n$

$\mathcal{O}(1)$ positive generator of $\mathrm{Pic}(X) \cong \mathbb{Z}$.

Assume $\mathcal{O}(1)$ very ample.

$c_1(X) > \dfrac{n+1}{2}$, $x \in X$ generic. Then,

$$0 \neq Z \in \Gamma(X, T_X) \Rightarrow \mathrm{ord}_x(Z) \leq 3 \ ;$$

$c_1(X) > \dfrac{2(n+2)}{3}$, $X \ncong \mathbb{P}^n$

$$\Rightarrow \dim(\mathrm{Aut}(X)) < n^2 + 2n \ .$$

Ideas of Proof

(1) A holomorphic vector field Z vanishing at $x \in X$ to the order ≥ 2 gives by power series expansion

$$Z = \sum_{i,j,k} A^k_{ij} z^i z^j \frac{\partial}{\partial z_k} + \text{ higher order terms}$$

$$A \in S^2 T^*_x \otimes T_x \text{ with the property that}$$

(†) for any $\alpha \in \tilde{\mathcal{C}}_x$, for

$$A_\alpha := \sum A^k_{\alpha j} dz^j \otimes \frac{\partial}{\partial z_k} \in \text{ End}(T_x) \ ,$$

$A_\alpha|_{\tilde{\mathcal{C}}_x}$ is tangent to $\tilde{\mathcal{C}}_x$.

Here we identify vector fields on T_x with endomorphisms.

(2) Taking $\alpha, \beta \in \tilde{\mathcal{C}}_x$; $\alpha, \beta \neq 0$

$$A_{\alpha\beta} = A_\alpha(\beta) = A_\beta(\alpha)$$

is tangent to $\tilde{\mathcal{C}}_x$ both at α and β, i.e.

$$A_{\alpha\beta} \in P_\alpha \cap P_\beta \ .$$

(3) The symmetry property on A forces (by letting $\beta \to \alpha$) that $A_{\alpha\alpha} \in \mathrm{Ker}(\sigma_\alpha)$ for the second fundamental form σ_α on $\tilde{\mathcal{C}}_x - \{0\}$. If $\mathcal{C}_x \subsetneq \mathbb{P}T_x$ is smooth and non-linear, $\mathrm{Ker}(\sigma_\alpha) = \mathbb{C}\alpha$ (Zak's Thm.), and

$$\overline{A} \in \Gamma(\mathcal{C}_x; \mathrm{Hom}(L^2, L)) = \Gamma(\mathcal{C}_x, L^*)$$

for the tautological line bundle L.

(4) We can get bounds for the dimension of Z with $\mathrm{ord}_x(Z) \geq 2$ if we know that

$$(*) \qquad \overline{A} = 0 \Rightarrow A = 0 \ .$$

Moreover, the latter is enough to prove the nonexistence of nontrivial Z with $\mathrm{ord}_x(Z) \geq 3$. If $\mathrm{ord}_x(Z) \geq 3$ start with

$$A \in S^3 T_x^* \otimes T_x \quad \text{such that}$$
$$A_{\alpha\beta\gamma} \in P_\alpha \cap P_\beta \cap P_\gamma \text{ for } \alpha, \beta, \gamma \in \tilde{\mathcal{C}}_x - \{0\}.$$

Then, we get

$$A_{\alpha\alpha\gamma} \in P_\alpha \cap P_\gamma \text{ for any } \alpha, \gamma \in \tilde{\mathcal{C}}_x - \{0\}$$
$$\Rightarrow A_{\alpha\alpha\gamma} = 0$$
$$\Rightarrow A \equiv 0 \text{ if } (*) \text{ holds.}$$

Prolongation of infinitesimal automorphisms of projective varieties

V complex vector space, $\dim V = n$

$\mathfrak{g} \subset End(V)$ Lie subalgebra

$\mathfrak{g}^{(k)} \subset S^{k+1}V^* \otimes V$, $\sigma \in \mathfrak{g}^{(k)} \Leftrightarrow$

> $\forall\ v_1, \ldots, v_k \in V$, writing
>
> $$\sigma_{v_1,\ldots,v_k}(v) = \sigma(v; v_1, \ldots, v_k)\ ,$$
>
> we have $\sigma_{v_1,\ldots,v_k} \in \mathfrak{g}$.

$\mathfrak{g}^{(k)} = k$-th prolongation of $\mathfrak{g}$; $\mathfrak{g}^{(0)} = \mathfrak{g}$.

$\mathfrak{g}^{(k)} = 0 \Rightarrow \mathfrak{g}^{(k+1)} = 0$.

$\mathfrak{h} \subset \mathfrak{g} \Rightarrow \mathfrak{h}^{(k)} \subset \mathfrak{g}^{(k)}$.

