GEOMETRIC STRUCTURES ON UNIRULED PROJECTIVE MANIFOLDS DEFINED BY THEIR VARIETIES OF MINIMAL RATIONAL TANGENTS

by

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Abstract. — In a joint research programme with Jun-Muk Hwang we have been investigating geometric structures on uniruled projective manifolds, especially Fano manifolds of Picard number 1, defined by varieties of minimal rational tangents associated to moduli spaces of minimal rational curves. In this article we outline a heuristic picture of the geometry of Fano manifolds of Picard number 1 with non-linear varieties of minimal rational tangents, taking as hints from prototypical examples such as those from holomorphic conformal structures. On an open set in the complex topology the local geometric structure associated to varieties of minimal rational tangents is equivalently given by families of local holomorphic curves marked at a variable base point satisfying certain compatibility conditions. Differential-geometric notions such as (null) geodesics, curvature and parallel transport are a source of inspiration in our study. Formulation of problems suggested by this heuristic analogy and their solutions, sometimes in a very general context and at other times applicable only to special classes of Fano manifolds, have led to resolutions of a series of well-known problems in Algebraic Geometry.

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Résumé. — Dans un programme de recherche avec Jun-Muk Hwang nous avons étudié des structures géométriques sur les variétés projectives uniréglées, en particulier les variétés de Fano de nombres de Picard egaux à 1, definies par les variétés de tangentes rationnelles minimales associées aux espaces de modules de courbes rationnelles minimales. Dans cet article nous esquissons un dessin heuristique sur la géométrie des variétés de Fano de nombres de Picard egaux à 1 dont les variétés de tangentes rationnelles minimales sont non linéaires, en prenant comme prototypes les exemples tels ques les structures conformes holomorphes. Dans un ouvert par rapport à la topologie complexe, la structure géométrique associée aux variétés de tangentes rationnelles minimales equivaut aux données de familles de courbes holomorphes locales marquées à un point de base variable vérifiant des conditions de compatibilité. Des notions de la géometrie différentielle comme les géodesiques (nulles), la courbure et le transport parallèle constituent une source d'inspiration dans notre étude. Des formulations de problèmes suggérés par cette analogie heuristique et leurs solutions, des fois dans un contexte très générale et des fois applicables seulement aux classes de variétés de Fano spéciales, ont conduit a des résolutions d'une série de problèmes bien connus en géométrie algébrique.

1. Introduction

1.1. Background and motivation. — In 1979, Mori [45] established the fundamental existence result on rational curves on a projective manifold where the canonical line bundle is not numerically effective, thereby resolving the Hartshorne Conjecture. When the manifold is Fano, Miyaoka-Mori [38] (1986) proved that the manifold is uniruled. In a joint research programme undertaken with Jun-Muk Hwang, we have been studying algebro-geometric and complex-analytic problems on uniruled projective manifolds basing on geometric objects arising from special classes of rational curves, viz., minimal rational curves. In this article the author would like to highlight some geometric aspects of the underlying theory.

Given a uniruled projective manifold X and fixing an ample line bundle L, by a minimal rational curve we will mean a free rational curve of minimal degree with respect to L among all free rational curves. A connected component $\mathcal{K}$ of the space of minimal rational curves will be called a minimal rational component. In practice we will fix a minimal rational component $\mathcal{K}$ and consider only minimal rational curves belonging to $\mathcal{K}$. Associated to $\mathcal{K}$, there is the universal family $\rho : \mathcal{U} \to \mathcal{K}$, $\mu : \mathcal{U} \to X$, where $\rho : \mathcal{U} \to \mathcal{K}$ is a holomorphic $\mathbb{P}^1$-bundle, and $\mu : \mathcal{U} \to X$ is the evaluation map. In connection with $\mathcal{U}$ there is the tangent map $\tau : \mathcal{U} \to \mathbb{P}T_X$. For a minimal rational curve C marked at $x \in X$ and immersed at the marking, and for α denoting a nonzero vector tangent to C at the marking, the tangent map associates to the marked point the element $[\alpha] \in \mathbb{P}T_x(X)$. For a general point $x \in X$ we define the variety of minimal rational tangents (VMRT) $\mathcal{C}_x$ at x to be the strict transform of the tangent map $\tau_x : \mathcal{U}_x \to \mathbb{P}T_x(X)$. The basic set-up of our study takes place on the total space of the double fibration given by the universal family $\rho : \mathcal{U} \to X$, $\mu : \mathcal{U} \to X$, equipped with the tangent map $\tau : \mathcal{U} \to \mathbb{P}T_x(X)$ and the fibered space $\pi : \mathcal{C} \to X$ of VMRTs. The overriding question is the extent to which a uniruled projective manifold X is determined by its VMRTs.

Given a uniruled projective manifold $(X, \mathcal{K})$ equipped with a minimal rational component $\mathcal{K}$, and a connected open subset $U \subset X$ in the complex topology, we consider $(U, \mathcal{C}|_U)$ as a complex manifold equipped with a geometric structure. Here the term 'geometric structure' is understood by analogy to standard examples. As a prototype in the context of smooth manifolds, a real m-dimensional Riemannian manifold (M, g) can be understood as one equipped with a reduction of the frame bundle from the structure group $\mathrm{GL}(m, \mathbb{R})$ to $\mathrm{O}(m)$. In the context of complex manifolds, a simplest example of a *holomorphic* geometric structure relevant to the study of uniruled projective manifolds is the case of holomorphic conformal structures, alias hyperquadric structures. A holomorphic conformal structure on an n-dimensional complex manifold X determines at every point $x \in X$ its null-cone, defining equivalently a holomorphic fiber subbundle $\mathcal{Q} \subset \mathbb{P}T_X$ consisting of fibers $\mathcal{Q}_x$ isomorphic to an $(n-2)$-dimensional hyperquadric. It corresponds to a reduction of the holomorphic frame bundle from $\mathrm{GL}(n; \mathbb{C})$ to $\mathbb{C}^* \cdot \mathrm{O}(n; \mathbb{C})$, and this reduction is completely determined by $\mathcal{Q} \subset \mathbb{P}T_X$. When $X = Q^n$, the n-dimensional hyperquadric, $\mathcal{Q}_x$ agrees with the VMRT $\mathcal{C}_x$, and by analogy we speak of the geometric structure on a uniruled projective manifold $(X, \mathcal{K})$ equipped with a minimal rational component as defined by its fibered space $\pi : \mathcal{C} \to X$ of VMRTs. As our geometric study of VMRTs are in many cases motivated by differential-geometric consideration, especially in relation to global properties that can be captured by local differential-geometric information, we will be considering a general point $x \in X$, and the local geometric structure defined by the germ of the fibered space $\pi : \mathcal{C} \to X$ at x, equivalently the restriction $\pi|_U : \mathcal{C}|_U :\to U$ to arbitrarily small Euclidean open neighborhoods U of x.

1.2. A heuristic picture. — While a substantial part of our programme applies generally to any uniruled projective manifold, our focus of investigation has been primarily on those of Picard number 1. These manifolds, which are necessarily Fano, are not amenable to further reduction by means of extremal rays in Mori theory, and as such they are called 'hard nuts' among Fano manifolds in Miyaoka [36]. Our geometric theory on uniruled projective manifolds based on VMRTs serve in particular as a basis for a systematic study of all Fano manifolds of Picard number 1. There emerges a dichotomy between those for which the VMRT at a general point is the union of finitely many projective linear subspaces and the rest. We will say that $(X, \mathcal{K})$ has linear VMRTs in the former case and non-linear VMRTs otherwise. The linear case includes those for which VMRT at a general point is 0-dimensional, where the fibered space $\pi : \mathcal{C} \to X$ gives rise to a geometry on X resembling that of web geometry. We will discuss in this article exclusively the non-linear case and refer the reader to Hwang-Mok [20] (2003) for results in the case of 0-dimensional VMRTs, and to Hwang [13] (2007) for a problem which necessitates the study of the hypothetical case of linear VMRTs of higher dimensions.

At this stage of investigation we have the following heuristic picture in the case of non-linear VMRTs. The universal $\mathbb{P}^1$-bundle $\rho : \mathcal{U} \to \mathcal{K}$ associated to the minimal rational component $\mathcal{K}$ gives rise via the tangent map to a tautological multi-foliation on the fibered space $\pi : \mathcal{C} \to X$ of VMRTs, and the 'local' geometric structure $(U, \mathcal{C}|_U)$ on open subsets $U \subset X$ in the complex topology corresponds to the data of families of local holomorphic curves marked at points $x \in U$. The local holomorphic curves are then solutions to a system of partial differential equations which in the case of holomorphic conformal structures correspond to the null geodesics. We may think of the local holomorphic curves as analogues of (null) geodesics. The fact that these 'geodesics' can be extended to minimal rational curves on $(X, \mathcal{K})$ should impose serious constraints on the underlying geometric structure. In the case of the holomorphic conformal structure on the hyperquadric, the splitting type of the tangent bundles on minimal rational curves is enough to force the vanishing of the holomorphic Bochner-Weyl tensor and thus to force flatness of the structure. In the general case of $(X, \mathcal{K})$, for a general $\mathcal{K}$-minimal rational curve the normal bundle has only direct summands of degree 1 or 0. Such a rational curve, to be called a standard rational curve, resembles minimal rational curves on a hyperquadric, and there ought to be partial 'flatness' of the geometric structure of $(X, \mathcal{C})$ along standard rational curves which places serious restrictions on geometric structures that can possibly arise from VMRTs. The heuristic analogy between minimal rational curves and (null) geodesics also goes further as the former should serve to propagate geometric information from a germ of geometric structure to the ambient Fano manifold X of Picard number 1. In this case, any two general points can be connected by a chain of minimal rational curves, and the bad set of 'inaccessible points' must be of codimension ≥ 2.

A further geometric concept that ought to play an important role in the study of geometric structures defined by VMRTs is the notion of parallel transport along a standard rational curve. In the special case of irreducible Hermitian symmetric spaces of the compact type the VMRTs are invariant under parallel transport with respect to any choice of a canonical Kähler-Einstein metric. For Fano manifolds of Picard number 1, endowed with geometric structures arising from VMRTs but without privileged local holomorphic connections the only general source for the notion of parallel transport arises from splitting types over minimal rational curves. In this direction it is found that for the germ of families of VMRTs along the tautological lifting $\widehat{C}$ of a standard rational curve, the second fundamental in the fiber directions can be identified as a section of a flat bundle over C, and as such one can speak of the parallel transport of second fundamental forms along a standard rational curve.

Other than geometric structures defined by VMRTs, in important classes of Fano manifolds X of Picard number 1 there are additional underlying structures with differential-geometric meaning. These are the cases where the VMRTs are positive-dimensional, irreducible and linearly degenerate at a general point. They span distributions which give rise to differential systems by taking Lie brackets. The study of this class of manifolds, which is particularly important for questions on deformation rigidity, reveals an intimate link between issues of integrability and projective-geometric properties of the VMRT at a general point.

1.3. Summary and presentation of results. — While some aspects of the overall heuristic picture on geometric structures defined by VMRTs can be confirmed to a large extent, other aspects are only beginning to be explored. In the research programme emphasis has been placed on solutions of concrete problems, and in some cases confirmation of some conjectural properties on VMRTs can lead to important consequences. Here we describe general results and highlights of applications that fall within the framework of the heuristic picture discussed.

For the prototypical examples of geometric structures on irreducible Hermitian symmetric spaces S of the compact type and of rank ≥ 2, Ochiai's result [47] (1970) can be interpreted as saying that a local VMRT-preserving holomorphic map necessarily extends to an automorphism of S. In Hwang-Mok [17] (1999), [18] (2001) we established the analogous phenomenon, which we call Cartan-Fubini extension, for Fano manifolds of Picard number 1 with positive-dimensional VMRTs under the additional assumption that the Gauss map of the VMRT is generically finite, proving at the same time that the tangent map at a general point is birational under the same assumption. In conjunction with the works of Kebekus [25] (2002) on the tangent map and Cho-Miyaoka-Shepherd-Barron [3] (2002) on a characterization of the projective space in terms of minimal rational curves we proved in Hwang-Mok [22] (2004) that the same results hold true for any Fano manifold of Picard number 1 with non-linear VMRTs at a general point, resulting in a new solution of the Lazarsfeld Problem in [32] (1984) regarding finite holomorphic maps on rational homogeneous spaces G/P of Picard number 1 (Hwang-Mok [22]). Cartan-Fubini extension has recently been extended to non-equidimensional VMRT-respecting local holomorphic maps between uniruled projective manifolds in Mok [43] and Hong-Mok [9] with applications to the characterization of certain submanifolds saturated with respect to minimal rational curves, in analogy to totally geodesic submanifolds in Riemannian geometry.

The idea of exploiting the splitting type of the tangent bundle over standard rational curves to prove vanishing theorems on curvature has given rise to a characterization of irreducible Hermitian symmetric spaces S of the compact type and of rank ≥ 2 as the unique uniruled projective manifolds admitting G-structures for reductive complex Lie groups G (Hwang-Mok [14] (1997), leading also to an analogous result of Hong [6] (2000) for geometric structures modeled after Fano homogeneous contact manifolds of Picard number 1. The idea of parallel transport of second fundamental forms was first used in relation to the Campana-Peternell Conjecture, leading to the characterization of Fano manifolds of Picard number 1 with 1-dimensional VMRTs and nef tangent bundle under the additional assumption that the fourth Betti number equals 1 (Mok [41], 2001), a condition that was removed in Hwang [12] (2007), resulting together with earlier works in the confirmation of the Campana-Peternell Conjecture for 4 dimensions. The same idea was further exploited to yield for rational homogeneous manifolds G/P of Picard number 1 defined by long simple roots a characterization of G/P by the VMRT at a general point (Mok [42] and Hong-Hwang [8]). The study of distributions spanned by irreducible linearly degenerate VMRTs has led to projective-geometric necessary conditions on such VMRTs (Hwang-Mok [15], 1998; [17], 1999), and applications of such results to deformation of complex structures are important in the final confirmation of rigidity of rational homogeneous manifolds G/P of Picard number 1 under Kähler deformation (Hwang-Mok [23] (2005) and references therein). Another important element in relation to deformation rigidity is the study of Lie algebras of holomorphic vector fields by means of prolongation theory for infinitesimal automorphisms of VMRTs.

In the current article results falling within the general geometric framework described revolving around the geometry of VMRTs will be stated and discussed, with (sketches of) proofs of special cases for the purpose of illustration, in an order different from the above that conforms more (but not strictly) to the chronology. The reader may consult Hwang-Mok [17], Hwang [11] (2000) for more systematic overviews at earlier stages of the programme, Mok [40] (1999) for aspects of the theory in relation to G-structures, Hwang-Mok [22] for general results on the tangent map, and Hwang [12] (2007) for an overview on rigidity of rational homogeneous manifolds. We have completely omitted the important role played by VMRTs on the geometry of moduli spaces of stable vector bundles on an algebraic curve, for which the reader is referred to Hwang-Ramanan [24] (2004) and the references contained therein.

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2. Varieties of minimal rational tangents

2.1. Minimal rational curves. — By a projective $\mathbb{P}^1$-fibered space $\nu : Z \to B$ we mean an irreducible reduced projective variety Z equipped with a surjective holomorphic map ν onto a projective variety B, such that the general fiber of ν is an algebraic curve of genus 0, i.e., isomorphic to the Riemann sphere $\mathbb{P}^1$. A projective manifold X is said to be uniruled if there exists a projective $\mathbb{P}^1$-fibered space $\nu : Z \to B$ and a dominant holomorphic map $\varphi : Z \to X$ onto X. By restricting ν to a properly chosen subvariety of B of dimension equal to $\dim(X) - 1$, without loss of generality we may assume that the dominant holomorphic map $\varphi : Z \to X$ is generically finite. Replacing Z by its normalization we may also assume that Z is a projective manifold. By Miyaoka-Mori [38] (1986) any Fano manifold is uniruled.

By a parametrized rational curve on a projective manifold X we mean a nonconstant holomorphic map $f : \mathbb{P}^1 \to X$ from the Riemann sphere $\mathbb{P}^1$ into X. We say that two parametrized rational curves f_1 and f_2 are equivalent if and only if they are the same up to a re-parametrization of $\mathbb{P}^1$, i.e., if and only if there exists $\gamma \in \mathrm{Aut}(\mathbb{P}^1)$ such that $f_2 = f_1 \circ \gamma$. By a rational curve we mean an equivalence class $[f]$ of parametrized rational curves $f : \mathbb{P}^1 \to X$ under this equivalence relation. We will sometimes also refer to the nontrivial image $f(\mathbb{P}^1) = C$ (as a cycle) as a rational curve.