$[\mathfrak{g}^{(k)}; \mathfrak{g}^{(\ell)}] \subset \mathfrak{g}^{(k+\ell)}$.

$Y \subset \mathbb{P}V$ projective subvariety, $dim Y = p$

$\tilde{Y} \subset V$ affine cone of Y. Define

$aut(Y) = \{A \in End(V) : exp(tA)(\tilde{Y}) \subset \tilde{Y}, t \in \mathbb{C}\}.$

X complex manifold, $dim\ X = n$

$\mathcal{C} \subset \mathbb{P}T(X)$ projective and flat over X

$\mathcal{C}_x \subset \mathbb{P}T_x(X)$ irreducible, reduced

$\mathfrak{f} :=$ germs of $\mathcal{C}$-preserving holomorphic vector fields at x

For $\ell \geq -1$, let

$$\mathfrak{f}^\ell = \{Z \in \mathfrak{f} : ord_x(Z) \geq \ell + 1\} .$$

Proposition. *For $k \geq 0$, identify $\mathfrak{f}^k/\mathfrak{f}^{k+1} \subset S^{k+1}T_x^*(X) \otimes T_x(X)$ by taking leading terms of Taylor expansions of the vector fields at x. Then*

$$\mathfrak{f}^k/\mathfrak{f}^{k+1} \subset aut(\mathcal{C}_x)^{(k)} \ ,$$

the k-th prolongation of the Lie algebra of infinitesimal automorphisms of the projective variety $\mathcal{C}_x$.

Proof. Z hol. vector field at x, defined on $U \subset X$, $ord_x Z \geq k+1$

$$j_x^{j+1}(Z) \in S^{k+1}T_x^*(X) \otimes T_x(X)$$

Z can be lifted canonically to Z' on $\mathbb{P}T(U)$: Z = inf. generator of $\{f_t\}$, germs of biholomorphism at x

$f_t : U \to X$ gives $F_t : T(U) \to T(X)$, where $F_t(x, \eta) = (f_t(x), df_t(x)(\eta))$.

$\eta \in T_x(X)$, $ord_\eta(Z') \geq k$,

$$j^k_\eta \in S^k T^*_\eta(T(X)) \otimes T_\eta(T(X)) \ .$$

For $k = 0$, $j^0_\eta \in T_\eta(T(X))$.

For $k \geq 1$, $Z'|_{T_x(X)} \equiv 0$,

$$j^k_\eta \in S^k N^*_\eta \otimes T_\eta(T(X)) \ ,$$

where N = normal bundle of $T_x(X)$ in $T(X)$, $N \cong \pi^* T(X)$. Since $ord_x(Z) \geq k+1$, $\pi_*(j^k_\eta(v_1, \dots, v_k)) = 0$ for $v_1, \dots, v_k \in T_x(X)$. Hence,

$$j^k_\eta(Z') \in S^k N^*_\eta \otimes T_\eta(T_x(X)) \cong S^k T^*_x(X) \otimes T_x(X) \ .$$

Straightforward calculations give

$$j^k_\eta(Z')(v_1, \dots, v_k) = j^{k+1}_x(Z)(v, v_1, \dots, v_k)$$

where we write η and v for the same thing, η when it is consider a point on the fiber $T_x(X)$, v when it is considered a tangent vector at x.

Leading Terms of Hol. Vector Fields

$0 \in \Omega \subset \mathbb{C}^n$; Z = hol. vector field on Ω

$\mathrm{ord}_0(Z) = p \geq 0$

$Z = \sum A^k_{i_1 \cdots i_p} z^{i_1} z^{i_2} \cdots z^{i_p} \frac{\partial}{\partial z_k} + O(|z|^{p+1})$

Principal term $\rho(Z)$ at o:

$$\rho(Z) = A \in S^p T_o^* \otimes T_o \ .$$

Lemma. *Z, W = germs of hol. vector fields at o, $ord_o(Z) = p$, $ord_o(Z) = q$. Then $ord_o[Z, W] \geq p + q - 1$. Suppose $ord_o[Z, W] = p + q - 1$, $p + q \geq 1$. Then,*

$\rho([Z, W])$ = bilinear expression in $\rho(Z)$, $\rho(W)$.

For $p = 1$, so that $\rho(Z) \in \mathrm{End}(T_o)$,

$$\rho([Z, W]) = \rho(Z)(\rho(W)) \ .$$

Symbolic Lie algebra of leading terms

Hermitian symmetric case

$$\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$$
$$= \mathfrak{m}^- \oplus \mathfrak{k}^{\mathbb{C}} \oplus \mathfrak{m}^+ \ .$$
$$[\mathfrak{m}^-, \mathfrak{m}^-] = [\mathfrak{m}^+, \mathfrak{m}^+] = 0$$
$$\mathfrak{m}^- \ = \{Z \in \Gamma(S, T_S) : \mathrm{ord}_o Z \geq 2\} \ .$$

All Lie brackets determined by principal terms:

$$[k, m^+], [k, m^-], [k, k'], [m^-, m^+] \ .$$

Deformation Rigidity

Given $\pi : \mathfrak{X} \to \Delta$

$$\mathfrak{g}^t = \mathfrak{aut}(X_t) \text{ for } t \neq 0$$
$$\mathfrak{g}^0 = \text{Limiting Lie algebra} \ .$$

More precisely,

$\mathcal{T}$ = relative tangent bundle

$\pi_*\mathcal{T} = \mathcal{O}(V)$, V hol. vector bundle on Δ

$\mathfrak{g}^t := V_t$, Lie alg. structure induced from $\mathcal{T}$.

Assume stability of $\mathcal{C}_{\sigma(t)}$ as $t \mapsto 0$. Define

$$J_t^{(k)} = \{Z \in \mathfrak{g}^t : \mathrm{ord}_{\sigma(t)}(Z) \geq k\}$$
$$I_t = \{Z \in \mathfrak{g}^t : Z(\sigma(t)) = 0\ ,\ A_Z \in \mathbb{C} \cdot id\}\ .$$

For $t \neq 0$, any $Z \in E_t$, $A_Z \not\equiv 0$ determines a $\mathbb{C}^*$-action. Since $\mathcal{C}_{\sigma(0)} \subset \mathbb{P}T_{\sigma(0)}(X_0)$ is conjugate to $\mathcal{C}_o \subset \mathbb{P}T_o(S)$

$$\dim E_0^{(2)} \leq n\ ,\ E_0^{(k)} = 0 \text{ for } k \geq 3$$
$$\dim I_0 \geq n + 1 \ \textit{(upper semicontinuity)}$$
$$\dim I_0 \leq n + 1 \text{ (VMRT)}\ .$$

Therefore, $\dim I_0 = n+1$ and $\exists$ a hol. vector bundle I of rank $n+1$, $\mathcal{I} = \mathcal{O}(I)$.
$\exists Z \in I_0$ such that $A_Z \not\equiv 0$, and we have a hol. family of $\mathbb{C}^*$-actions T_t.
$T_t = \{e^{\lambda E_t}\}$, period $2\pi i$.

$$\mathfrak{g}_i^t \stackrel{\text{def}}{=} \{Z \in \mathfrak{g}^t : [E_t, Z] = iZ\}$$
$$\mathfrak{g}^t = \mathfrak{g}_{-1}^t \oplus \mathfrak{g}_0^t \oplus \mathfrak{g}_1^t \ .$$

For $t \neq 0$,

$$\mathfrak{g}_0^t \cong \{A \in \mathrm{End}_{\sigma(t)}(T_{\sigma(t)}) : A|_{\tilde{\mathcal{C}}_{\sigma(t)}} \text{ is tangent to } \tilde{\mathcal{C}}_{\sigma(t)}\} \ .$$

Dimension count forces the same for $t = 0$. $[\mathfrak{g}_1^0, \mathfrak{g}_1^0] = [\mathfrak{g}_{-1}^0, \mathfrak{g}_{-1}^0] = 0$. Lie algebra structure on $\mathfrak{g}^0$ completely determined by leading terms. Hence $X_0 = G/P \cong S$.

Uniqueness of tautological foliation:

$\rho : \mathcal{U} \to \mathcal{K}$, $\mu : \mathcal{U} \to X$ universal family

$\pi : \mathcal{C} \to X$ family of VMRTs

$\mathcal{F} = 1 -$ dim. multi-foliation on $\mathcal{C}$

defined by *tautological* liftings $\hat{C}$ of C,

$\mathcal{F} :=$ *tautological foliation*

For C standard $T_X|_C \cong \mathcal{O}(2) \oplus \mathcal{O}(1)^p \oplus \mathcal{O}^q$.

Write $T_x C = \mathbb{C}\alpha$, $P_\alpha = (\mathcal{O}(2) \oplus \mathcal{O}(1)^p)_x$.

$$\mathcal{P}_{[\alpha]} = \{\eta \in T_{[\alpha]}(\mathcal{C}) : d\pi(\eta) \in P_\alpha\}.$$

As $T_{[\alpha]}(\mathcal{C}_x) \cong P_\alpha/\mathbb{C}\alpha$, $\mathcal{P}$ is defined by $\mathcal{C}$.