Let X be a uniruled projective manifold and fix an ample line bundle L on X. By the degree of an algebraic curve C on X will will mean the degree of C with respect to L, i.e., the integral of a (positive) curvature form of L over C. Let $\varphi : Z \to X$ be a generically finite dominant holomorphic map from a projective $\mathbb{P}^1$-fibered space $\nu : Z \to X$ onto X where Z is nonsingular. From the surjectivity of $\varphi : Z \to X$ it follows that for a general $\mathbb{P}^1$-fiber E of $\nu : Z \to X$, $\lambda : \mathbb{P}^1 \cong E$, and for the parametrized rational curve $f : \mathbb{P}^1 \to X$ defined by $f := \varphi \circ \lambda$, the holomorphic vector bundle f^*T_X must be spanned by global sections at a general point. By the Grothendieck Splitting Theorem any holomorphic vector bundle over $\mathbb{P}^1$ splits into the direct sum of holomorphic line bundles, and it follows that f^*T_X is nonnegative in the sense that it is a direct sum of holomorphic line bundles of degree ≥ 0.

By a free rational curve on X we mean the equivalence class of a nonconstant holomorphic map $f : \mathbb{P}^1 \to X$ such that f^*T_X is nonnegative. From the above discussion it follows that any uniruled projective manifold admits a free rational curve. Conversely, if a projective manifold X admits a free rational curve parametrized as $f : \mathbb{P}^1 \to X$, then $H^0(\mathbb{P}^1, f^*T_X)$ is spanned by global sections, and $H^1(\mathbb{P}^1, f^*T_X) = 0$ since $H^1(\mathbb{P}^1, \mathcal{O}(k)) = 0$ whenever $k \geq -1$, so that there is no obstruction in the deformation of $f : \mathbb{P}^1 \to X$ as a parametrized rational curve. By deforming f and considering Chow spaces it follows readily that there exists a projective $\mathbb{P}^1$-fibered space $\nu : Z \to B$ such that Z dominates X. As a consequence, a projective manifold X is uniruled if and only if X admits a free rational curve.

By a minimal rational curve on X we will mean a free rational curve of minimal degree among all free rational curves on X. The set of minimal rational curves can be given naturally the structure of a complex manifold, a connected component of which will be called a minimal rational component $\mathcal{K}$. A rational curve belonging to $\mathcal{K}$ will sometimes be called a $\mathcal{K}$-curve. The degree of $\mathcal{K}$, to be denoted by $\deg(\mathcal{K})$, is the degree of one and hence any $\mathcal{K}$-curve.

For a general reference on rational curves in Algebraic Geometry we refer the reader to Kollár [29]. The reader may also consult Hwang-Mok ([15], §2; [17], (1.1)) for basic facts on the deformation theory of rational curves relevant to our discussion.

2.2. The universal family of $\mathcal{K}$-curves and the canonical double fibration. — Associated to $(X, \mathcal{K})$ there is the universal family $\rho : \mathcal{U} \to \mathcal{K}$ of $\mathcal{K}$-curves, where $\mathcal{U}$ is smooth and $\rho : \mathcal{U} \to \mathcal{K}$ is a holomorphic $\mathbb{P}^1$-bundle, constructed as follows. Let $\mathcal{H}$ be the connected component of the space of all parametrized free rational curves $f : \mathbb{P}^1 \to X$ such that $\mathcal{K} = \mathcal{H}/\mathrm{Aut}(\mathbb{P}^1)$. Since f^*T_X is nonnegative, the obstruction group $H^1(\mathbb{P}^1, f^*T_X) = 0$, hence $\mathcal{H}$ carries naturally the structure of a complex manifold with tangent spaces $T_f(\mathcal{H}) = H^0(\mathbb{P}^1, f^*T_X)$. Recall that $\mathcal{K}$ is the quotient of $\mathcal{H}$ by the group $\mathrm{Aut}(\mathbb{P}^1)$, which acts on $\mathcal{H}$ by setting $\gamma(f) = f \circ \gamma$ for $\gamma \in \mathrm{Aut}(\mathbb{P}^1)$ and $f \in \mathcal{H}$. By the minimality of $\mathcal{K}$ any $f \in \mathcal{H}$ must be generically injective, from which it follows that $\mathrm{Aut}(\mathbb{P}^1)$ acts effectively on $\mathcal{H}$, so that $\mathcal{K}$ inherits the structure of a complex manifold with $T_{[f]}(\mathcal{K}) = H^0(\mathbb{P}^1, f^*T_X)\big/df\big(H^0\big(\mathbb{P}^1, T_{\mathbb{P}^1}\big)\big)$. The canonical projection $p : \mathcal{H} \to \mathcal{K}$ realizes $\mathcal{H}$ as a principal $\mathrm{Aut}(\mathbb{P}^1)$-bundle over $\mathcal{K}$. $\mathrm{Aut}(\mathbb{P}^1) \cong \mathrm{SL}(2, \mathbb{C})/\{\pm I\}$ is a 3-dimensional complex Lie group which acts transitively on $\mathbb{P}^1$, and we can represent $\mathbb{P}^1 \cong \mathrm{Aut}(\mathbb{P}^1)/\mathrm{Aut}(\mathbb{P}^1; 0))$, as a homogeneous space, where $\mathrm{Aut}(\mathbb{P}^1; 0)) \subset \mathrm{Aut}(\mathbb{P}^1)$

is the (2-dimensional) isotropy subgroup at $0 \in \mathbb{P}^1$. Define $\mathcal{U} := \mathcal{H}/\mathrm{Aut}(\mathbb{P}^1; 0)$. Associated to the principal $\mathrm{Aut}(\mathbb{P}^1)$-bundle $p : \mathcal{H} \to \mathcal{K}$ we have thus a holomorphic bundle of homogeneous spaces $\rho : \mathcal{U} \to \mathcal{K}$ with fibers $\mathrm{Aut}(\mathbb{P}^1)/\mathrm{Aut}(\mathbb{P}^1; 0)) \cong \mathbb{P}^1$, which gives the universal family $\pi : \mathcal{U} \to \mathcal{K}$.

It can be proven that as a complex manifold $\mathcal{K}$ is biholomorphic to a quasi-projective manifold. In fact, there is a canonical injective holomorphic map from $\mathcal{K}$ into the Chow space of X whose image is a dense Zariski-open subset $\mathcal{K}_0$ of a projective subvariety $\mathcal{Q}$ of some irreducible component of the Chow space of X. Thus, $\mathcal{K}$ can be identified as the normalization of $\mathcal{K}_0$ and must itself be quasi-projective. From this identification the universal $\mathbb{P}^1$-bundle $\rho : \mathcal{U} \to \mathcal{K}$ can be compactified to a projective $\mathbb{P}^1$-fibered space. In particular, $\mathcal{U}$ is also quasi-projective.

The fiber $\rho^{-1}(\kappa) \cong \mathbb{P}^1$ of a point $\kappa \in \mathcal{K}$ gives a copy of the Riemann sphere $\mathbb{P}^1$ corresponding to the rational curve represented by κ. From any choice of parametrization $f : \mathbb{P}^1 \to X$ of κ, a point on $\rho^{-1}(\kappa)$ gives a point of the cycle $C = f(\mathbb{P}^1) \subset X$ determined by κ, and we have in fact a canonical holomorphic map $\mu : \mathcal{U} \to X$ which we call the evaluation map. From the nonnegativity of f^*T_X it follows readily that $\mu : \mathcal{U} \to X$ must be a holomorphic submersion. Thus, the universal family comes equipped with a canonical double fibration $\rho : \mathcal{U} \to \mathcal{K}$, $\mu : \mathcal{U} \to X$ such that $\mu(\mathcal{U})$ must contain a dense Zariski-open subset of X. As X is of Picard number 1, any $\mathcal{K}$-curve must intersect any nontrivial divisor D, hence $\mathcal{K}$-curves must cover the complement of a subvariety $Z \subset X$ of codimension ≥ 2; i.e., $\mu(\mathcal{U}) \supset X - Z$.

2.3. $\mathcal{K}$-curves marked at a point. — Fix a point $x \in X$ and consider the set $\mathcal{H}_x$ of all holomorphic maps $f : \mathbb{P}^1 \to X$ belonging to $\mathcal{H}$ such that $f(0) = x$. As a space of free rational curves marked at x, $\mathcal{H}_x$ carries naturally the structure of a complex manifold, as follows. The infinitesimal deformation of $f \in \mathcal{H}_x$ as a parametrized rational curve marked at x is given by $H^0(\mathbb{P}^1, f^*T_X \otimes \mathcal{I}_0)$, while the obstruction group to the deformation of f fixing the marking at x is given by $H^1(\mathbb{P}^1, f^*T_X \otimes \mathcal{I}_0)$, where $\mathcal{I}_0$ stands for the ideal sheaf defined by the reduced point $0 \in \mathbb{P}^1$. Since f^*T_X is nonnegative, $f^*T_X \otimes \mathcal{I}_0 \cong f^*T_X \otimes \mathcal{O}(-1)$ is a direct sum of holomorphic line bundles of degree ≥ -1, and we still have $H^1(\mathbb{P}^1, f^*T_X \otimes \mathcal{I}_0) = 0$. Again $\mathrm{Aut}(\mathbb{P}^1; 0)$ acts effectively on $\mathcal{H}_x$, and we have a nonsingular quotient manifold $\mathcal{K}_x = \mathcal{H}_x/\mathrm{Aut}(\mathbb{P}^1; 0)$ serving as the base manifold of a holomorphic principal $\mathrm{Aut}(\mathbb{P}^1; 0)$-bundle $q_x : \mathcal{H}_x \to \mathcal{K}_x$. Through a general point $x \in X$ any rational curve of degree $\leq \deg(\mathcal{K})$ must be free. It follows that $\mathcal{K}$-curves marked at such a point x cannot be decomposed into two or more irreducible components under deformations fixing the base point x. Thus, $\mathcal{K}_x$ must be compact, hence projective for a general point $x \in X$.

For a point $x \in X$, although the complex structures on $\mathcal{H}_x$ and $\mathcal{H}$ arise from two distinct classification problems, set-theoretically $\mathcal{H}_x$ can still be identified with a subset of the complex manifold $\mathcal{H}$. For every $f \in \mathcal{H}_x$ the canonical inclusion $i : \mathcal{H}_x \subset \mathcal{H}$ identifies the tangent space $T_f(\mathcal{H}_x) = H^0(\mathbb{P}^1, f^*T_X \otimes \mathcal{I}_0)$ as a vector subspace of $H^0(\mathbb{P}^1, f^*T_X) = T_f(\mathcal{H})$ so that $i : \mathcal{H}_x \subset \mathcal{H}$ is a holomorphic immersion, hence an embedding. We can therefore identify $\mathcal{H}_x$ as a complex submanifold of $\mathcal{H}$. After this identification, in the construction of the universal family $\rho : \mathcal{U} \to \mathcal{K}$, $\mu : \mathcal{U} \to X$ the μ-fiber $\mathcal{U}_x$ over any $x \in X$ is nothing other than $\mathcal{H}_x/\mathrm{Aut}(\mathbb{P}^1; 0)$, so that $\mathcal{K}_x$ can be identified with $\mathcal{U}_x$. On the other hand, $\rho|_{\mathcal{U}_x} : \mathcal{U}_x \to \mathcal{K}$ need not be an embedding. In fact, it need not be bijective as *a priori* the cycle $C = f(\mathbb{P}^1)$ underlying $f \in \mathcal{H}_x$ may be locally reducible at x. At the same time, a simple calculation also shows that $\rho|_{\mathcal{U}_x}$ is an immersion at $u \in \mathcal{U}$ precisely when the $\mathcal{K}$-curve $\kappa = \rho(u)$ is immersed at $x := \mu(u)$. Thus, it fails to be an immersion at a point $u \in \mathcal{U}_x$ corresponding to a cusp on the minimal rational curve $\kappa = \rho(u)$.

2.4. The tangent map and varieties of minimal rational tangents. — By Mori's Breaking-up Lemma, on a projective manifold X there does not exist any nontrivial algebraic family of rational curves fixing 2 distinct points. In fact, to each nontrivial algebraic 1-parameter family of rational curves fixing two distinct points one can associate a ruled surface $\pi : S \to B$ over an algebraic curve B equipped with two disjoint holomorphic sections Γ_0 and Γ_∞ corresponding to the two distinct fixed points. On the one hand, each of the two sections must have negative self-intersection number as it is an exceptional divisor on S. On the other hand, $\Gamma_0^2 = -\Gamma_\infty^2$ as disjoint sections of a ruled surface, thus leading to a contradiction.

Let now $(X, \mathcal{K})$ be a uniruled projective manifold equipped with a minimal rational component. For $x \in X$ denote by $\mathcal{K}_x$ the moduli space of $\mathcal{K}$-curves marked at x. From Mori's Breaking-up Lemma one deduces (cf. Mok [39], Lemma (2.4.3), pp.203ff.)

Lemma 1. — *For a general point $x \in X$, a general member $[f] \in \mathcal{K}_x$ is standard in the sense that $f^*T_X \cong \mathcal{O}(2) \oplus [\mathcal{O}(1)]^p \oplus \mathcal{O}^q$ for some nonnegative integers p and q.*

Proof. — Suppose otherwise. Then, a general $\mathcal{K}$-curve is not standard. Hence there exists a nonempty open subset $\mathcal{W} \subset \mathcal{H}$ and a holomorphic vector field $\mathcal{Z}$ on $\mathcal{W}$ such that for every $f \in \mathcal{W}$, $\mathcal{Z}(f)$ vanishes at $0, \infty \in \mathbb{P}^1$ and does not belong to $df(H^0(\mathbb{P}^1, T_{\mathbb{P}^1}))$. Integrating $\mathcal{Z}$ and descending from $\mathcal{H}$ to $\mathcal{K}$ we obtain some nontrivial holomorphic 1-parameter family $\{\Phi_t : t \in \Delta\}$ of $\mathcal{K}$-curves passing through two distinct points $x, y \in X$. Identifying $\mathcal{K}$ as the normalization of a Zariski-open subset $\mathcal{K}_0$ of a projective subvariety $\mathcal{Q}$ of the Chow space of X, the set of $\mathcal{K}$-curves passing through x and y is naturally endowed the structure of a quasi-projective variety. The existence of a nontrivial holomorphic 1-parameter family of such curves implies therefore that there also exists a nontrivial algebraic 1-parameter family $\{\Psi_t : t \in B\}$ of such curves. We may choose x such that any rational curve passing through x of degree $\leq \deg(\mathcal{K})$ must be free, in which case any $\mathcal{K}$-curve passing through x cannot decompose under deformation fixing x, and the base curve B can be taken to be projective, leading to a contradiction with Mori's Breaking-up Lemma. □

We have the following important notion of the tangent map and the associated varieties of minimal rational tangents.

***Definition 1* (the tangent map & VMRTs).** — *Let $(X, \mathcal{K})$ be a uniruled projective manifold equipped with a minimal rational component $\mathcal{K}$. Over a general point $x \in X$ we have a rational map called the tangent map $\tau_x : \mathcal{K}_x \to \mathbb{P}T_x(X)$ defined by assigning each rational curve $[f]$ marked and immersed at x to the complex line $\mathbb{C}df(T_0(\mathbb{P}^1)) \subset T_x(X)$. The total transform $\mathcal{C}_x := \overline{\tau_x(\mathcal{K}_x)} \subset \mathbb{P}T_x(X)$ is called the variety of minimal rational tangents, alias* VMRT, *of $(X, \mathcal{K})$ at x.*

Note that a standard rational curve is immersed, since the natural map $\nu : \mathcal{O}(2) \cong T_{\mathbb{P}^1} \to f^*T_X \cong \mathcal{O}(2) \oplus \mathcal{O}(1)]^p \oplus \mathcal{O}^q$ is injective at every point. For $x \in X$ a general point and $[\alpha] \in \mathcal{C}_x$ a smooth point such that α is tangent to a standard $\mathcal{K}$-curve ℓ, assumed embedded for convenience, we write P_α for the positive part $(\mathcal{O}(2) \oplus \mathcal{O}(1)^p)_x \subset T_x(X)$ at x with respect to a splitting of $T_X|_\ell$. The following result highlights the role of standard rational curves in relation to the tangent map.