$\mathcal{W} =$ distribution on $\mathcal{K}$ defined by

$$\mathcal{W}_{[C]} = \Gamma(C, \mathcal{O}(1)^p) \subset \Gamma(C, N_{C|X}) \cong T_{[C]}(\mathcal{K}).$$

We have

$$\mathcal{P} = \rho^{-1}\mathcal{W} \ , \quad \mathcal{F} = \rho^{-1}(0) \Rightarrow [\mathcal{F}, \mathcal{P}] \subset \mathcal{P} \ .$$

Proposition

Assume Gauss map on a generic VMRT $\mathcal{C}_x$ to be injective at a generic $[\alpha] \in \mathcal{C}_x$. Then, $[v, \mathcal{P}] \subset \mathcal{P} \Rightarrow v \in \mathcal{F}$, *i.e.*,

$$\textit{Cauchy Char.}\ (\mathcal{P}) = \mathcal{F}.$$

Corollary

Assume $U \subset X$, $U' \subset X'$, $f : U \xrightarrow{\cong} U'$, $[df]^*\mathcal{C}' = \mathcal{C}|_U$. Then,

f maps open pieces of mrc on X to open pieces of mrc on X.

Proof. Write $f^*\mathcal{C}'$ for $[df]^*\mathcal{C}'$, etc. Then, $f^*\mathcal{C}' = \mathcal{C}|_U$ implies $f^*\mathcal{P}' = \mathcal{P}|_U$. Thus,

$$[f^*\mathcal{F}', \mathcal{P}] = [f^*\mathcal{F}', f^*\mathcal{P}']$$
$$= f^*[\mathcal{F}', \mathcal{P}'] \subset f^*\mathcal{P}' = \mathcal{P}.$$

Proposition implies $f^*\mathcal{F}' = \mathcal{F}$. $\square$

Theorem. (Hwang-Mok, JMPA 2001, AJM 2004)

X projective uniruled, $b_2(X) = 1$,

$\mathcal{K}$ minimal rational component on X.

Assume

(†) for a general point $x \in X$, $\dim \mathcal{C}_x = \phi > 0$ and Gauss map on $\mathcal{C}_x$ generically finite.

(♯) more generally if $\mathcal{C}_x$ is *non-linear*, i.e., not a finite union of projective linear subspaces

Then,

$(X, \mathcal{K})$ has the Cartan-Fubini Extension Property

Examples:

(1) $X = G/P \neq \mathbb{P}^N$, G simple, P maximal parabolic.

(2) $X \subset \mathbb{P}^N$ smooth complete intersection, Fano with $\dim(X) \geq 3$, $c_1(X) \geq 3$.

Ideas of proof of CF *under the assumption* (†)

(1) $f : (X, \mathcal{K}) \to (X', \mathcal{K}')$ gen. finite surj. map, $f^*\mathcal{C}' = \mathcal{C}$ (*i.e.*, VMRT-preserving).

Uniqueness of tautological foliation
$\Rightarrow f$ preserves tautological foliation

(2) Analytic continuation along mrc, obtained by passing to moduli spaces of mrc:

$f : X \to X'$ induces $f^\# : \mathcal{V} \to \mathcal{K}'$ on some open subset $\mathcal{V} \subset \mathcal{K}$.

Now, interpret a point $x \in X$ as the intersection of C, $[C] \in \mathcal{K}_x$, to do analytic continuation.

(3) $(X, \mathcal{K})$ is rationally connected, Analytic cont. along chains of mrc defines a multi-valued map $F : X \to X'$.

(4) $b_2(X) = 1 \Rightarrow$ any mrc C intersects any hypersurface $H \subset X$.

Analytic cont. along C forces univalence of F, *viz.*, F is a birational map preserving VMRTs

(5) birational + VMRT-preserving
$\Rightarrow$ biholomorphic

(a) VMRT-preserving
$\Rightarrow R(F) = \emptyset, \quad R:$ ramification divisor

(b) Embed X to $\mathbb{P}^N$ by $K_X^{-\ell}$, X being Fano, etc. $R(F) = \emptyset$ gives hol. extension of F^*s for sections s of $K_X^{-\ell}$,
$F : X \to X'$ is the restriction of some projective linear isomorphism of $\mathbb{P}^N$.

Local rigidity of holomorphic maps

$\pi : \mathfrak{X} \to \triangle$ regular family

X_t Fano, $\mathrm{Pic}(X_t) \cong \mathbb{Z}$

X_0 carries a rational curve C, with *trivial* normal bundle

X' projective manifold

$f_t : X' \to X_t$ holomorphic family of generically finite surjective holomorphic maps. Then,

$$\boxed{\begin{array}{c}\text{There exist } \varphi_t : X_0 \xrightarrow{\cong} X_t \\ \text{such that } f_t \equiv \varphi_t \circ f_0\end{array}}$$

Application of Cartan-Fubini

Theorem (Hwang-Mok, AJM 2004)

X Fano manifold; $b_2(X) = 1$

$\mathcal{K}$: minimal rational component

$\mathcal{C}_x$: VMRT of $(X, \mathcal{K})$, $x \in X$ generic

Y projective manifold

$f_t : Y \to X$ one-parameter family
of surjective finite holomorphic maps.