Lemma 2. — *Let $(X, \mathcal{K})$ be a uniruled projective manifold equipped with a minimal rational component. Suppose $x \in X$, and $\lambda \in \mathcal{K}_x$ is a marked $\mathcal{K}$-curve which is immersed at its marking at x. Then, the tangent map $\tau_x : \mathcal{K}_x \to \mathbb{P}T_x(X)$ is a holomorphic immersion at λ if and only if the underlying $\mathcal{K}$-curve is standard. Moreover, writing $\tau_x(\lambda) = \mathbb{C}\alpha$, in the latter case we have $T_{[\alpha]}(\mathcal{C}_x) = P_\alpha/\mathbb{C}\alpha$.*

Proof. — Parametrize λ by $f : \mathbb{P}^1 \to X$ such that $f(0) = x$. A tangent vector in $T_\lambda(\mathcal{H}_x)$ is equivalently a holomorphic section $\sigma \in H^0(\mathbb{P}^1, f^*T_X \otimes \mathcal{I}_0)$. Write $\overline{\sigma} := \sigma \bmod df(T_0(\mathbb{P}^1) \otimes \mathcal{I}_0)$. Let $\eta \in T_0(\mathbb{P}^1)$ and write $\alpha := df(\eta) \in T_x(X)$. Let $\Gamma \subset X$ be a germ of holomorphic curve at $x \in X$ which is the image under f of the germ of $\mathbb{P}^1$ at 0. The germ of s at 0 corresponds to a section s in $H^0(\Gamma, T_X)$ vanishing at x. Extend s to a holomorphic vector field $\widetilde{s}$ on a neighborhood of x in X. Choose any holomorphic coordinate system at $x \in X$ and denote by ∇ the flat connection defined by it. $\nabla_\alpha(\widetilde{s})$ is independent of the extension $\widetilde{s}$, and it is further independent of the choice of holomorphic coordinates since $s(x) = 0$. The differential of the tangent map $d\tau_x$ at $s \in T_\lambda(\mathcal{K}_x)$ is an element of $\mathrm{Hom}\big(T_\lambda(\mathcal{K}_x), T_{[\alpha]}(\mathbb{P}T_x(X))\big)$. Now $T_{[\alpha]}(\mathbb{P}T_x(X)) \cong \mathrm{Hom}\big(\mathbb{C}\alpha, T_x(X)/\mathbb{C}\alpha\big)$, so that we can interpret $d\tau_x$ as an element of $\mathrm{Hom}\big(T_\lambda(\mathcal{K}_x) \otimes \mathbb{C}\alpha, T_{[\alpha]}(\mathbb{P}T_x(X))/\mathbb{C}_\alpha\big)$ canonically. In local coordinates we have

$$d\tau_x(\overline{s})(\alpha) = \nabla_\alpha(\widetilde{s}) \bmod \mathbb{C}\alpha.$$

Thus $\overline{s} \in \mathrm{Ker}(d\tau_x)$ if and only if $\nabla_\alpha(\widetilde{s}) \in \mathbb{C}\alpha$, which is the case if and only if s vanishes to the order ≥ 2 at x modulo $\mathbb{C}\alpha$. Hence $\mathrm{Ker}(d\tau_x) = 0$ if and only if $f \in \mathcal{H}_x \subset \mathcal{H}$ is standard. The last statement in Lemma 2 follows readily from the proof. □

By a line on a projective subvariety $S \subset \mathbb{P}^N$ we will mean a projective line lying on S. Regarding minimal rational components and their VMRTs on a projective submanifold $X \subset \mathbb{P}^N$ uniruled by lines we have

Lemma 3. — *Let $X \subset \mathbb{P}^N$ be a projective submanifold equipped with the polarization inherited from the projective space, and $\mathcal{K}$ be a minimal rational component of X corresponding to a uniruling of X by lines. Then, at a general point $x \in X$, the variety of minimal rational tangents $\mathcal{C}_x \subset \mathbb{P}T_x(X)$ is nonsingular, and the tangent map $\tau_x : \mathcal{K}_x \to \mathbb{P}T_x(X)$ is a biholomorphism onto $\mathcal{C}_x$.*

Proof. — A $\mathcal{K}$-curve is a line ℓ on X, and we have $T_X|_\ell \subset T_{\mathbb{P}^n}|_\ell \cong \mathcal{O}(2) \oplus \mathcal{O}(1)^{N-1}$. When ℓ is a free rational curve on X, $T_X|_\ell$ is a direct sum of holomorphic line bundles of degree ≥ 0. Since $\mathcal{O}(2) \cong T_\ell \subset T_X|_\ell$, we conclude that $T_X|_\ell \cong \mathcal{O}(2) \oplus \mathcal{O}(1)^p \oplus \mathcal{O}^q$ for some nonnegative integers p and q. Now every $\mathcal{K}$-curve passing through a general point x is free, and the moduli space $\mathcal{K}_x$ of $\mathcal{K}$-curves marked at x is projective. By Lemma 2 the tangent map $\tau_x : \mathcal{K}_x \to \mathbb{P}T_x(X)$ is a holomorphic immersion. On the other hand, for each nonzero vector $\xi \in T_x(X) \subset T_x(\mathbb{P}^N)$ there is at most one line ℓ on X tangent to ξ, so that τ_x must be injective. In other words, $\tau_x : \mathcal{K}_x \to \mathbb{P}T_x(X)$ is a biholomorphism onto its image $\mathcal{C}_x$, the VMRT at x, as desired. □

While for projective submanifolds $X \subset \mathbb{P}^N$ uniruled by lines the tangent map at a general point is an isomorphism, and the same remains true for all known examples, on a theoretical level the behavior of the tangent map on an abstract uniruled projective manifold $(X, \mathcal{K})$ is far from being fully understood. In Hwang-Mok [17] (1999) it was proven that the tangent map $\tau_x : \mathcal{K}_x \to \mathbb{P}T_x(X)$ is birational under an additional non-degeneracy assumption on the Gauss map of the VMRT. On the other hand, the tangent map τ_x is holomorphic whenever every $\mathcal{K}$-curve marked at x is immersed at the marking. In 2002, Kebekus [25] showed by studying cusps of rational curves on X that this is indeed the case at a general point $x \in X$. He proved in fact that the tangent map is a finite holomorphic map at a general point $x \in X$. In conjunction with [25] and Cho-Miyaoka-Shepherd-Barron [3], we proved

***Theorem 1* (Hwang-Mok [22]).** — *Let $(X, \mathcal{K})$ be a uniruled projective manifold equipped with a minimal rational component $\mathcal{K}$ and x be a general point on X. Then, ($\mathcal{K}_x$ is projective and) the tangent map $\tau_x : \mathcal{K}_x \to \mathcal{C}_x$ is a finite birational holomorphic map onto its image. In other words, $\mathcal{K}_x$ is the normalization of the variety of minimal rational tangents $\mathcal{C}_x$ at a general point $x \in X$.*

Remarks. — The results on the tangent map apply to a rational component $\mathcal{K}$ whenever the variety of $\mathcal{K}$-tangents is projective at a general point. In the literature $\mathcal{K}$ is referred to as a non-splitting family of rational curves on X. One may extend the notion in (2.1) of a minimal rational component to mean a rational component $\mathcal{K}$ such that the variety of $\mathcal{K}$-tangents at a general point is projective. In this article we use the term 'minimal' to mean minimality of degrees among free rational curves, but statements of results remain valid for the extended meaning of 'minimality'.

2.5. Examples. — As first examples we consider the n-dimensional Fermat hypersurface X of degree d in $\mathbb{P}^{n+1}$, where $1 \le d \le n$. Thus,

$$X := \left\{[z_0, z_1, \cdots, z_{n+1}] \in \mathbb{P}^{n+1} : z_0^d + z_1^d + \cdots + z_{n+1}^d = 0\right\}$$

To determine the VMRT at a general point $x = [z_0, z_1, \ldots, z_{n+1}] \in X$, it is equivalent to find all $(w_0, w_1, \ldots, w_{n+1})$ such that for every $t \in \mathbb{C}$, $[z_0 + tw_0, z_1 + tw_1, \ldots, z_{n+1} + tw_{n+1}] \in X$. In other words, we have

$$\begin{gathered}(z_0 + tw_0)^d + \cdots + (z_{n+1} + tw_{n+1})^d = 0, \quad \text{i.e.,} \\ (z_0^d + \cdots + z_n^d) + t(z_0^{d-1}w_0 + \cdots + z_{n+1}^{d-1}w_{n+1}) \cdot d \\ + t^2(z_0^{d-2}w_0^2 + \cdots + z_{n+1}^{d-2}w_{n+1}^2) \cdot \frac{d(d-1)}{2} + \cdots + t^d(w_0^d + w_1^d + \cdots + w_{n+1}^d) = 0.\end{gathered}$$

When $(z_0, z_1, \ldots, z_{n+1})$ is fixed, we get $d+1$ homogeneous equations given by

$$(\flat)_k \qquad z_0^{d-k}w_0^k + \cdots + z_{n+1}^{d-k}w_{n+1}^2; \quad 0 \le k \le d$$

The equation $(\flat)_0$ says that $x = [z_0, z_1, \cdots, z_{n+1}]$ lies on X. The equation $(\flat)_1$ says that the vector $(w_0, w_1, \cdots, w_{n+1})$ mod $\mathbb{C}(z_0, z_1, \cdots, z_{n+1})$ is tangent to X at x. The

$d-1$ other equations describe $\mathcal{C}_x$ as the intersection of $d-1$ hypersurfaces of degree $2, 3, \cdots, d$ in $\mathbb{P}T_x(X) \cong \mathbb{P}^{n-1}$. Geometrically the system of equations $(\flat)_k$, $0 \le k \le d$, says that a line ℓ touching X at x to the order $\ge d$ must necessarily lie on X. By Lemma 3, $\mathcal{C}_x$ is smooth for a general point $x \in X$. The anti-canonical line bundle of $\mathbb{P}^{n+1}$ is isomorphic to $\mathcal{O}(n+2)$. Since $X \subset \mathbb{P}^{n+1}$ is of degree d, the normal bundle $N_{X|\mathbb{P}^{n+1}}$ on X is isomorphic to the restriction of $\mathcal{O}(d)$ to X. By the Adjunction Formula, $\det(T_X) \cong \mathcal{O}(n+2-d)|_X$. Over a line $\ell \subset X \subset \mathbb{P}^{n+1}$ which is free as a rational curve we have $T_X|_\ell \cong \mathcal{O}(2) \oplus (\mathcal{O}(1))^{n-d} \oplus \mathcal{O}^{d-1}$ by the proof of Lemma 3, so that the VMRT at a general point of X is of dimension $n-d$. It follows that for $1 \le d \le n$, the degree-d Fermat hypersurface $X \subset \mathbb{P}^{n+1}$ is uniruled by lines such that the VMRT at a general point is the $(n-d)$-dimensional smooth complete intersection of $d-1$ hypersurfaces on $\mathbb{P}T_x(X) \cong \mathbb{P}^{n-1}$, which is necessarily connected whenever $n-d > 0$. With exactly the same argument the VMRT at a general point of *any* smooth Fano hypersurface of $\mathbb{P}^{n+1}$ of degree $d \le n-1$ must necessarily be a (connected) smooth complete intersection of dimension $n-d \ge 1$.

Note that in general for any smooth hypersurface $X \subset \mathbb{P}^{n+1}$, $K_X^{-1} \cong \mathcal{O}(n+2-d)$ is in fact ample for $1 \le d \le n+1$. In the case where $d = n+1$, the minimal rational curves are however no longer lines. They are quadric curves C of $\mathbb{P}^{n+1}$ which lie on X, and $T_X|_C \cong \mathcal{O}(2) \oplus \mathcal{O}^{n-1}$, so that VMRTs are 0-dimensional at a general point.

The following table gives a description of the (smooth) VMRT at a general point of a smooth Fano hypersurface of degree $\le n$ in $\mathbb{P}^{n+1}$ highlighting some examples of special interest. Here we denote by X_d^n a smooth hypersurface of degree d in $\mathbb{P}^{n+1}$.

VMRT at a general point for smooth hypersurfaces of degree $d \le n$ in $\mathbb{P}^{n+1}$

X	VMRT $\mathcal{C}_x$ at a general point
$\mathbb{P}^n$	$\mathbb{P}^{n-1}$
Q^n	$Q^{n-2} \subset \mathbb{P}^{n-1}$
smooth cubic $\subset \mathbb{P}^{n+1}$	quadric $\cap$ cubic in $\mathbb{P}^{n-1}$
$X_3^5 \subset \mathbb{P}^6$	K^3 − surfaces
$X_n^n \subset \mathbb{P}^{n+1}$	$n!$ points
$X_d^n \subset \mathbb{P}^{n+1},\ d \le n$	codim$-(d-1)$ complete intersection $\subset \mathbb{P}^{n-1}$ of hypersurfaces of degrees $2, \ldots, d$

The first problem that we treated in our programme is the question of rigidity of irreducible Hermitian symmetric spaces under Kähler deformation (Hwang-Mok [15]) by a study of deformation of their VMRTs. The following table, taken from [15], ((2.1), p.440), gives their VMRTs. In this table G stands for the identity component of the isometry group of (S, g), where g is a canonical Kähler-Einstein metric on S, and $K \subset G$ denotes the isotropy subgroup at $0 \in S$. $G(p, q)$ stands for the Grassmannian of p-planes in $\mathbb{C}^{p+q}$, $G^{II}(n, n) \subset G(n, n)$ the complex submanifold of n-planes in $\mathbb{C}^{2n}$ isotropic with respect to a non-degenerate symmetric form, $G^{II}(n, n) \subset G(n, n)$ the complex submanifold of n-planes in $\mathbb{C}^{2n}$ isotropic with respect to a symplectic form. $\mathbb{O}$ stands for the octonions.

Table of irreducible Hermitian symmetric spaces S of the compact type and their VMRTs $\mathcal{C}_0$

Type	G	K	$G/K = S$	$\mathcal{C}_0$	Embedding
I	$SU(p+q)$	$S(U(p) \times U(q))$	$G(p,q)$	$\mathbb{P}^{p-1} \times \mathbb{P}^{q-1}$	Segre
II	$SO(2n)$	$U(n)$	$G^{II}(n,n)$	$G(2,n-2)$	Plücker
III	$Sp(n)$	$U(n)$	$G^{III}(n,n)$	$\mathbb{P}^{n-1}$	Veronese
IV	$SO(n+2)$	$SO(n) \times SO(2)$	Q^n	Q^{n-2}	by $\mathcal{O}(1)$
V	E_6	$\mathrm{Spin}(10) \times U(1)$	$\mathbb{P}^2(\mathbb{O}) \otimes_{\mathbb{R}} \mathbb{C}$	$G^{II}(5,5)$	by $\mathcal{O}(1)$
VI	E_7	$E_6 \times U(1)$	exceptional	$\mathbb{P}^2(\mathbb{O}) \otimes_{\mathbb{R}} \mathbb{C}$	Severi

3. Linearly degenerate VMRTs

3.1. Distributions and differential systems generated by VMRTs. — Let $(X, \mathcal{K})$ be a uniruled projective manifold equipped with a minimal rational component. Suppose the VMRT $\mathcal{C}_x$ at a general point $x \in X$ is irreducible and linearly degenerate. Then, it spans a meromorphic distribution $W \subsetneq T_X$. The singularity set $\mathrm{Sing}(W)$ is of codimension ≥ 2 in X. Suppose W is integrable, then a leaf L of W is quasi-projective, and its compactification $\overline{L}$ can be obtained as follows. Pick a point $x \in X - \mathrm{Sing}(W)$. Consider the subvariety $\mathcal{V}_1(x)$ swept out by all $\mathcal{K}$-curves passing through x. Enlarge $\mathcal{V}_1(x)$ to obtain $\mathcal{V}_2(x)$ by adjoining all minimal rational curves passing through general points on $\mathcal{V}_1(x)$ and taking topological closure. Repeating this process a finite number of times, we obtain a compactification of the leaf L_x through x (Hwang-Mok [15], Proposition 11). By definition, any $\mathcal{K}$-curve ℓ_0 emanating from x lies on $\overline{L_x}$, and by the deformation theory of rational curves ℓ_0 can always be deformed to avoid the set $\mathrm{Sing}(W)$ which is of codimension ≥ 2 in X, yielding a $\mathcal{K}$-curve ℓ disjoint from a hypersurface $\mathcal{H} \subset X$ swept out by compactifications of leaves of W. This is possible only if X is of Picard number ≥ 2. We have in fact

Proposition 1. — *Let $(X, \mathcal{K})$ be a Fano manifold of Picard number 1. Suppose for a general point $x \in X$ the associated variety of minimal rational tangents $\mathcal{C}_x$ is irreducible and linearly degenerate. Then, the distribution W spanned at a general point by $\widetilde{\mathcal{C}}_x$ cannot be integrable. More generally, any proper distribution D on X containing W cannot be integrable.*

In general, from $W \subsetneq X$ one can derive a finite series of distributions $W = W_1 \subsetneq W_2 \subsetneq \cdots \subsetneq W_k = W_{k+1} = \cdots$ where W_i is defined by induction by setting $\mathcal{W}_{j+1} = [\mathcal{W}_j, \mathcal{W}_j]$ as sheaves. We have thus the weak derived system generated by W. In case X is of Picard number 1, Proposition 1 applies to $D = W_k$ to show that the tangent bundle can be recovered from W by successively taking Lie brackets.

3.2. Integrability of distributions via projective geometry of VMRTs. — While [(3.1), Proposition 1] forces a distribution spanned by the VMRT at a general point to be non-integrable when the uniruled projective manifold X is of Picard number 1, we prove on the other hand that sufficient conditions for integrability of W can be deduced from projective-geometric properties of VMRTs. The argument goes as follows. The lack of integrability of W is encoded in the Frobenius form $\varphi : \Lambda^2 W \to T_X/W$, and integrability amounts to the vanishing of φ by the Frobenius Theorem. To prove that W is integrable it suffices to produce at a general point $x \in X$ enough elements of $\mathrm{Ker}(\varphi_x)$ to span $\Lambda^2 W_x$. In particular, if Σ is a germ of complex-analytic integral surface of W passing through x and $T_x(\Sigma)$ is spanned by η_1 and η_2, then $\eta_1 \wedge \eta_2 \in \mathrm{Ker}(\varphi_x)$. We consider a standard $\mathcal{K}$-curve ℓ passing through x and smooth at x, and take a smooth point $x_0 \in \ell$ distinct from x. Then, any pencil of rational curves emanating from x_0 including ℓ and smooth along ℓ produces a germ of surface Σ at x. Since the pencil fixes $y, T_x(\Sigma)$ is spanned by $T_x\ell = \mathbb{C}\alpha$ and a vector belonging to P_α. Thus $T_x(\Sigma) \subset P_\alpha \subset \mathrm{Span}(\widetilde{\mathcal{C}_x}) = W_x$. An analogous statement holds for any $y \in \Sigma$ sufficiently close to x, implying that Σ is a germ of integral surface of W at x. By linear algebra as explained in Hwang-Mok ([14], §2) we derived the following sufficient conditions for the integrability of W in terms of projective-geometric properties of VMRTs. For the formulation, given a finite-dimensional complex vector space V and any irreducible subvariety $Z \subset \mathbb{P}V$, its tangent variety $\mathcal{T} \subset \mathbb{P}(\Lambda^2 V)$ is by definition the closure of the set of elements $[\alpha \wedge \beta]$ where α is a smooth point of $\widetilde{Z}$ and $\beta \in T_\alpha(\widetilde{Z})$. We have

Proposition 2. — *The distribution W is integrable if the tangent variety $\mathcal{T}_x \subset \mathbb{P}(\Lambda^2 W_x)$ of $\mathcal{C}_x$ is linearly non-degenerate for a general point $x \in X$. The latter is in particular the case whenever the second fundamental form $\sigma_{[\alpha]} : T_{[\alpha]}(\mathcal{C}_x) \times T_{[\alpha]}(\mathcal{C}_x) \longrightarrow N_{\mathcal{C}_x|\mathbb{P}W_x,[\alpha]}$ at a general smooth point $[\alpha]$ of $\mathcal{C}_x$ is surjective.*

Proposition 3. — *Suppose at a general point $x \in X$ the variety of minimal rational tangents $\mathcal{C}_x \subset \mathbb{P}W \subset \mathbb{P}T_x(X)$ is irreducible and smooth and $\dim(\mathcal{C}_x) > \frac{1}{2}\mathrm{rank}(\mathcal{W}) - 1$. Then, W is integrable.*

3.3. Fano homogeneous contact manifolds. — From the perspective of geometric structures associated to VMRTs, after the irreducible Hermitian symmetric spaces of the compact type one naturally turns to rational homogeneous manifolds $S = G/P$ of Picard number 1. Here G is simple and $P \subset G$ is a maximal parabolic, corresponding to the choice of a simple root in the Dynkin diagram of the Lie algebra $\mathfrak{g}$ of G. For the background on rational homogeneous manifolds, especially root space decompositions, graded Lie algebras and G-invariant distributions we refer the reader to Hwang-Mok ([16], (3.3)-(3.4)). Among them, the Fano homogeneous contact manifolds were studied in relation to rigidity under Kähler deformation in Hwang [10] (1997). On a complex manifold X of dimension ≥ 2, a holomorphic distribution $W \subset T_X$ is said to be a contact distribution if and only if W is of co-rank 1 and the Frobenius form $\varphi : \Lambda^2 W \to T_X/W$ is non-degenerate at every point $x \in X$.

For the classification of Fano homogeneous contact manifolds we follow Boothby [1]. In the case of $\mathfrak{g} = A_k, k \geq 2$, S is of Picard number 2, $S \cong \mathbb{P}T^*_{\mathbb{P}^k}$. For the case of $\mathfrak{g} = C_k$ we have $S \cong \mathbb{P}^{2k-1}$ as a complex manifold. These cases will be excluded. For any other simple complex Lie algebra $\mathfrak{g}$ there is a unique choice of a long simple root in the Dynkin diagram of $\mathfrak{g}$, corresponding to a choice of a maximal parabolic subalgebra $\mathfrak{p} \subset \mathfrak{g}$, such that the associated rational homogeneous manifold $S = G/P$ is of contact type. We write $S = K(\mathfrak{g})$. In the following table we list the relevant Fano homogeneous contact manifolds of Picard number 1 according to the classification of $\mathfrak{g}$, with information on the Levi factor $\mathfrak{q} \subset \mathfrak{p}$, and a description of the VMRT $\mathcal{C}_0 \subset \mathbb{P}W_0$ as given in Hwang ([10], Proposition 5).

Table of Fano contact homogeneous spaces $S \ncong \mathbb{P}^{2n-1}$ of Picard number 1 and their varieties of minimal rational tangents

$\mathfrak{g}$	$\mathfrak{q}$	$\mathcal{C}_0$	Embedding
B_k	$A_1 \times B_{k-2}$	$\mathbb{P}^1 \times Q^{2k-5}$	Segre*
D_k	$A_1 \times D_{k-2}$	$\mathbb{P}^1 \times Q^{2k-6}$	Segre*
G_2	A_1	$\mathbb{P}^1$	by $\mathcal{O}(3)$
F_4	C_3	$G^{II}(3,3)$	by $\mathcal{O}(1)$
E_6	A_5	$G(3,3)$	by $\mathcal{O}(1)$
E_7	D_6	$G^{II}(6,6)$	by $\mathcal{O}(1)$
E_8	E_7	exceptional**	by $\mathcal{O}(1)$

* Here $k \geq 3$ for $\mathfrak{g} = B_k, k \geq 4$ for $\mathfrak{g} = D_k$. The embedding arises from the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^m$ into $\mathbb{P}^{2m+1}$ and the canonical embedding $Q^{m-1} \subset \mathbb{P}^m$.

** In this case $\mathcal{C}_0$ is biholomorphic to the irreducible compact Hermitian symmetric space of type VI pertaining to E_7, of dimension 27.

As examples of Fano homogeneous contact manifolds described in geometric terms consider those arising from hyperquadrics as follows. For the hyperquadric Q^n of dimension $n \geq 5$ consider the minimal rational component $\mathcal{K}(Q^n)$, i.e., the moduli space of lines ℓ on Q^n, which is a rational homogeneous manifold. We have $T_{Q^n}|_\ell \cong \mathcal{O}(2) \oplus (\mathcal{O}(1))^{n-2} \oplus \mathcal{O}$ for every $\ell \in \mathcal{K}(Q^n)$. The normal bundle $N_{\ell|Q^n} \cong (\mathcal{O}(1))^{n-2} \oplus \mathcal{O}$. At any $\ell \in \mathcal{K}(Q^n)$ the tangent space $T_\ell(\mathcal{K}(Q^n))$ can be identified with the vector space $H^0(\ell, N_{\ell|Q^n})$ and it contains a vector subspace $H^0(\ell, (\mathcal{O}(1))^{n-2})$ of codimension 1 which defines, as ℓ varies, a holomorphic distribution $\mathcal{D} \subset T_{\mathcal{K}(Q^n)}$ of co-rank 1. Since $n \geq 5$, $\mathcal{C}_x \cong Q^{n-2}$ is of Picard number 1, and the base manifold $\mathcal{K}(Q^n)$ of the double fibration $\mu : \mathcal{U} \to Q^n$, $\rho : \mathcal{U} \to \mathcal{K}(Q^n)$ is also of Picard number 1. For any $x \in Q^n$ any vector α tangent to $\mathcal{C}_x$ arises from an element of $H^0(\ell, N_{\ell|Q^n})$ vanishing at x, thus taking values in $(\mathcal{O}(1))^{n-2}$, and $\mathcal{C}_x$ projects under the canonical map $\rho' : \mathcal{C} \to \mathcal{K}$ to a submanifold $\mathcal{Q}_x \in \mathcal{K}(Q^n)$ which is tangent to $\mathcal{D}$. The VMRT $\mathcal{C}_x$ is isomorphic to $Q^{n-2} \cong \mathbb{P}^{n-1}$, and it contains a projective line λ whose image under ρ' gives a minimal rational curve on $\mathcal{K}(Q^n)$. (For this $n \geq 4$ is enough.) Thus, any minimal rational curve on $\mathcal{K}(Q^n)$ is tangent to $\mathcal{D}$. From [(3.1), Proposition 1], $\mathcal{D} \subset T_{\mathcal{K}(Q^n)}$ is not integrable. Q^n is associated to the classical groups G of type B_k or D_k, for which every rational homogeneous manifold $S = G/P$ of Picard number 1 has at most 1 proper G-invariant distribution. Hence, denoting by $\varphi : \Lambda^2\mathcal{D} \to T_{\mathcal{K}(Q^n)}/\mathcal{D}$ the Frobenius form, the kernel $\mathrm{Ker}(\varphi) \subsetneq \mathcal{D}$ must be trivial, and we conclude that the Frobenius form φ defines a twisted symplectic form on the distribution $\mathcal{D}$, and $\mathcal{K}(Q^n)$ is a Fano homogeneous contact manifold of Picard number 1.

For any $\ell \in \mathcal{K}(Q^n)$, any $x \in \ell, T_x(\ell) = \mathbb{C}\alpha$, $\mathbb{P}T_\ell(\mathcal{Q}_x) \cap \mathbb{P}T_\ell(\mathcal{K}(Q^n))$ parametrizes the space of lines on $\mathcal{C}_x$ passing through $[\alpha]$, and it defines a hyperquadric in $\mathbb{P}T_{[\alpha]}(\mathcal{C}_x)$, of dimension $n-4$. As the point x varies over ℓ, we recover a $\mathbb{P}^1$-family of disjoint $(n-4)$-dimensional hyperquadrics which exhausts the VMRT $\mathcal{C}'_\ell \in \mathbb{P}T_\ell(\mathcal{K}(Q^n))$. This family is actually isomorphic to the product $\mathbb{P}^1 \times Q^{n-4}$. (This product structure can be explained in terms of the parallel transport of second fundamental forms along ℓ to be given in (6.2).) For $n = 2k-1$ with $k \geq 3$, $\mathcal{K}(Q^n) = K(B_k)$ and $\mathcal{C}'_\ell \cong \mathbb{P}^1 \times Q^{2k-5}$; for $n = 2k-2$ with $k \geq 4$ we have $\mathcal{K}(Q^n) = K(D_k)$ and $\mathcal{C}'_\ell \cong \mathbb{P}^1 \times Q^{2k-6}$.

Excepting for $\mathbb{P}^{2n-1}$ of dimension ≥ 3, which we exclude, for any Fano homogeneous contact manifold (S, D) of Picard number 1, $\dim(S) = 2s+1$, the line bundle $L := T_S/D$ is isomorphic to $\mathcal{O}(1)$, the positive generator of the Picard group $\mathrm{Pic}(S)$. Thus for any minimal rational curve ℓ on S, $T_\ell \cong \mathcal{O}(2)$ must project to 0 on $L = T_S/D$, so that ℓ is tangent to D. Over a minimal rational curve ℓ on S we have $D|_\ell = \mathcal{O}(2) \oplus (\mathcal{O}(1))^p \oplus \mathcal{O}^p \oplus \mathcal{O}(-1)$ by root space decomposition. All known Fano contact manifolds are homogeneous. The question of characterization of Fano contact manifolds (X, D) is known to be reducible to the essential case where X is of Picard number 1 and where $L := T_X/D \cong \mathcal{O}(1)$ (Kebekus-Peternell-Sommese-Wiśniewski [27] (2000)). Kebekus [25] (2001) proved in this case that X is uniruled by degree-1 curves. From elementary consideration involving splitting types and the non-degeneracy of the Frobenius form $\varphi : \Lambda^2 D \to L$ one deduces readily that all minimal rational curves ℓ passing through a general point x are standard. In [25] it was proven that ℓ is actually smooth. Thus, $\mathcal{C}_x \subset \mathbb{P}T_x(X)$ is a Lagrangian submanifold with respect to the symplectic form φ_x. It is tempting to believe that the complex structure of X can be recovered from its VMRTs.

Conjecture 1. — *Let X be a Fano contact manifold. Then, X is biholomorphic to a Fano homogeneous contact manifold.*

Confirmation of Conjecture 1 would imply the same for the LeBrun-Salamon Conjecture (LeBrun [34], 1995), according to which a compact quaternionic Kähler manifold (M, h) of positive scalar curvature is Riemannian symmetric. The link is given by the twistor construction, by which one obtains from (M, h) a twistor space X which admits the structure of a Fano contact manifold. We note that for a Fano contact manifold X of Picard number 1 other than $\mathbb{P}^{2n-1}$, the contact structure is unique since the contact distribution is spanned at a general point by the VMRT.

Among Fano homogeneous contact manifolds of Picard number 1 other than $\mathbb{P}^{2n-1}$, the one of smallest dimension is $K(G_2)$, of dimension 5, where the VMRT is the cubic rational curve in $\mathbb{P}D_x \cong \mathbb{P}^3$ for the contact distribution D of rank 4. Other than the projective plane $\mathbb{P}^2$ and the 3-dimensional hyperquadric Q^3, $K(G_2)$ is the only rational homogeneous manifold of Picard number 1 with 1-dimensional VMRTs.

3.4. Applications to rigidity under Kähler deformation. — Regarding the problem of rigidity of rational homogeneous manifolds $S = G/P$ of Picard number 1 under Kähler deformation, the first result was established for the special case of irreducible Hermitian symmetric spaces of the compact type in Hwang-Mok ([15], 1998). After a series of articles we have now settled the problem, as follows.

***Theorem 2* (Hwang-Mok [23]).** — *Let $S = G/P$ be a rational homogeneous manifold of Picard number 1. Let $\pi : \mathcal{X} \to \Delta := \{t \in \mathbb{C}, |t| < 1\}$ be a regular family of projective manifolds such that $X_t := \pi^{-1}(t)$ is biholomorphic to S for $t \neq 0$. Then, X_0 is also biholomorphic to S.*

$S = G/P$ is determined by the choice of a simple root in the Dynkin diagram. When it is a long root, considerations on integrability of distributions spanned by or derived from VMRTs enter in an essential way. In the case of irreducible Hermitian symmetric spaces S, excluding the obvious case of $\mathbb{P}^n$, we make use of S-structures (cf. (4.2)). An S-structure on a complex manifold M can be equivalently defined by the varieties of highest weight tangents $\pi : \mathcal{W}(M) \to M$, and in the case of $M = S$, the latter agrees with the fibered space $\pi : \mathcal{C} \to S$ of VMRTs. The idea is to consider the VMRT $\mathcal{C}_{x_0}(X_0)$ at a general point of X_0. Suppose $\mathcal{C}_{x_0}(X_0) \subset \mathbb{P}T_{x_0}(X_0)$ is congruent to the model $\mathcal{C}_0 \subset \mathbb{P}T_0(S)$. From closedness of the flatness condition (cf. (4.3)) the S-structure at $x_0 \in X_0$ is flat. By Matsushima-Morimoto [35] the moduli space of projective submanifolds $\mathcal{A} \subset \mathbb{P}V$ congruent to $\mathcal{C}_0 \subset \mathbb{P}T_0(S)$ is isomorphic to an affine algebraic variety. Let $E \subset X_0$ be the singularity set of the S-structure defined at general points of X_0. Since $E \subset \mathcal{X}$ is of codimension ≥ 2 we have by [35] Hartogs extension of S-structures on the relative tangent bundle of $\pi : \mathcal{X} \to \Delta$, and X_0 carries a flat S-structure, implying that X_0 is isomorphic to the model space S from Ochiai's Theorem [47] on S-structures (cf. (4.2) here) and the method of developing maps.

Thus it remains to identify the VMRT at a general point $x_0 \in X_0$ with that of the model space. For $t \in \Delta$, at $x_t \in X_t$ denote by $\mathcal{K}_{x_t}$ the moduli space of minimal rational curves marked at x_t. For a generic choice of holomorphic section $\sigma : \Delta \to \mathcal{X}$, as t varies over Δ, $\{\mathcal{K}_{\sigma(t)}\}$ constitutes a regular family of projective manifolds such that $\mathcal{K}_{\sigma(t)} \cong \mathcal{C}_0(S)$ for $t \neq 0$. Noting that $\mathcal{C}_0(S)$ is itself a Hermitian symmetric space (cf. (2.5)), irreducible except in the case of the Grassmannian, by an inductive argument coupled with cohomological considerations in the case of the Grassmannian, $\mathcal{K}_{\sigma(0)}$ remains biholomorphically equivalent to $\mathcal{C}_0(S)$. To reconstruct an S-structure on X_0 it remains to examine the tangent map $\tau_{\sigma(0)} : \mathcal{K}_{\sigma(0)} \to \mathbb{P}T_{\sigma(0)}(X_0)$. From the rigidity of $\mathcal{K}_{\sigma(t)}$ at $t = 0$, degeneration of VMRTs can only arise from a linear projection on the model $\mathcal{C}_0(S)$. If this happens at a general point of X_0, we obtain a distribution $W \subsetneq T_{X_0}$ generated at a general point by its VMRT. On the one hand, by [(3.1), Proposition 1] W is not integrable since X_0 is of Picard number 1. On the other hand, from the description of $\mathcal{C}_0(S)$ as the closure of the graph of a vector-valued quadratic polynomial, at any $[\alpha] \in \mathcal{C}_0(S)$ the second fundamental form σ is surjective.

By linear projection the same remains true for $\mathcal{C}_{\sigma(0)}(X_0)$, and by [(3.2), Proposition 2] the distribution $\mathcal{W} \subset T_{X_0}$ is integrable, yielding a contradiction and proving that the VMRT is linearly non-degenerate at a general point of X_0, implying $X_0 \cong S$.