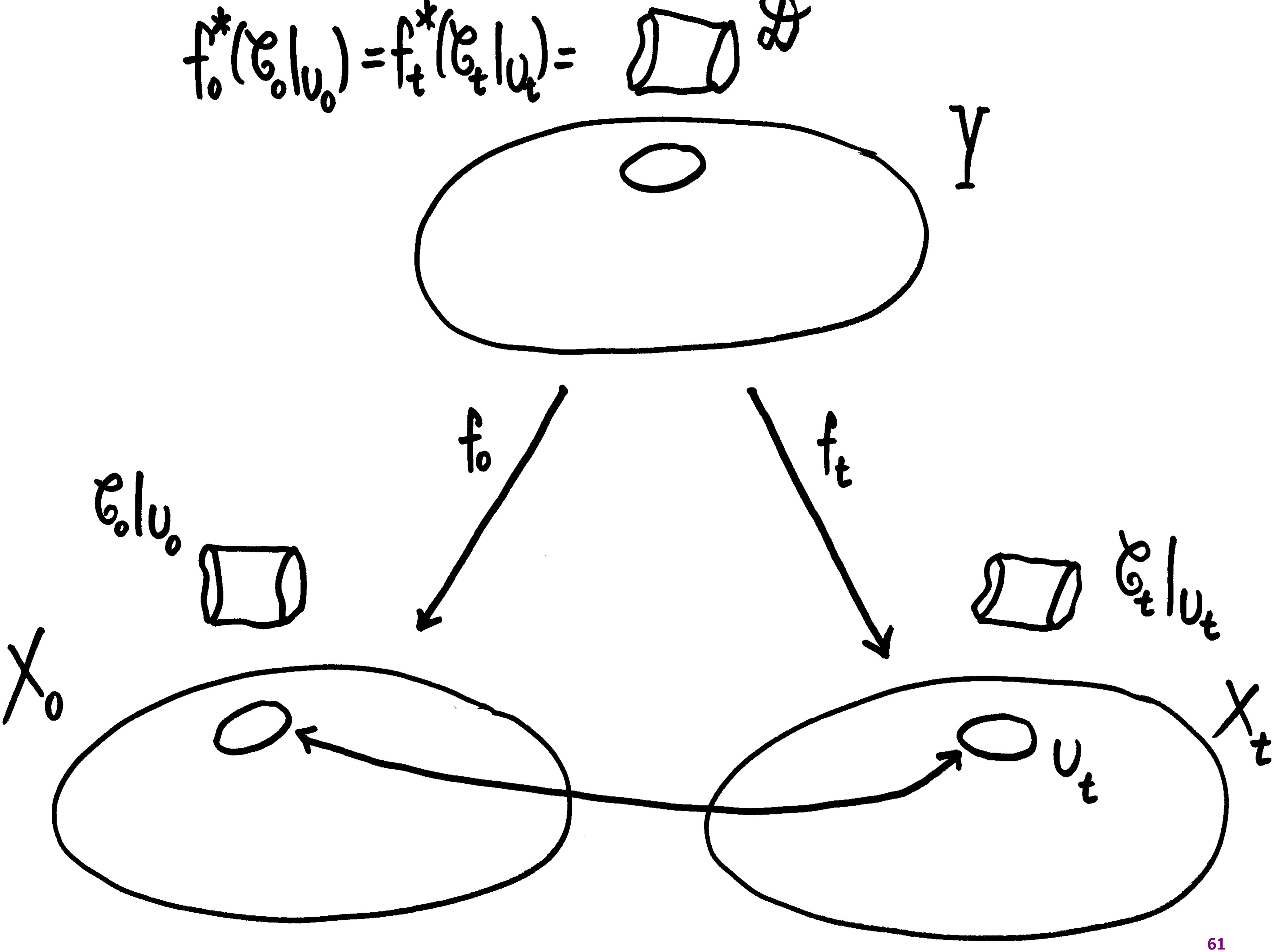
Assume $\dim \mathcal{C}_x := p > 0$, and

$\mathcal{C}_x \subset \mathbb{P}T_x(X)$ is non-linear. Then,

---

$\exists \Phi_t \in \mathrm{Aut}(X)$ such that

$f_t \equiv \Phi_t \circ f_0;\ \Phi_0 = id.$



$f_0^*(\mathcal{C}_0|_{U_0}) = f_t^*(\mathcal{C}_t|_{U_t}) =$
$\mathcal{D}$
Y
f_0
f_t
$\mathcal{C}_0|_{U_0}$
$\mathcal{C}_t|_{U_t}$
X_0
X_t
U_t

Theorem (Hwang-Mok 2004, AJM). *Local rigidity for* $f_t : Y \to X_t$ *remains valid under the assumption that* X_0 *carries a minimal component* $\mathcal{K}_0$ *whose general VMRT is non-linear.*

<u>New solution of Lazarsfeld Problem</u>

$Y = G/P$ G simple, P maximal parabolic

Take $X_t = X$, $f : Y \to X$.