In the case where S is a Fano homogeneous contact manifold other than $\mathbb{P}^{2n-1}$, the VMRT $\mathcal{C}_0(S) \subset \mathbb{P}D_0$, where $D \subset T_S$ is the contact distribution. The kernel of the Frobenius form $\varphi_0 : \Lambda^2 D_0 \to T_0(S)/D_0 \cong \mathbb{C}$ is of codimension 1. Theorem 2 for the contact case was established in Hwang [10]. Following the same scheme as in the Hermitian symmetric case, the problem reduces to showing that for a generic choice of a holomorphic section $\sigma : \Delta \to \mathcal{X}$, the linear span $\mathcal{W}_{\sigma(0)}$ of $\mathcal{C}_{\sigma(0)}(X_0)$ is of codimension 1, and $\mathcal{C}_{\sigma(0)}(X_0) \subset W_{\sigma(0)}$ is congruent to the model $\mathcal{C}_0(S) \subset \mathbb{P}D_0$. In fact, granting this one can recover the structure of a Fano contact manifold on the central fiber X_0, and we have $X_0 \cong S$ by the local rigidity result of LeBrun [34] for Fano contact manifolds. It remains to rule out degeneration of VMRTs at a general point $x_0 \in X_0$ corresponding to a proper linear projection of $\mathcal{C}_0(S)$. Such a linear projection cannot occur, because the second fundamental form σ_0 of $\mathcal{C}_0 \subset \mathbb{P}D_0$ at $[\alpha] \in \mathcal{C}_0(S)$ has image of codimension 1, and any proper linear projection χ of $\mathcal{C}_0(S)$ renders the second fundamental form $\sigma_{[\beta]}$ surjective at a general point $[\beta]$ of the image $\chi(\mathcal{C}_0(S))$. In other words, if the VMRT at a general point on X_0 were more linearly degenerate than the model case, the distribution $\mathcal{W}$ on X_0 would become integrable, violating [(3.2), Proposition 2].

Given a distribution on a complex manifold, one can define a differential system by successively taking Lie brackets. On a uniruled projective manifold $(X, \mathcal{K})$ with an irreducible and linearly degenerate VMRT a general point, the distribution W spanned by VMRTs gives rise to such a differential system. When $S = G/P$ is defined by a long simple root but is neither of the symmetric nor of the contact type, Theorem 2 was solved by Hwang-Mok ([19], 2001). We make use of the work of Yamaguchi [51] on symbol algebras arising from differential systems on rational homogeneous manifolds. Following the same scheme of proof for Theorem 2 as above and making use of [51], the key issue is to prove that the differential system on the central fiber derived from the VMRTs is isomorphic to that of the model space. The VMRTs are tangents to minimal rational curves, and the argument using pencils of minimal rational curves in (3.2) produces elements in the kernel of the Frobenius form $\varphi_x : \Lambda^2 W_{x_0} \to T_{x_0}(X_0)/W_{x_0}$ at a general point $x_0 \in X_0$. We can consider the universal Lie algebra defined by taking elements of W_{x_0} as generators, and by taking the relations to be those generated by the argument of pencils of minimal rational curves in (3.2). Using Serre relations, we show that this universal Lie algebra is isomorphic to the symbol algebra at $0 \in S$ defined by $T_0(S)$ as a nilpotent algebra. In particular, proper linear projection of $\mathcal{C}_0(S)$ will yield a distribution such that the maximal distribution obtained by successively taking Lie brackets, which is by definition integrable, remains a proper distribution $\mathcal{W}^\sharp \subsetneq T_{X_0}$. This violates [(3.2), Proposition 1] and solves the key difficulty of Theorem 2 for the long root case being considered.

The method of using distributions associated to VMRTs does not in general work for the short root case. In all remaining cases one imitates the same scheme of proof, but in a typical case defined by a short root the key difficulty occurs *after* we already know that the VMRT at a general point of the central fiber agrees with that of the model space. New ideas are needed to complete the proof of Theorem 2. In (4.4) we will examine the degeneration of the Lie algebras of holomorphic vector fields associated to $\pi : \mathcal{X} \to \Delta$ by resorting to a study of prolongation of algebras of infinitesimal automorphisms associated to VMRTs.

4. Holomorphic G-structures and prolongations associated to VMRTs

4.1. Holomorphic conformal structures. — By a holomorphic metric on a complex manifold M we mean a nowhere degenerate holomorphic symmetric 2-tensor. In local holomorphic coordinates (z_i), we have $g = \sum g_{ij}(z) dz^i \otimes dz^j$ such that $\det(g_{ij})(z)$ is nowhere zero. For $x \in M$, a tangent vector $\alpha \in T_x(M)$ is called a null vector if and only if $g(\alpha, \alpha) = 0$. The space $\mathcal{N}_x$ of null vectors at x is called the null cone at x. It corresponds to a hyperquadric $\mathcal{Q}_x \subset \mathbb{P}T_x(M)$ which we call the variety of null tangents. On (M, g) there is a unique holomorphic torsion-free connection ∇ such that $\nabla g = 0$ on M, analogous to the Levi-Civita connection in Riemannian geometry, given by the same formula

$$\Gamma^k_{ij} = \frac{1}{2} \sum_\ell g^{k\ell} \Big(\frac{\partial g_{i\ell}}{\partial z_j} + \frac{\partial g_{j\ell}}{\partial z_i} - \frac{\partial g_{ij}}{\partial z_\ell} \Big)$$

for the Riemann-Christoffel symbols (Γ^k_{ij}). On a complex manifold M two holomorphic metrics g and $\widetilde{g}$ on are said to be conformally equivalent to each other if and only if there exists a nowhere vanishing holomorphic function λ such that $\widetilde{g} = \lambda g$. The Riemann-Christoffel symbols $(\widetilde{\Gamma}^k_{ij})$ of $\widetilde{g}$ are related to those of g by

$$\begin{aligned}
\widetilde{\Gamma}^k_{ij} &= \sum_\ell \frac{1}{2\lambda} g^{k\ell} \Big(\frac{\partial}{\partial z_j}(\lambda g_{i\ell}) + \frac{\partial}{\partial z_i}(\lambda g_{j\ell}) - \frac{\partial}{\partial z_\ell}(\lambda g_{ij}) \Big) \\
&= \Gamma^k_{ij} + \frac{1}{2}\delta^k_i \frac{\partial}{\partial z_j} \log\lambda + \frac{1}{2}\delta^k_j \frac{\partial}{\partial z_i}\log\lambda - \frac{1}{2}\Big(\sum_\ell g^{k\ell} \frac{\partial}{\partial z_\ell}\log\lambda \Big) g_{ij} \ .
\end{aligned}$$

A (parametrized) complex geodesic on M is a nonconstant holomorphic map $\gamma : D \to M$ defined on some domain $D \subset \mathbb{C}$ satisfying in analogy to geodesics in Riemannian geometry the second order differential equation

$$\frac{\partial^2 \gamma}{\partial t^2} + \Gamma_{\dot\gamma\dot\gamma} = 0 \ .$$

Replacing g by $\widetilde{g} = \lambda g$ we have

$$\frac{\partial^2 \gamma}{\partial t^2} + \widetilde{\Gamma}_{\dot\gamma\dot\gamma} = (\partial_{\dot\gamma} \log\lambda)\dot\gamma - \frac{1}{2}\Big(\sum_\ell g^{k\ell} \frac{\partial}{\partial z_\ell} \log\lambda \Big) g_{\dot\gamma\dot\gamma} \ .$$

where $\dot\gamma$ stands for $\frac{\partial\gamma}{\partial t}$. In invariant form the differential equation is given by $\nabla_{\dot\gamma}\dot\gamma = 0$. A complex geodesic γ is called a null geodesic if and only if $\dot\gamma(t)$ lies on the null cone $\mathcal{N}_{\gamma(t)}$ for every $t \in D$. Since $\nabla g = 0$, for a complex geodesic $g_{\dot\gamma\dot\gamma}$ is a constant. In particular, γ is a null geodesic if and only if $\dot\gamma$ is a null vector at one point. Suppose γ is a null geodesic on (M,g). Then, with respect to the holomorphic metric $\widetilde g$ we have

$$\frac{\partial^2\gamma}{\partial t^2} + \widetilde\Gamma_{\dot\gamma\dot\gamma} = (\partial_{\dot\gamma}\log\lambda)\dot\gamma \ .$$

Write $f(t) := \partial_{\dot\gamma}\log\lambda(t)$. At a point $t_0 \in D$, making a local holomorphic change of variable $s = s(t)$ at t_0 and writing $t = \varphi(s), \gamma(t) = \mu(s)$, we have

$$\frac{\partial^2\mu}{\partial s^2} + \widetilde\Gamma\Big(\frac{\partial\mu}{\partial s}, \frac{\partial\mu}{\partial s}\Big) = \varphi'(s)^2\frac{\partial^2\gamma}{\partial t^2} + \varphi''(s)\frac{\partial\gamma}{\partial t} + \varphi'(s)^2\widetilde\Gamma_{\dot\gamma\dot\gamma} = \varphi''(s) + \big(\varphi'(s)^2 f(\varphi(t))\big)\frac{\partial\gamma}{\partial t} \ .$$

Thus, making a change of variables by solving by means of power series the second order differential equation $\varphi''(s) + \big(\varphi'(s)^2 f(\varphi(s)) = 0$ which admits a unique solution subject to a choice of $s_0 = \varphi^{-1}(t_0)$ and a choice of $\varphi'(s_0)$. In other words, a germ of null geodesic on (M,g) can be re-parametrized to give a germ of null geodesic on $(M,\widetilde g)$. We will sometimes speak of a complex geodesic to mean the image of a parametrized complex geodesic. In this sense, the space of null geodesics on (M,g) is a property of the conformal equivalence class of g.

By a holomorphic conformal structure on M we will mean a holomorphic line subbundle $\Lambda \subset S^2T^*_M$, generated at each point by a non-degenerate holomorphic symmetric 2-tensor. Equivalently, it is given by the data $(U_\alpha, g_\alpha)_{\alpha\in A}$ consisting of holomorphic metrics g_α on open subsets U_α covering M such that over the non-empty overlaps $U_{\alpha\beta} = U_\alpha \cap U_\beta$, g_α and g_β are conformally equivalent. A holomorphic conformal structure on M is equivalently defined by the fibered space of varieties of null tangents $\pi : \mathcal{Q} \to M$, and we will speak of $(M,\mathcal{Q})$ as a complex manifold equipped with a holomorphic conformal structure. Each null geodesic lifts to a local holomorphic curve on $\mathcal{Q}$ by sending a point $\gamma(t)$ to $[\dot\gamma(t)] \in \mathcal{Q}_{[\gamma(t)]}$, which we call the tautological lifting, and we have a 1-dimensional holomorphic foliation on $(M,\mathcal{Q})$ by liftings of null geodesics. In Riemannian geometry, for computations at a given base point one often makes use of local coordinates with respect to which the Riemann-Christoffel symbols (Γ^k_{ij}) vanish at the base point 0. The proof of existence of such coordinates works verbatim in the holomorphic situation. Starting with a given holomorphic local coordinate system (z_i) at a point $x \in M$, $z(x) = 0$, such that $g_{ij}(0) = \delta_{ij}$, we introduce a new holomorphic coordinate system (w_j) such that $w(0) = 0$ and $\frac{\partial w^k}{\partial z_i}(0) = \delta^k_i$. Writing

$$\sum_{i,j} g_{ij}dz^i \otimes dz^j = \sum_{k,\ell} h_{k\ell}dw^k \otimes dw^\ell \ , \quad h_{k\ell} = \sum_{i,j} g_{ij}\frac{\partial z^i}{\partial w_k}\ \frac{\partial z^j}{\partial w_\ell} \ ;$$

$$\frac{\partial h_{k\ell}}{\partial w_s}(0) = \frac{\partial g_{k\ell}}{\partial z_s}(0) + \frac{\partial^2 z^\ell}{\partial w_s\partial w_k}(0) + \frac{\partial^2 z^k}{\partial w_s\partial w_\ell}(0) \ .$$

Now choose (w_k) such that $z^k = w^k + \sum_{s,\ell} c^k_{s\ell}w^s w^\ell$, where $c^k_{s\ell} = c^k_{\ell s}$. Then, setting

$$c^k_{s\ell} = -\frac{1}{2}\Big(\frac{\partial g_{k\ell}}{\partial z_s} + \frac{\partial g_{ks}}{\partial z_\ell} - \frac{\partial g_{s\ell}}{\partial z_k}\Big)(0) = -\Gamma^k_{s\ell}(0) \ .$$

we conclude that $\frac{\partial h_{k\ell}}{\partial w_s}(0) = 0$, and as a consequence $\widetilde\Gamma^k_{ij}(0) = 0$ in w coordinates.

In Riemannian geometry for a given base point x there is a privileged coordinate system adapted to x given by the geodesic normal coordinates in terms of which in particular the Riemann-Christoffel symbols vanish at x. The notion of geodesic normal coordinates generalizes in the setting of holomorphic metrics.

To start with we note that complex geodesics can be re-parametrized by a rescaling of the domain variable. Let $D \subset \mathbb{C}$ be a domain containing 0, $x \in M$, and $\gamma : D \to M$ be a parametrized complex geodesic such that $\gamma(0) = x$. Then, given any nonzero complex number $\lambda \in \mathbb{C}$, the function $\delta : \frac{1}{\lambda} D \to M$ defined by $\delta(t) = \gamma(\lambda t)$ is again a parametrized complex geodesic, as can be seen from the defining equation for a complex geodesic. On the total space $\pi : L \to \mathbb{P}T_x(M)$ of the tautological line bundle over $\mathbb{P}T_x(M)$, for a sufficiently small neighborhood U of $\mathbb{P}T_x(M)$ one can define a holomorphic map $\Phi_0 : U \to M$, as follows. For $[\alpha] \in \mathbb{P}T_x(M)$ and $\eta \in L_{[\alpha]} = \mathbb{C}\alpha$, $\eta = t\alpha$ sufficiently small, let $\Phi_0(\lambda)$ be $\gamma_\alpha(t)$, where γ is the unique germ of complex geodesic at $0 \in \mathbb{C}$ such that $\gamma(0) = x$ and $\frac{\partial \gamma}{\partial t}|_{t=0} = \alpha$. If we replace α by $\lambda\alpha$ for some nonzero λ, then $\gamma_{\lambda\alpha}(\frac{t}{\lambda}) = \gamma_\alpha(t)$ from uniqueness of geodesics with fixed initial value and initial first derivative. It follows that $\Phi(\eta)$ is well-defined, and we have a holomorphic map $\Phi_0 : U \to M$ which collapses $\mathbb{P}T_x(M)$ to x, from which it follows readily that Φ_0 descends to a holomorphic map $\Phi : \Omega \to M$, where Ω is a neighborhood of 0 in T_x. From the construction we have readily $d\Phi(0) = \mathrm{id}$. φ is the holomorphic exponential map, and it defines holomorphic geodesic normal coordinates at x. With respect to these coordinates, obviously the Riemann-Christoffel symbols vanish at 0. Moreover, by the same proof as in Riemannian geometry, the holomorphic metric admits a power series expansion at 0 in terms of the curvature tensor and its covariant derivatives at x. In particular, if the curvature vanishes identically, the holomorphic geodesic normal coordinates define a coordinate system with respect to which the holomorphic metric tensor (g_{ij}) is of constant coefficients. We may take g_{ij} to be δ_{ij}.

Exactly as in Riemannian geometry, the curvature tensor ${R_{ijk}}^{\ell}$ of (M, g) admits a decomposition ${R_{ijk}}^{\ell} = {A_{ijk}}^{\ell} + {W_{ijk}}^{\ell}$, where $W = ({W_{ijk}}^{\ell}) \in H^0(M, \Lambda^2 T_M^* \otimes \mathrm{End}(T_M))$ is the Bochner-Weyl tensor, which is unchanged when a holomorphic metric is modified by a conformal factor. A holomorphic metric is by definition conformally flat if and only if $W = 0$. A conformally flat holomorphic metric g is conformally equivalent to a holomorphic metric h with vanishing curvature, i.e., $R_h = 0$. Using holomorphic geodesic normal coordinates for h, we have seen that g is conformally flat if and only of it is given locally by $g_{ij} = \lambda\delta_{ij}$ for an appropriate choice of holomorphic coordinates and for some non-zero holomorphic function λ.

4.2. G-structures associated to irreducible Hermitian symmetric spaces of rank ≥ 2. — The model space of a holomorphic conformal structure is the hyperquadric $Q^n, n \geq 3$. In terms of Harish-Chandra coordinates on an open Schubert cell $\mathbb{U} \subset Q^n$, the Euclidean translations on $\mathbb{U}$ extend to automorphisms of Q^n, and the null-cones $\widetilde{\mathcal{N}}$ on Q^n form a constant family since they are invariant under automorphisms of Q^n, showing that the the holomorphic conformal structure on $\mathbb{U} \subset Q^n$ is defined by the equivalence class of a holomorphic metric of constant coefficients. Holomorphic conformal structures will also be referred to as hyperquadric structures, or Q^n-structures, in a sense that applies in general to Hermitian symmetric spaces S of the compact type and of rank ≥ 2. In this general context the hyperquadric structure on Q^n is said to be flat (or integrable) in the sense that there exists local holomorphic coordinates (the Harish-Chandra coordinates) with respect to which the null cones $\mathcal{N} \subset T_{Q^n}$ form constant families over the coordinate charts.

The notion of a hyperquadric structure generalizes to S-structures for any irreducible Hermitian symmetric space of rank ≥ 2. For the fibered space of null cones $\pi : \mathcal{N} \to M$ of a complex manifold M equipped with a holomorphic conformal structure, there is an underlying complex Lie group consisting of linear transformations preserving a model light cone $\mathcal{N}_0 \subset V := T_0(Q^n)$. The group is precisely the reductive complex Lie subgroup $\mathbb{C} \cdot \mathrm{O}(n;\mathbb{C}) \subset \mathrm{GL}(V)$. In general for any complex Lie subgroup G of $GL(V)$ for a finite-dimensional complex vector space we have the notion of a (holomorphic) G-structure. For its formulation let n be a positive integer, V be an n-dimensional complex complex vector space, and M be any n-dimensional complex manifold. In what follows all bundles are understood to be holomorphic. The frame bundle $\mathcal{F}(M)$ is a principal $\mathrm{GL}(V)$-bundle with the fiber at x defined as $\mathcal{F}(M)_x = \mathrm{Isom}(V, T_x(M))$.

***Definition 2* (G-structure).** — *Let* $\mathrm{G} \subset GL(V)$ *be any complex Lie subgroup. A holomorphic* G*-structure is a* G*-principal subbundle* $\mathcal{G}(M)$ *of* $\mathcal{F}(M)$*. An element of* $\mathcal{G}_x(M)$ *will be called a* G*-frame at* x*. For* $\mathrm{G} \subsetneq \mathrm{GL}(V)$ *we say that* $\mathcal{G}(M)$ *defines a holomorphic reduction of the tangent bundle to* G*.*

We have in general the notion of a flat G-structure, as follows.

***Definition 3* (flat G-structure).** — *In terms of Euclidean coordinates we identify* $\mathcal{F}(U_\alpha)$ *with the product* $\mathrm{GL}(V) \times U_\alpha$*. We say that a* G*-structure* $\mathcal{G}(M)$ *on* M *is flat if and only if there exists an atlas of charts* $\{\varphi_\alpha : U_\alpha \to V\}$ *such that the restriction* $\mathcal{G}(U_\alpha)$ *of* $\mathcal{G}(M)$ *to* U_α *is the product* $\mathrm{G} \times U_\alpha \subset \mathrm{GL}(V) \times U_\alpha$*.*

Let (S, g) be an irreducible Hermitian symmetric space of the compact type and of rank ≥ 2. Write G_c for the identity component of the isometry subgroup of (S, g), and $K \subset G_c$ be the isotropy subgroup at a reference point $0 \in S$. As a rational homogeneous manifold $S = G/P$, where G is a complexification of G_c and $P \subset S$ is a maximal parabolic subgroup. We have the Harish-Chandra decomposition of the Lie algebra $\mathfrak{g}$ of G, $\mathfrak{g} = \mathfrak{m}^+ \oplus \mathfrak{k}^{\mathbb{C}} \oplus \mathfrak{m}^-$, in which $\mathfrak{k}^{\mathbb{C}}$ is the complexification of the Lie algebra $\mathfrak{k}$ of K. Regarding $\mathfrak{g}$ as the Lie algebra of holomorphic vector fields on S, $\mathfrak{m}^-$ stands for the vector space of holomorphic vector fields vanishing to the order ≥ 2 at 0. P admits a Levi decomposition $P = K^{\mathbb{C}} \cdot M^-$. Here $K^{\mathbb{C}} = \exp(\mathfrak{k}^{\mathbb{C}})$ is the reductive group consisting of automorphisms of S fixing 0, identified with a complex linear subgroup of $\mathrm{GL}(T_0(S))$ where $\gamma \in K^{\mathbb{C}}$ is mapped to $d\gamma(0)$, and $M^- = \exp(\mathfrak{m}^-)$. S then carries a G-structure with $\mathrm{G} = K^{\mathbb{C}}$. Regarding S-structures we have

***Theorem 3* (Ochiai [47]).** — *Let* S *be an irreducible Hermitian symmetric space of the compact type and of rank* ≥ 2*. Let* X *be a compact simply-connected complex manifold with a flat* S*-structure. Then,* X *is biholomorphic to* S*.*

$K^{\mathbb{C}}$ acts irreducibly on the model vector space $V = T_0(S)$, and its highest weight orbits define a rational homogeneous manifold $\mathcal{W}_0 \subset \mathbb{P}T_0(S)$, leading to a fibered space of highest weight tangents $\pi : \mathcal{W} \to M$ on any complex manifold equipped with a $K^{\mathbb{C}}$-structure. Let $(M_1, \mathcal{G}_1)$ resp. $(M_2, \mathcal{G}_2)$ be two complex manifolds equipped with G-structures, $\mathrm{G} = K^{\mathbb{C}}$, with fibered spaces of highest weight tangents $\pi_1 : \mathcal{W}_1 \to M_1$ resp. $\pi_2 : \mathcal{W}_2 \to M_2$. A biholomorphism $f : M_1 \to M_2$ preserves the G-structures if and only if it preserves the fibered spaces of highest weight tangents, i.e., $f_*\mathcal{W}_1 = \mathcal{W}_2$.

Denote by $\mathcal{O}(1)$ the ample line bundle on S which is the positive generator of the Picard group of S. S can be embedded into the projective space by $\mathcal{O}(1)$, e.g., the Grassmannian is embedded by the Plücker embedding. With respect to this embedding, S is uniruled by lines. When S itself is considered as the underlying space of an S-structure, the variety of highest weight tangents $\mathcal{W}_x$ agrees with the VMRT $\mathcal{C}_x$ at any $x \in S$. This follows from the construction of lines on S by means of $\mathrm{SL}(2, \mathbb{C})$ orbits highest weight vectors (cf. Mok [40], (1.4)) for a verification in the case of Grassmannians). To give a proof of Ochiai's Theorem using VMRTs, the starting point is the following result on local VMRT-preserving holomorphic maps.

Lemma 4. — *Let S be an irreducible Hermitian symmetric space of the compact type and of rank ≥ 2. Let $D, D' \subset S$ be nonempty connected open subsets of S and $f : D \to D'$ be a* VMRT*-preserving biholomorphic map. Then, for any line on S intersecting D, $f(L \cap D)$ is an open subset of some line L' of S*

Proof. — Denote by $\pi : \mathcal{C} \to S$ the fibered space of VMRTs over S and by $\mathcal{F}$ the tautological foliation on $\mathcal{C}$. By assumption $[df](\mathcal{C}|_D) = \mathcal{C}|_{D'}$. We have to show that for any line $L \subset S$ such that $L \cap D \neq \emptyset$, $[df](\widehat{L} \cap \mathcal{C}|_D)$ is an integral curve of the tautological foliation on S. This is the case if and only if $f^*\mathcal{F}$ agrees with $\mathcal{F}$ on $\mathcal{C}|_D$, i.e., if and only if the image under $[df]$ of each $\widehat{L} \cap \mathcal{C}|_D$ is tangent at every point to the tautological lifting $\widehat{L'}$ of some line L'. Equivalently this means that the image of each $L \cap D$ is tangent at every point to a line on S up to the second order. To prove Lemma 4 it suffices therefore to show that $\partial^2 f(\alpha, \alpha)$ is proportional to $df(\alpha)$ for any minimal rational tangent $[\alpha]$. In these coordinates $\pi : \mathcal{C} \to S$ is a constant family. Let α, β be vectors in $\widetilde{\mathcal{C}}_0$. (For a projective subvariety $\mathcal{A} \subset \mathbb{P}^N$ we denote by $\widetilde{\mathcal{A}} \subset \mathbb{C}^{N+1} - \{0\}$ its homogenization.) Then, $\partial^2 f(\alpha, \beta) = \partial_\alpha(df(\beta^\sharp))$, where $\beta^\sharp$ stands for the constant vector field on D such that $\beta^\sharp(0) = \beta$. Thus, $\partial^2 f(\alpha, \beta)$ is the tangent at β to some holomorphic curve on $\widetilde{\mathcal{C}}_0$, so that $\partial^2 f(\alpha, \beta) \in P_\beta = T_\beta(\widetilde{\mathcal{C}}_0)$. By symmetry we have $\partial^2 f(\alpha, \beta) \in P_\alpha \cap P_\beta$.

It remains to derive that for any $\alpha \in \widetilde{C}$, $\partial^2 f(\alpha, \alpha) = \lambda\alpha$ for some λ. On a non-linear projective submanifold, by Zak's Theorem (Zak [52]) the Gauss map is non-degenerate at a general point. Thus, the kernel of the second fundamental form σ is trivial at a general point. In the case of $\mathcal{C}_0 \subset \mathbb{P}T_0(S)$, which is homogeneous as a projective submanifold, $\mathrm{Ker}(\widetilde{\sigma}) = 0$ everywhere. Equivalently, lifting to homogenizations, $\mathrm{Ker}(\widetilde{\sigma}_\alpha) = \mathbb{C}\alpha$ for the (Euclidean) second fundamental form $\widetilde{\sigma}_\alpha$ at any $\alpha \in \widetilde{\mathcal{C}}_0$, and it remains for the proof of Lemma 4 to show that $\partial^2 f(\alpha, \alpha) \in \mathrm{Ker}(\widetilde{\sigma}_\alpha)$ for any such α. Fix now $\alpha \in \widetilde{\mathcal{C}}_0$ and let $\beta = \alpha(t), \alpha(0) = \alpha$, vary holomorphically on $\widetilde{\mathcal{C}}_0$ in the complex parameter t. Writing $\xi = \partial_t(\alpha)(0)$, from $\partial^2 f(\alpha, \alpha(t)) \in P_\alpha$ it follows that

$\partial^2 f(\alpha,\xi) \in P_\alpha$. On the other hand $\partial_t(\partial^2 f(\alpha(t),\alpha(t))|_{t=0} = 2\partial^2 f(\alpha,\xi)$, and hence $\nabla_\xi(\partial^2 f(\alpha(t),\alpha(t))|_{t=0} \in P_\alpha$ in terms of the Euclidean flat connection ∇ on $T_0(S)$. It follows that $\widetilde{\sigma}_\alpha(\xi,\partial^2 f(\alpha,\alpha)) = 0$. Since ξ can be chosen to be any tangent vector in $T_\alpha(\widetilde{\mathcal{C}}_0) = P_\alpha$, we conclude that $\partial^2 f(\alpha,\alpha) \in \mathrm{Ker}(\widetilde{\sigma}_\alpha)$, and we are done. □

By means of Lemma 4 the mapping $f : D \to D'$ can be analytically continued to give an automorphism of S. The idea is to pass to the moduli space $\mathcal{K}$ of lines. For each $x \in S$ denote by $\mathcal{Q}_x \subset \mathcal{K}$ the projective submanifold consisting of lines passing through x. We may assume D to be convex in Harish-Chandra coordinates. For any $\ell \in \mathcal{K}$ sufficiently close to $\mathcal{Q}_x$, $\ell \cap D$ is non-empty and connected, and $f(\ell \cap D)$ is an open subset of some line ℓ'. Thus, for a sufficiently small open neighborhood $\mathbb{U}$ of $\mathcal{Q}_x$ in $\mathcal{K}$, f induces a holomorphic map $f^\sharp : \mathbb{U} \to \mathcal{K}$. The problem of analytic continuation can be solved first by meromorphically extending $f^\sharp$ to $F^\sharp : \mathcal{K} \to \mathcal{K}$ and then by recovering $F : S \to S$ by considering a point $y \in S$ as the intersection of all lines passing through it, and by defining $f(y) := \bigcap \left\{ f^\sharp(\ell) : y \in \ell \right\}$ for a general point $y \in S$. The meromorphic extension of $f^\sharp$ to $F^\sharp$ is plausible because $\mathbb{U}$ is a 'big' open set in an analytic sense, as it contains the projective subvarieties $\mathcal{Q}_y$ for y sufficiently close to x. This latter extension problem can be solved by methods of Hartogs extension as done in Mok-Tsai [44]. The extension $F : S \to S$ thus obtained may have singularities, but they are proven to be removable by arguments involving deformation theory of rational curves (cf. Mok [40], (2.4)).

4.3. Flatness of G-structures via VMRTs. — Let V be a fixed n-dimensional complex vector space and $\mathrm{G} \subset \mathrm{GL}(V)$ be a connected complex Lie subgroup. Let X be an n-dimensional complex manifold endowed with a G-structure $\mathcal{G} \subset \mathcal{F}(X)$. We examine necessary and sufficient conditions for the G-structure to be flat. Recall that the G-structure $\mathcal{G}$ is flat if local holomorphic trivializations of $\mathcal{G}$ can be realized by choices of local holomorphic coordinates on X. Flatness imposes therefore differential constraints on $(X,\mathcal{G})$. The problem of identifying flat G-structures was solved in terms of obstructions to prolongations of G-structures (cf. Guillemin [4]).

Given a G-structure $(X,\mathcal{G})$ and a biholomorphic map $f : X \to Y$ onto another complex manifold Y, we have an induced G-structure $(Y, f_*\mathcal{G})$. Let $(X,\mathcal{G})$ and $(X',\mathcal{G}')$ be two complex manifolds endowed with G-structures. For $x \in X$ denote by (X,x) the germ of complex manifolds defined by X at x, etc. A germ of local biholomorphism $f : (X,x) \to (X',x')$ is said to be (0-th order) structure-preserving if $(f_*\mathcal{G})_{x'} = \mathcal{G}'_{x'}$. For k a positive integer, f is said to be k-th order structure-preserving if furthermore $f_*\mathcal{G}$ is tangent to $\mathcal{G}'$ along $\mathcal{G}'_x$ to an order $\geq k$. This notion depends only on the $(k+1)$-jet of f. For $k \geq 0$ the G-structure $(X,\mathcal{G})$ is said to be k-flat at x if there exists a local biholomorphism $f : (X,x) \to (V,0)$ which is k-th order structure-preserving, when V is endowed with the trivial G-structure $V \times G$.

When $(X,\mathcal{G})$ is uniformly k-flat, i.e., k-flat at every point $x \in X$, one can define in a canonical way some structure function c^k on some prolongation bundle over $\mathcal{G}$, such that $c^k \equiv 0$ if and only if $(X,\mathcal{G})$ is uniformly $(k+1)$-flat (Guillemin [5], Cor. to Theorem 4.1). By the Cartan-Kähler Theorem (Singer-Sternberg [49]) a G-structure is flat if and only if it is k-flat for every integer $k \geq 0$. In the case where G is reductive, the structure functions can be translated as obstruction tensors $\theta_k \in H^0\big(X, \mathrm{Hom}(\Lambda^2 T_X, T_X \otimes S^k T_X^*)\big)$. In the case of S-structures (cf. (4.2)) corresponding to $\mathrm{G} = K^{\mathbb{C}}$ it is known that $(X, \mathcal{G})$ is flat if and only if it is uniformly 2-flat. When S is Q^n, $n \geq 3$, given a point $x \in X$ the fibered space $\pi : \mathcal{Q} \to X$ of null tangents is always tangent at x to that of the flat Q^n-structure in terms of holomorphic normal coordinates at x. Thus, the only obstruction tensor is $\theta_1 \in H^0\big(X, \mathrm{Hom}(\Lambda^2 T_X, \mathrm{End}(T_X))\big)$, which agrees with the Bochner-Weyl tensor $({W_{ijk}}^{\ell})$ of the holomorphic conformal structure.

***Theorem 4* (Hwang-Mok [14]).** — *Let X be a uniruled projective manifold admitting an irreducible reductive* G*-structure,* $\mathrm{G} \subsetneq \mathrm{GL}(V)$*. Then, X is biholomorphic to an irreducible Hermitian symmetric space of the compact type and of rank ≥ 2.*

Outline of Proof. — Associated to a G-structure with $\mathrm{G} \subsetneq GL(V)$ reductive, there is on X the fibered space $\lambda : \mathcal{W} \to X$ of highest weight tangents. We show first of all that the latter agrees with the fibered space $\pi : \mathcal{C} \to X$ of VMRTs. The proof makes use of Grothendieck's classification of G-principal bundles on $\mathbb{P}^1$ in [4]. Then, we show that the G-structure is flat by proving successively the vanishing of the structure functions c^k. Finally, we identify the candidates of VMRTs on X to show that they correspond to S-structures in the Hermitian symmetric case, and conclude that $X \cong S$ by observing that X is rationally connected, hence simply connected.