Assume generic $\mathcal{C}_x \subset \mathbb{P}T_x(X)$ non-linear.

Local rigidity $\Rightarrow$ Any holomorphic vector field $\mathcal{Z}$ on Y descends to a holomorphic vector field $\mathcal{W}$ on X such that $f : Y \to X$ is equivariant w.r.t. 1-parameter groups generated by $\mathcal{Z}$ and $\mathcal{W}$.

$R :=$ ramification divisor of f

$B := f(R)$

Then, $\mathcal{W}$ is tangent to B.
Hence, $\mathcal{Z}$ is tangent to R,
contradicting homogeneity of $Y = G/P$!

Bounding degrees of holomorphic maps

X' projective manifold

$\mathcal{F}_0 = \{X \text{ Fano: } \mathrm{Pic}(X) \cong \mathbb{Z}; \exists \text{ rat. curve}$
$C \subset X \text{ with } \textit{trivial} \text{ normal bundle}\}$

Then,

There exists a constant $C(X')$ such that
$\forall f : X' \to X,\ X \in \mathcal{F}_0$
generically finite, surjective hol. map
$\deg(f) \leq C(X')$.

Finiteness Theorem

Given X', there exists at most *finitely many* pairs (X, f) of such maps $f : X' \to X$.

Finiteness Theorem in 3 dimensions

Y Fano manifold, $\mathrm{Pic}(Y) \cong \mathbb{Z}$, $\dim Y = 3$.

Then, there are at most *finitely many* projective manifolds X for which there exists a surjective holomorphic map

$$f : Y \to X \ .$$

Proof.
From sol'n to Lazarsfeld's Problem,

$$Y \cong \mathbb{P}^3 \Rightarrow X \cong \mathbb{P}^3;$$
$$Y \cong Q^3 \Rightarrow X \cong Q^3 \text{ or } \mathbb{P}^3.$$

Otherwise, Y carries a rational curve with trivial normal bundle, from Iskovskih's classification. Then,

$$X \cong \mathbb{P}^3,\ Q^3 \text{ or}$$

a *finite* no. of possibilities in $\mathcal{F}_0$.

Webs on a Fano manifold

$\mathcal{F}_0 = \{X$ Fano: $Pic(X) \cong \mathbb{Z}; \exists$ a rat. curve $C \subset X$ with *trivial* normal bundle$\}$

$X \in \mathcal{F}_0$, $C \subset X$, $N_{C|X} \cong \mathcal{O}^{n-1}$

$\mathcal{K} =$ minimal rational component, $[C] \in \mathcal{K}$.

$\mu : \mathcal{U} \to X$, $\rho : \mathcal{U} \to \mathcal{K}$ universal family

$X \in \mathcal{F}_0 \Leftrightarrow$ For $\pi : \mathcal{C} \to X$ of VMRTs, $\dim \mathcal{C}_x = 0$ for x generic.

$\mathcal{R} \subset \mathcal{U}$ ramification divisor,
$M = \mu(\mathcal{R}) \subset X$ branching divisor

$M :=$ *discriminantal divisor* of $\mathcal{K}$.

$L \subset X$ smallest hypersurface such that $\pi : \mathcal{C} \to X$ is unramified over $X - L - Z$ for some $Z \subset X$ of codim. ≥ 2, $M \subset L$.

$L :=$ *extended discriminantal divisor* of $\mathcal{K}$

Principal properties on webs

♦ $f : X' \to X$ gen. finite surj. hol. map, $\mathcal{K}$ web of rational curves on X
 $\Rightarrow$ $f^{-1}\mathcal{K}$ finite union of webs of rational curves on X'.

♦ $f^{-1}\mathcal{K} := \mathcal{K}' = \mathcal{K}_1 \cup \cdots \cup \mathcal{K}_m$
$L' := L_{\mathcal{K}_1} \cup \cdots \cup L_{\mathcal{K}_m}$, etc.
Then,

$$f^{-1}(L) \subset L' \ .$$

Solution to the Frankel Conjecture:

Theorem (Siu-Yau 1980).
(X, g) *compact Kähler, Bisect* $(X, g) > 0$
$\Rightarrow X \cong \mathbb{P}^n$.