To prove the vanishing of the structure functions c^k it suffices to prove the vanishing of the obstruction tensors $\theta = \theta_k$, which give in the reductive case sections in $H^0\big(X, \mathrm{Hom}(\Lambda^2 T_X, T_X \otimes S^k T_X^*)\big)$. Let ℓ be a standard rational curve, assumed embedded for convenience, so that $T_X|_\ell \cong \mathcal{O}(2) \oplus (\mathcal{O}(1))^p \oplus \mathcal{O}^q$. Each direct summand of $(T_X \otimes S^k T_X^*)|_\ell$ is of degree ≤ 2. If we fix $x \in X$, then $\theta_x(\alpha, \xi) = 0$ whenever $\alpha \in \widetilde{\mathcal{C}}_x$ and $\xi \in T_\alpha(\widetilde{\mathcal{C}}_x) = P_\alpha$, since $\alpha \wedge \xi$ belongs to a direct summand of degree 3. By [(3.2), Proposition 3], such elements generate $\Lambda^2 T_x(X)$, and we conclude that $\theta \equiv 0$. □

In the same vein Hong ([6], Proposition (3.1.4)) established the following characterization of Fano homogeneous contact manifolds of Picard number 1. The statement here is a slight modification of the original one which is implicit from the proof there.

***Theorem 5* (Hong [6]).** — *Let S be a Fano homogeneous contact manifold of Picard number 1 different from an odd-dimensional projective space. Let $\mathcal{C}_0 \subset \mathbb{P}T_0(S)$ be the* VMRT *of S at a reference point $0 \in S$. Let X be a Fano manifold of Picard number 1 whose* VMRT *$\mathcal{C}_x \subset \mathbb{P}T_x(S)$ at $x \in X$ is isomorphic to $\mathcal{C}_0 \subset \mathbb{P}T_0(S)$ as a projective subvariety for x lying outside a subvariety $Z \subset X$ of codimension ≥ 2. Denoting by D the distribution on X spanned by* VMRTs*, assume that the Frobenius form $\varphi : \Lambda^2 D \to T_X/D$ is everywhere non-degenerate on $X - Z$. Suppose furthermore that at every point $x \in X - Z$, a general minimal rational curve passing through x lies on $X - Z$. Then, X is biholomorphic to S.*

4.4. Prolongation of linear subalgebras of infinitesimal automorphisms of VMRTs. — Let $(X, \mathcal{K})$ be a uniruled projective manifold equipped with a minimal rational component with non-linear VMRTs, and $x \in X$ be a general point. Regarding the VMRTs in a neighborhood of x as defining a germ of geometric structure at x, we are interested to study its germs of infinitesimal automorphisms vanishing at x. By Cartan-Fubini extension, as to be explained in §5, this is the same as studying holomorphic vector fields on X vanishing at x. As a preparation we have

Lemma 5. — *Let X be a complex manifold, $x \in X$ be any point, $m \geq 1$ be a positive integer and Z be a holomorphic vector field vanishing at x to the order $\geq m$. Let $\{\varphi_t\}$ be the complex 1-parameter group of automorphisms on X generated by Z. Let $E \subset \mathbb{P}T_X$ be an irreducible subvariety invariant under the induced automorphisms $\{\Phi_t\}$ on $\mathbb{P}T_X$. Assume that $\pi|_E : E \to X$ is a holomorphic submersion at a general smooth point of $E_x := E \cap \mathbb{P}T_x$. In terms of local holomorphic coordinates (z_i) at x; $z_i(x) = 0$; write $Z = \sum_{i_1 \ldots i_m; k} A^k_{i_1 \cdots i_m} z^{i_1} \cdots z^{i_m} \frac{\partial}{\partial z_k} + O(|z|^{m+1})$, where the Taylor coefficients $A^k_{i_1, \cdots, i_m}$ are symmetric in $i_1, \cdots i_m$. Then, regarding the Taylor coefficients of m-th order terms as coefficients of a homomorphism $A : S^m T_x \to T_x$; for any choice of $m-1$ tangent vectors $\eta_1, \cdots, \eta_{m-1}$; the linear vector field $\sum_i w^i A(\eta_1, \cdots, \eta_{m-1}, \frac{\partial}{\partial w_i})$ on T_x is tangent to $\widetilde{E}_x$ at its smooth points.*

Proof. — Write $\varphi^k_t(z) = z + \sum B^k_{i_1 \cdots i_m}(t) z^{i_1} \cdots z^{i_m} + O(|z|^{m+1})$ for z lying on a small neighborhood of x and for t sufficiently small, where the summation is over $(i_1, \cdots, i_m)$. We have $\frac{\partial}{\partial t} B^k_{i_1 \cdots i_m}(t)|_{t=0} = A^k_{i_1 \cdots i_m}$. Writing (w_i) for fiber coordinates for T_X induced by (z_i), the induced automorphism Φ_t on T_X is given by

$$\Phi_t(z, w) = \big(\varphi_t(z); d\varphi_t(z)(w)\big) =$$

$$\sum \Big(B^k_{i_1 \cdots i_m}(t) z^{i_1} \cdots z^{i_m} e_k + O(|z|^{m+1}); m B^k_{i_1 \cdots i_m}(t) z^{i_1} \cdots z^{i_{m-1}} w^{i_m} \epsilon_k + O(|z|^m |w|) \Big)$$

Here $e_k = \frac{\partial}{\partial z_k}$ and $\epsilon_k = \frac{\partial}{\partial w_k}$. Since φ_t preserves the subvariety $\widetilde{E}$, the infinitesimal automorphism $\widetilde{Z} = \frac{\partial}{\partial t} \Phi_t|_{t=0}$ is tangent to $\widetilde{E}$ at smooth points. It is given by

$$\widetilde{Z} = \sum \Big(A^k_{i_1 \cdots i_m} z^{i_1} \cdots z^{i_m} e_k + O(|z|^{m+1}); m A^k_{i_1 \cdots i_m} z^{i_1} \cdots z^{i_{m-1}} w^{i_m} \epsilon_k + O(|z|^m |w|) \Big),$$

showing that the latter vanishes on T_x to the order $\geq m-1$. Taking partial derivatives $m-1$ times against horizontal constant vector fields $\eta_1, \cdots \eta_{m-1}$. we obtain $\sigma := \sum_{i,k} A^k_{\eta_1 \cdots \eta_{m-1} i} w^i \frac{\partial}{\partial w_k} = \sum_i w^i A\big(\eta_1, \eta_2, \cdots, \eta_{m-1}, \frac{\partial}{\partial w_i}\big)$. When $m = 1$ no differentiation is involved, and σ is simply the restriction of $\widetilde{Z}$ to T_x. Since at a smooth point of $\widetilde{E}_x$, σ is both tangent to $\widetilde{E}$ and to T_x, it must be tangent to $\widetilde{E}_x$, as desired. □

Lemma 6. — *Let X be an n-dimensional uniruled projective manifold admitting a minimal rational component whose* VMRT *$\mathcal{C}_x \subset \mathbb{P}T_x$ at a general point x is p-dimensional; $0 < p < n-1$; nonsingular and linearly non-degenerate. Given a general point $x \in X$, let Z be a holomorphic vector field vanishing at x to the order ≥ 2. In terms of local holomorphic coordinates (z_i) in a neighborhood of x; $z_i(x) = 0$; write $Z = \sum_{i,j,k} A^k_{ij} z^i z^j \frac{\partial}{\partial z_k} + O(|z|^3)$, where $A^k_{ij} = A^k_{ji}$. Then, regarding A^k_{ij} as coefficients of a linear homomorphism $A : S^2 T_x \to T_x$ we have $A_{\alpha\alpha} \in \mathbb{C}\alpha$ for any $\alpha \in \widetilde{\mathcal{C}}_x$.*

Proof. — By Lemma 5, for any $\eta \in T_x$ and any nonzero $\alpha \in \widetilde{\mathcal{C}}_x$ we have $A_{\alpha\eta} \in T_\alpha(\widetilde{\mathcal{C}}_x) = P_\alpha$. In particular, if η is itself a nonzero vector in $\widetilde{\mathcal{C}}_x$, we have from the symmetry of A the property that $A_{\alpha\beta} \in P_\alpha \cap P_\beta$. The rest of the proof is the same as in Lemma 2. (Here $A_{\alpha\beta}$ plays the same role as $\partial^2 f(\alpha, \beta)$ there.) □

Proposition 4. — *Under the hypothesis in Lemma 6 and in the notations there, suppose for the holomorphic vector field Z vanishing at x to the order ≥ 2 we have $A_{\alpha\alpha} = 0$ for any $\alpha \in \mathcal{C}_x$, then $A \equiv 0$.*

Sketch of proof. — Fixing $\eta \in T_x(X)$, $A_{\alpha\eta} \in P_\alpha$. From $A_{\alpha\alpha} = 0$ for every $\alpha \in \mathcal{C}_x$, varying $\alpha = \alpha(t)$ holomorphically and differentiating against t we conclude that $A_{\alpha\xi} = 0$ for every $\xi \in P_\alpha$. Thus, regarding A_α as an endomorphism of $T_x(X)$ given by $A_\alpha(\eta) = A_{\alpha\eta}$, we have $\mathrm{Im}(A_\alpha) \subset P_\alpha \subset \mathrm{Ker}(A_\alpha)$, so that $A_\alpha^2 = 0$. Thus, choosing sufficiently general points $\alpha, \beta \in \mathcal{C}_x$, the closure of the orbit of $[\alpha]$ under $\exp(tA_\beta)$ is a line joining $[\alpha]$ to $[\xi]$, where $\xi := A_{\alpha\beta} \neq \alpha, \beta$; likewise with α and β interchanged. Hence $\mathcal{C}_x$ is rationally 2-connected by lines. Proposition 4 is proven inductively. We denote by $\mathcal{K}'$ a minimal rational component consisting of lines on $\mathcal{C}_x$, and $\mathcal{C}'_{[\alpha]}$ the associated VMRT at $[\alpha]$. For induction we replace x by $[\alpha]$, X by $\mathcal{C}_x$, and consider the VMRT $\mathcal{C}'_{[\alpha]}$ at $[\alpha] \in \mathcal{C}_x$. Given a holomorphic vector field Z vanishing at x to the order ≥ 2 for which $A_{\alpha\alpha} = 0$ for every $\alpha \in \mathcal{C}_x$, we derive a holomorphic vector field $\mathcal{Z}$ on $\mathcal{C}_x$ vanishing at $[\alpha]$ to the order ≥ 2 such that $\mathcal{A}_{\mu\mu} = 0$ for every $\mu \in \mathcal{C}'_\alpha$.

Starting with the data $\big(X, \mathcal{K}, x, \mathcal{C}_x, Z, (A_{ij})\big)$ we derive $\big(\mathcal{C}_x, \mathcal{K}', [\alpha], \mathcal{C}'_{[\alpha]}, \mathcal{Z}, (\mathcal{A}_{k\ell})\big)$, noting that $\mathcal{C}'_{[\alpha]}$ is nonsingular at a general point $[\alpha] \in \mathcal{C}_x$, by Lemma 3. To be able to proceed by induction on the dimension, it remains to prove that $\mathcal{C}'_{[\alpha]} \subset \mathbb{P}T_{[\alpha]}(\mathcal{C}_x)$ is linearly non-degenerate. From the fact that $\mathcal{C}'_{[\alpha]}$ is rationally 2-connected by lines, it follows that $\dim(\mathcal{C}'_{[\alpha]}) \geq \frac{1}{2}\dim(\mathbb{P}T_{[\alpha]}(\mathcal{C}_x))$, and by [(3.2), Proposition 3] it would follow that $\mathcal{C}'_{[\alpha]}$ is linearly non-degenerate in $\mathbb{P}T_{[\alpha]}(\mathcal{C}_x)$, if we knew that $\mathcal{C}_{[\alpha]}$ is of Picard number 1. However, the latter need not be the case. Nonetheless, the proof of Proposition 3 still works since we know that the VMRT is rationally 2-connected by lines as explained, making it possible to prove Proposition 4 by induction.

Write $\mathfrak{f}$ for the germs of $\mathcal{C}$-preserving holomorphic vector fields at x. For $\ell \geq -1$, write $\mathfrak{f}^\ell$ for the vector subspace of all $Z \in \mathfrak{f}$ vanishing to the order $\geq \ell + 1$ at x. Then Proposition 4 says that, under the assumption that the VMRT $\mathcal{C}_x \subsetneq \mathbb{P}T_x(X)$ is irreducible, nonsingular and linearly non-degenerate, there is an injection of $\mathfrak{f}^1$ into $\Gamma(\mathcal{C}_x, \mathrm{Hom}(L^2, L)) = \Gamma(\mathcal{C}_x, L^*)$, where L stands for the tautological line bundle over $\mathbb{P}T_x(X)$. If furthermore $\mathcal{C}_x$ is linearly normal in $\mathbb{P}T_x(X)$, i.e., the embedding of $\mathcal{C}_x \subset \mathcal{P}T_x(X)$ is defined by a complete linear system, then $\dim(\mathfrak{f}^1) \leq n$. From the proof of Proposition 4 it follows readily that $\mathfrak{f}^\ell = 0$ for $\ell \geq 2$, i.e., there does not exist any nontrivial holomorphic vector field vanishing at x to the order ≥ 3. In fact, if a $\mathcal{C}$-preserving germ of holomorphic vector field Z vanishes at x to the order ≥ 2, and A_{ijk} are the coefficients of the third order terms of the Taylor expansion of Z at x, then for any $\gamma \in T_x(X)$, $B_{\alpha\beta} = A_{\alpha\beta\gamma}$ defines a 2-tensor for which the arguments apply, and from there the vanishing of A_{ijk} follows easily. The same arguments apply to the leading terms of any nontrivial holomorphic vector field Z vanishing at x to the order $s \geq 3$, and we have a contradiction unless $Z \equiv 0$. □

Lemma 5 can be stated in the language of prolongation theory for Lie subalgebras of $\mathrm{End}(T_x(X))$, as follows. Let V complex vector space, $\dim V = n$, and $\mathfrak{g} \subset \mathrm{End}(V)$ be a Lie subalgebra. For $k \geq -1$ denote by $\mathfrak{g}^{(k)} \subset S^{k+1}V^* \otimes V$ the vector subspace consisting of all $\sigma \in S^{k+1}V^* \otimes V$ such that, writing $\sigma_{v_1,\cdots,v_k}(v) = \sigma(v; v_1, \ldots, v_k)$, we have $\sigma_{v_1,\ldots,v_k} \in \mathfrak{g}$. Let now $Y \subset \mathbb{P}V$ be a projective subvariety, and $\widetilde{Y} \subset V$ be its lifting to V. We write $\mathfrak{aut}(Y) := \{A \in \mathrm{End}(V) : \exp(tA)(\widetilde{Y}) \subset \widetilde{Y}$ for all $t \in \mathcal{C}\}$. Then for every $\ell \geq 0$, $\mathfrak{f}^\ell \subset \mathfrak{aut}(Y)^{(\ell)}$. The argument in the proof of Proposition 4 applies to elements of $\mathfrak{aut}(\mathcal{C}_x)^{(\ell)}$ to imply that that $\dim(\mathfrak{aut}(\mathcal{C}_x)^{(1)} \leq \dim \Gamma(\mathcal{C}_x, L^*)$, and hence that $\mathfrak{aut}(\mathcal{C}_x)^{(\ell)} = 0$ whenever $\ell \geq 2$. In relation to holomorphic vector fields on a Fano manifold of Picard number 1 there are the following conjectures and results.

Conjecture 2. — *Let X be a Fano manifold of Picard number 1. Then, at a general point $x \in X$ there does not exist any nontrivial holomorphic vector field Z vanishing at x to the order ≥ 3.*

Conjecture 3. — *Let X be an n-dimensional Fano manifold of Picard number 1, Then,* $\dim(\mathrm{Aut}(X)) \leq n^2 + 2n$. *Moreover, equality holds if and only if $X \cong \mathbb{P}^n$.*

***Theorem 6* (Hwang-Mok [23])**. — *Let $(X, \mathcal{K})$ be a uniruled projective manifold equipped with a minimal rational component. Suppose the variety of minimal rational tangents $\mathcal{C}_x \subset \mathbb{P}T_x(X)$ at a general point $x \in X$ is irreducible, nonsingular and linearly non-degenerate. Then, at a general point $x \in X$ there does not exist any nontrivial holomorphic vector field vanishing at x to the order ≥ 3. If furthermore $\mathcal{C}_x \subset \mathbb{P}T_x(X)$ is linearly normal, then $\dim(\mathfrak{aut}(\mathcal{C}_x)^{(1)}) \leq n$, and equality holds if and only if $\mathcal{C}_x \subset \mathbb{P}T_x(X)$ is congruent to $\mathcal{C}_0 \subset \mathbb{P}T_0(S)$ for the variety of minimal rational tangents of an irreducible Hermitian symmetric space of the compact type. Furthermore,* $\dim(\mathrm{Aut}(X)) \leq n^2 + 2n$, *and equality holds if and only if $X \cong \mathbb{P}^n$.*

Remarks. — As will be seen in (6.3) the statement that $\mathcal{C}_x \subset \mathbb{P}T_x(X)$ is congruent to $\mathcal{C}_0 \subset \mathbb{P}T_0(S)$ implies that X is biholomorphic to S.