Solution to the Generalized Frankel Conjecture:

Theorem (Mok 1988).
(X, g) *compact Kähler, Bisect* $(X, g) \geq 0$
$\Rightarrow \tilde{X} \cong \mathbb{C}^m \times$ *Hermitian symmetric space of compact type.*

For X *Fano, we have*

$X \cong$ *Hermitian symmetric space of compact type.*

Solution to the Harshorne Conjecture:

Theorem (Mori 1979).
X projective manifold, T_X ample
$\Rightarrow X \cong \mathbb{P}^n$.

How about a "Generalized Hartshorne Conjecture"?

Conjecture (Campana-Peternell 1991).
X Fano manifold, T_X numerically effective
$\Rightarrow X \cong$ *rational homogeneous space*

Solved for dim ≤ 3 independently by Campana-Peternell and Fangyuan Zheng:

Case of 3 dimensions:

$X \cong \mathbb{P}^3$, Q^3 , $\mathbb{P}^1 \times \mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ or $\mathbb{P}(T_{\mathbb{P}^2})$

Theorem (Mok 2002, Trans. AMS).

X projective manifold

$b_2(X) = b_4(X) = 1$,

$T_X \geq 0$ *(numerically effective).*

Suppose $\dim \mathcal{C}_x = 1$ *for x generic.*

Then,

$$X \cong \mathbb{P}^2 \ , \ Q^3 \quad or \quad K(G_2) \ ,$$

where $K(G_2)$ = 5-dimensional Fano contact homogeneous manifold associated to the exceptional Lie group G_2.

Theorem (Hwang 2004).

The condition $b_4 = 1$ can be dropped.

Campana-Peternell 1993

Their conjecture is valid in dimension 4 except for the possible exception of a Fano manifold X of Picard number 1 with nef tangent bundle such that $c_1(X) = 1$ (i.e. positive generator of $\mathrm{Pic}(X) \cong \mathbb{Z}$).

Elimination of the exceptional case $c_1 = 1$

$p = 0$ implies the existence of a 1-dim (hence integrable) distribution spanned by VMRTs, contradicting $b_2 = 1$

$p = 1$ ruled out by Mok + Hwang's improvement

$p = 2$ would contradict Miyaoka's characterization of the hyperquadric

$p = 3$ ruled out by the characterization of projective spaces of Cho-Miyaoka-Shepherd-Barron, Kebekus

Theorem (Mok 2006, Hong-Hwang 2007). *Let $S = G/P$ be a rational homogeneous manifold of Picard number 1 corresponding to a long simple root α. (We say that S is of type $(\mathfrak{g}, \alpha)$), $S \ncong \mathbb{P}^n$.*

Let X be a Fano manifold of Picard number 1 admitting a component $\mathcal{K}$ of minimal rational tangents. Write

$$\mathcal{C}_0(S) \subset \mathbb{P}T_o(S) \ , \quad o \in S \quad \textit{reference point} \ ;$$
$$\mathcal{C}_x(\mathcal{K}) \subset \mathbb{P}T_x(X) \ , \quad x \in X \quad \textit{general point}$$

for varieties of minimal tangents. Then,

$$\boxed{\begin{array}{l} \mathcal{C}_x(\mathcal{K}) \subset \mathbb{P}T_o(X) \quad \textit{congruent to} \\ \mathcal{C}_0(S) \subset \mathbb{P}T_o(S) \\ \qquad \Rightarrow \quad \boxed{X \cong S} \end{array}}$$

Ideas of proof

- parallel transport along tautological liftings $\hat{C}$ of minimal rational tangents
- behavior of second fundamental forms σ of $\mathcal{C}_x \subset \mathbb{P}T_x(X)$ invariant under parallel transport, hence kernels, images, etc. are invariant.
- $\mathcal{C}_o \subset \mathbb{P}T_o(S)$ are quadratic or cubic Hermitian symmetric subspaces. If irreducible and of rank > 1 the G-structure on $\mathcal{C}_o$ is determined by second and third fundamental forms σ and κ, which determine $\mathcal{C}_{[\alpha]}(\mathcal{C}_o)$.
- In the reducible case transversal foliations are preserved by parallel transport.
- The special case of the second Veronese embedding of a projective space can be recovered from the surjectivity of the second fundamental form σ.

Theorem (Hwang-Mok 2004, JAG).
X Fano manifold, $\mathrm{Pic}(X) \cong \mathbb{Z}$*.*
M an irreducible component of the space of minimal rational curves.
$M^x \subset M$ subset of members of M passing through a general point $x \in X$.
If M^x is irreducible, and $\dim(M^x) \geq 2$*.*
Then, $Aut_0(X) = Aut_0(M)$.

Remarks. Theorem fails when $\dim(M^x) = 0, 1$.

Examples:

(a) $\dim(M^x) = 0$. Take $X = \mathrm{codim} - 3$ general linear section of $G(2, 3)$, $M \cong \mathbb{P}^2$

$$\mathrm{Aut}_0(X) \cong \mathbb{P}SL(2, \mathbb{C});$$
$$\mathrm{Aut}_0(M) \cong \mathbb{P}SL(3, \mathbb{C}).$$

(b) $\dim(M^x) = 1$. Take $X = Q_3$, $M \cong \mathbb{P}^3$

$$\mathrm{Aut}_0(X) \cong \mathbb{P}SO(5, \mathbb{C});$$
$$\mathrm{Aut}_0(X) \cong \mathbb{P}SL(4, \mathbb{C}).$$

Most recent results:

Holomorphic Lagrangian fibrations (Hwang 2007)

Let Z be a projective irreducible symplectic manifold and $\pi : Z \to X$ a surjective holomorphic map onto a projective manifold with connected and positive-dimensional fibers. Then $X \cong \mathbb{P}^n$.

Remark

By the work of Matsushita a general fiber is an Abelian variety, the underlying subvariety of every fiber of f is Lagrangian, and X is a Fano manifold of Picard number 1. In particular, X can be studied by means of geometric structures associated to VMRTs.

Generalized Lazarsfeld Problem (Lau, to appear in J. Alg. Geom.)

Resolution of the generalized Lazarsfeld Problem for surjective holomorphic maps $\pi : G/Q \to X$, X smooth, G any semisimple complex Lie groups, $Q \subset G$ any parabolic subgroup.
(*Currently generalized also to semisimple* G.)

Non-equidimensional Cartan-Fubini extension (Hong-Mok 2008)

(a) Analytic continuation for germs of holomorphic maps $f : (Z, z_0) \to (X, x_0)$ for uniruled projective manfiolds equipped with VMRTs such that

$$df(\widetilde{\mathcal{C}}_{Z,z}) = \widetilde{\mathcal{C}}_{X,f(z)} \cap T_{f(z)}(X)$$

for z on some neighborhood U of z_0. We say that f *respects* VMRTs.

(b) Application of non-equidimensional Cartan Fubini to the characterization of certain holomorphic embeddings $G/P \hookrightarrow G'/P'$ in terms of the VMRT-respecting property.

Open Problems

(1) *Irreducibility of VMRTs*

Conjecture: X uniruled, projective $\mathcal{K}$ minimal rational component, $p(X, \mathcal{K}) > 0$. Then, $\mathcal{C}_x$ is *irreducible* for generic in X.

Special case:

If $\mathcal{C}_x$ is a union of projective linear subspaces and $p(X, \mathcal{K}) > 0$, then $\mathcal{C}_x$ is irreducible.

Consequence of special case

$f : X' \to X$ a generically finite map onto a Fano manifold X of Picard number 1, $X \not\cong \mathbb{P}^n$. Then f is locally rigid when X' is fixed and X is allowed to vary.

(2) *Contact Fano manifolds*

Conjecture: X Fano, $\text{Pic}(X) \cong \mathbb{Z}$, equipped with a contact structure

$\Rightarrow$ X rational homogeneous.

(3) *Finite holomorphic maps*

Conjecture: X, Y n-dim. Fano manifolds of Picard number 1, $X, Y \not\cong \mathbb{P}^n$. $f : X \to Y$ finite holomorphic map. Then, $\deg(f) \leq \mathrm{Const}(X, Y)$.

Consequence

$X \not\cong \mathbb{P}^n$, $\mathrm{Pic}(X) \cong \mathbb{Z} \;\Rightarrow\; \mathrm{End}(X) = \mathrm{Aut}(X)$

(4) *Vector Fields*

Conjecture: X Fano, $\mathrm{Pic}(X) \cong \mathbb{Z}$. Then,

(a) At a general point $\nexists$ holomorphic vector fields vanishing to the order ≥ 3.

(b) $\dim\big(\mathrm{Aut}(X)\big) < n^2 + 2n$ unless $X \cong \mathbb{P}^n$.

(5) *Moduli space of minimal rational curves*

X Fano manifold of Picard number 1, $\mathcal{K}$ a minimal rational component, $x \in X$ general point, $\rho : \mathcal{U} \to \mathcal{K}$ universal family, $\mathcal{C}_x =$ VMRT at x.

Conjecture (on pseudoconcavity):

Suppose every $\mathcal{K}$-curve through x is standard and embedded. Write $\mathcal{Q}_x := \rho(\mathcal{C}_x) \subset \mathcal{K}$. Then, every meromorphic function defined on some open neighborhood $\mathbb{U} \supset \mathcal{Q}_x$ extends meromorphically to $\mathcal{K}$.