Corollary 1. — *Let X be an n-dimensional Fano manifold of Picard number 1, and denote by $\mathcal{O}(1)$ the positive generator of* $\mathrm{Pic}(X) \cong \mathbb{Z}$. *Assume that $\mathcal{O}(1)$ is very ample. Suppose $c_1(X) > \frac{n+1}{2}$. Then, for a general point $x \in X$ there does not exist any nontrivial holomorphic vector field vanishing at x to the order* ≥ 3. *Suppose X satisfies the stronger condition $c_1(X) > \frac{2(n+2)}{3}$, then* $\dim(\mathrm{Aut}(X)) \leq n^2 + 2n$, *and equality holds if and only if $X \cong \mathbb{P}^n$.*

In relation to VMRTs in general the following conjecture summarizes what one can optimistically hope as compared to known results in [(2.4), Theorem 1].

Conjecture 4. — *Let $(X, \mathcal{K})$ be a uniruled projective manifold equipped with a minimal rational component, and $\pi : \mathcal{C} \to X$ be the fibered space of varieties of minimal rational tangents associated to $\mathcal{K}$. Then, at a general point $x \in X$, either $\mathcal{C}_x$ is finite, or it is irreducible, nonsingular and linearly normal in its linear span $\mathbb{P}W_x \subset \mathbb{P}T_x(X)$.*

Regarding Conjectures 2 and 3, the fundamental assumption in the partial result (Theorem 6) is the linear non-degeneracy of the VMRT $\mathcal{C}_x$ at a general point. At least the statement regarding vanishing order of holomorphic vector field is accessible whenever an irreducible component of $\mathcal{C}_x$ is linearly non-degenerate.

4.5. Applications to rigidity under Kähler deformation. — We return to the question of rigidity of rational homogeneous manifolds $S = G/P$ of Picard number 1 under Kähler deformation, as given in [(3.4), Theorem 2]. In (3.4) we explained that for the case of $P \subset G$ defined by a long simple root, the problem is solved by studying the integrability of distributions spanned by or derived from VMRTs. In Hwang-Mok [21] we settled the problem for $G = F_4$ for the 20-dimensional F_4-homogeneous space associated to a short root. There we have the nilpotent graded algebra $\mathfrak{n} = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \mathfrak{g}_3 \oplus \mathfrak{g}_4$. As opposed to the long root case the VMRT does not lie in $\mathbb{P}D_1$ for the minimal proper G-invariant distribution D_1, but it remains linearly degenerate, spanning the proper G-invariant distribution $D_2 \neq T_S$, and the method using distributions spanned by VMRTs and Yamaguchi [51] is still applicable.

What remain are the cases of $S = G/P$ defined by short simple roots in the cases of C_n, and the 15-dimensional case of type F_4. In both cases we have $\mathfrak{n} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$, the VMRT $\mathcal{C}_x$ at any point $x \in S$ is almost homogeneous with two orbits corresponding to highest weight vectors in $\mathfrak{g}_1$ resp. $\mathfrak{g}_2$, and $\mathcal{C}_x \subset \mathbb{P}T_x(S)$ is linearly non-degenerate. The problem is solved in Hwang-Mok [23] (2005). To proceed we showed that the VMRT at a general point of the central fiber X_0 of $\pi : \mathcal{X} \to \Delta$ remains isomorphic to that of the model space $\mathcal{C}_0 \subset \mathbb{P}T_0(S)$. On X_0 we still have a 2-step filtration $0 \subset D^1 \subset D^2 = T_{X_0}$, but $\mathcal{C}_{x_0} \cap \mathbb{P}D^1_{x_0}$ does not have an algebro-geometric meaning, and the methods involving distributions spanned by VMRTs do not apply.

To solve the problem we examine the Lie algebra of holomorphic vector fields on X_0 which occur as limits of those on $X_t, t \neq 0$, with an aim to recuperating the Lie algebra $\mathfrak{g}$ on X_0. For illustration we consider the Hermitian symmetric case and sketch a proof in the last step using holomorphic vector fields in place of Ochiai's Theorem on S-structures. We assume already known that, over a suitably chosen holomorphic section $\sigma : \Delta \to \mathcal{X}$, the VMRTs of $\mathcal{C}_{\sigma(t)} \subset \mathbb{P}T_{\sigma(t)}(X_t)$ on $X_{\sigma(t)}$ form a holomorphically trivial family of projective submanifolds all congruent to $\mathcal{C}_0 \subset T_0(S)$ on the model space S. Writing $\mathcal{T}$ for the relative tangent sheaf of $\pi : \mathcal{X} \to \Delta$, the direct image $\mathcal{V} = \pi_*\mathcal{T}$ is the sheaf of germs of sections of a holomorphic vector bundle V on Δ, where for $t \neq 0, \mathfrak{g}^t := V_t$ carries naturally the structure of a Lie algebra isomorphic to the Lie algebra $\mathfrak{g}$ of $G = \mathrm{Aut}_0(S)$, and our aim is to prove that this remains true at $t = 0$. The idea is to reconstruct the Lie algebra structure from data that can be recovered along $\sigma : \Delta \to \mathcal{X}$. For the model space $S = G/P$ we have the decomposition of the Lie algebra $\mathfrak{g}$ of G as a graded Lie algebra, and equivalently the Harish-Chandra decomposition (in the notations of (4.2)) given by

$$\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 = \mathfrak{m}^- \oplus \mathfrak{k}^{\mathbb{C}} \oplus \mathfrak{m}^+;$$

$$[\mathfrak{m}^-, \mathfrak{m}^-] = [\mathfrak{m}^+, \mathfrak{m}^+] = 0, \quad \text{where} \quad \mathfrak{m}^- = \{Z \in \Gamma(S, T_S) : \mathrm{ord}_0 Z \geq 2\}.$$

For $k, k' \in \mathfrak{k}^{\mathbb{C}}$, $m^+ \in \mathfrak{m}^+$ and $m^- \in \mathfrak{m}^-$ the Lie brackets $[k, m^+] \in \mathfrak{m}^+, [k, m^-] \in \mathfrak{m}^-, [k, k'], [m^-, m^+] \in \mathfrak{k}^{\mathbb{C}}$ are completely determined by the leading terms of the Lie algebra elements at 0. Here the leading term stands for the 0-th order term for m^+, the first-order term for k and k', and the second-order term for m^-. For a holomorphic vector field Z on X_t vanishing at $\sigma(t)$ we denote by A_Z the coefficient matrix for the linear term of Z, which defines an element of $\mathrm{End}(T_{\sigma(t)}(X_t))$. Define

$$J_t^{(k)} = \{Z \in \mathfrak{g}^t : \mathrm{ord}_{\sigma(t)}(Z) \geq k\}; \quad I_t = \{Z \in \mathfrak{g}^t : Z(\sigma(t)) = 0\ ,\ A_Z \in \mathbb{C} \cdot \mathrm{id}\}\ .$$

For $t \neq 0$ we have $\dim J_t^{(2)} = n; \dim J_t^{(k)} = 0$ for any $k \geq 3$, and $\dim I_t = n+1$, and any $Z \in I_t$, $A_Z \not\equiv 0$ determines a $\mathbb{C}^*$-action. Since $\mathcal{C}_{\sigma(0)} \subset \mathbb{P}T_{\sigma(0)}(I_t)$ is conjugate to $\mathcal{C}_0 \subset \mathbb{P}T_0(S)$, by [(4.4), Theorem 6] we have

$$\dim J_0^{(2)} \leq n\ ,\ J_0^{(k)} = 0 \text{ for } k \geq 3\ .$$

Thus, $\dim I_0 \leq n+1$ while $\dim I_0 \geq n+1$ by upper semicontinuity of $\dim I_t$ in $t \in \Delta$. Therefore, $\dim I_0 = n+1$, so that there exists $Z \in I_0$ such that $A_Z \not\equiv 0$ and such that $e^{\lambda Z}$ defines a $\mathbb{C}^*$-action on X_0 of period $2\pi i$ in λ. This $\mathbb{C}^*$-action on X_0 can be extended to a holomorphic family T_t of $\mathbb{C}^*$-actions on X_t, of period $2\pi i$ in λ, given by $T_t(\lambda) = e^{\lambda E_t}$, $E_0 = Z$. Finally, defining

$$\mathfrak{g}_i^t := \{Z \in \mathfrak{g}^t : [E_t, Z] = iZ\}; \quad \text{we have} \quad \mathfrak{g}^t = \mathfrak{g}_{-1}^t \oplus \mathfrak{g}_0^t \oplus \mathfrak{g}_1^t\ .$$

For $t \neq 0$,

$$\mathfrak{g}_0^t \cong \{A \in \mathrm{End}_{\sigma(t)}(T_{\sigma(t)}) : A|_{\widetilde{\mathcal{C}}_{\sigma(t)}} \text{is tangent to } \widetilde{\mathcal{C}}_{\sigma(t)}\}.$$

Dimension count forces the same for $t = 0$. The Lie algebra structure on $\mathfrak{g}^0$ is determined by leading terms at $\sigma(0)$ of elements in $\mathfrak{g}_{-1}^0$, $\mathfrak{g}_0^0$ and $\mathfrak{g}_1^0$. Clearly, the rules for taking Lie brackets by means of the leading terms at σ_0 agrees with those at $0 \in S$ for the model space, and we have shown that $X_0 = G/P \cong S$.

Let $n \geq 2$ and W be a $2n$-dimensional complex vector space equipped with a symplectic form ν. For $1 < k < n$ we denote by $S_{k,n}$ the symplectic Grassmannian of k-planes V in W isotropic with respect to ν. The symplectic Grassmannian $S = S_{k,n}$ is clearly homogeneous under the group G of symplectic transformations of W, $G \cong \mathrm{Sp}(n, \mathbb{C})$. It is a complex submanifold of the Grassmannian $\mathrm{Gr}(k, W)$ of k-planes in W. With respect to the Plücker embedding $p : \mathrm{Gr}(k, W) \to \mathbb{P}^N$, a line ℓ on S passing through the point $[V] \in S$, where $V = V^{(k)}$, is defined by the choices of a $(k-1)$-plane $E^{(k-1)}$ and a $(k+1)$-plane $F^{(k+1)}$ such that $E^{(k-1)} \subset V^{(k)} \subset F^{(k+1)}$. There are precisely two distinct isomorphism classes of lines with respect to the action of $\mathrm{Sp}(W)$, according to whether $\nu|_F$ is isotropic or otherwise. The VMRT $\mathcal{C}_0$ at $0 \in S$ is only *almost* homogeneous with precisely two orbits. Since $S \subset \mathrm{Gr}(k, W) \subset \mathbb{P}^N$ is uniruled by lines, $\mathcal{C}_0 \subset \mathbb{P}T_0(S)$ is non-singular. As a rational homogeneous manifold $S_{k,n}$ is of type C_n, corresponding to a short simple root α_k, $1 < k < n$. The tangent bundle of $T_{S_{k,n}}$ has exactly one proper invariant distribution, and we have a decomposition $\mathfrak{g} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$. From this description the $\mathrm{SL}(2, \mathbb{C})$-orbit of a highest weight vector of $\mathfrak{g}_1$ gives a highest weight line which is a minimal rational curve. Such a line corresponds to a line $\ell \subset S_{k,n}$ arising from the choice of some $F^{(k+1)} \supset E^{(k)}$ for which $\nu|_F \equiv 0$. When $S = G/P$ is defined by a short simple root, the $\mathrm{SL}(2, \mathbb{C})$-orbit C_s defined by a highest weight vector of $\mathfrak{g}_s$ need not be of degree s. In the case of $s = 2$ for $S = S_{k,n}$, $C = C_2$ is in fact a line, and it corresponds to the generic choice of $F^{(k+1)}$ so that $\nu|_F \not\equiv 0$. From this description the VMRT $\mathcal{C}_0 \subset \mathbb{P}T_0(S_{k,n})$ is linearly non-degenerate, and the question of rigidity under Kähler deformation of symplectic

Grassmannians $S_{k,n}, 1 < k < n$ is therefore susceptible to be studied by means of the method of prolongation of infinitesimal automorphisms of VMRTs, as is the case of irreducible Hermitian symmetric spaces of rank ≥ 2.

The proof of deformation rigidity for $S_{k,n}$ and also for the remaining 15-dimensional case of type F_4 were settled along the line of arguments as sketched for the Hermitian symmetric case. For the graded Lie algebra $\mathfrak{g} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$ of the model space, the summands $\mathfrak{g}_i$ can be described in terms of conditions on vanishing orders and leading terms of holomorphic vector fields, and the multiplication table of $\mathfrak{g}$ as a Lie algebra can be determined to a good extent from the leading terms. For instance, denoting by $D \subset T_S$ the proper invariant distribution $D \subset T_S$, $\mathfrak{g}_{-1}$ consists of holomorphic vector fields Z vanishing at $0 \in S$ to the order ≥ 1 with leading term corresponding to $A_Z \in \mathrm{End}(T_0(S))$ satisfying $A|_{D_1} \equiv 0$, and $\mathfrak{g}_{-2} \subset \mathfrak{g}$ is the subspace consisting of holomorphic vector fields vanishing to the order ≥ 2. Nonetheless, as opposed to the Hermitian symmetric case, the structure of the Lie algebra $\mathfrak{g}$ thus obtained is incomplete. In the case of the symplectic Grassmannian $S = S_{k,n}$ the missing element is some symplectic form appearing implicitly in the Frobenius form $\varphi : \Lambda^2 D \to T_S/D$. From $\pi : \mathcal{X} \to \Delta$ we are able to identify the structure of the Lie algebra $\mathfrak{g}^0$ of limit holomorphic vector fields at the central fiber X_0, thereby showing that X_0 is obtained by blowing down some holomorphic fiber bundle, and the final step is achieved by showing that, in the event that there is actually a degeneration of the Lie algebra structure, singularities must occur in the blown-down space, contradicting the starting point that $\pi : \mathcal{X} \to \Delta$ is a regular family.

5. Analytic continuation of VMRT-preserving maps

5.1. Characterization of the tautological foliation under a non-degeneracy condition on the Gauss map. — Let $x \in X$ be a general point and $u \in \mathcal{U}_x$ be a point such that $\kappa := \rho(u) \in \mathcal{K}$ is a standard rational curve. Then, the tangent map τ is a holomorphic immersion at u, and it maps some open neighborhood $\mathbb{W}$ of u in $\mathcal{U}$ biholomorphically onto some locally closed complex submanifold Ω of $\mathbb{P}T_X$. Ω gives the germ of some irreducible branch of $\mathcal{C}$ at $[\alpha]$. Choosing x and $u \in \mathcal{U}_x$ sufficiently general and $\mathbb{W}$ sufficiently small we assume furthermore that $[\alpha] \in \mathcal{C}$ is a smooth point and that Ω is a neighborhood of $[\alpha]$ in $\mathcal{C}$.

On Ω we define a distribution $\mathcal{P}$, as follows. Let $f : \mathbb{P}^1 \to \mathcal{U}$ be a parametrization of κ. The base point $x \in X$ is a smooth point of the support $C := \mu(\rho^{-1}(\kappa))$ of the standard rational curve κ. The decomposition $f^*T_X \cong \mathcal{O}(2) \oplus (\mathcal{O}(1))^p \oplus \mathcal{O}^q$ over $\mathbb{P}^1$ gives a filtration $T_{\mathbb{P}^1} \subset Q \subset f^*T_X$ of f^*T_X over $\mathbb{P}^1$, where $Q = \mathcal{O}(2) \oplus (\mathcal{O}(1))^p$ is the positive part of f^*T_X, which is well-defined since $Q \otimes \mathcal{O}(-1) \subset f^*T_X \otimes \mathcal{O}(-1)$ is the vector subbundle spanned by global sections. At the point $x = f(0)$ we have correspondingly a filtration $T_x(C) \subset P_x \subset T_x(X)$, where $P_x = df(Q_0)$. Define now $\mathcal{P}_{[\alpha]} \subset T_{[\alpha]}(\mathcal{C})$ to be the vector subspace consisting of all tangent vectors η such that $d\pi(\eta) \in P_x$. The tangent vector η is equivalently the image under $d\tau$ of some $\overline{\sigma}$, where $\sigma \in H^0(\mathbb{P}^1, f^*T_X)$, and $\overline{\sigma} := \sigma \bmod df(H^0(\mathbb{P}^1, T_{\mathbb{P}^1} \otimes \mathcal{I}_0))$. For the universal family $\rho : \mathcal{U} \to \mathcal{K}$ we have $d\rho(\overline{\sigma}) = \sigma \bmod df(H^0(\mathbb{P}^1, T_{\mathbb{P}^1})$. Equivalently, writing $\widehat{\rho} := \rho \circ \tau^{-1}$ over

Ω, where $\tau^{-1} : \Omega \to \mathcal{W}$, we have $d\widehat{\rho}(\eta) = \sigma \bmod df(H^0(\mathbb{P}^1, T_{\mathbb{P}^1}))$. The assumption that $d\pi(\eta) \in P_x$ means precisely that $\sigma(0) \in Q_0$, thus $\sigma' := \sigma \bmod Q \in H^0(\mathbb{P}^1, \mathcal{O}^q)$ must vanish at 0 and hence on all of $\mathbb{P}^1$, showing that $\sigma \in H^0(\mathbb{P}^1, Q)$. On an open neighborhood $\mathbb{U}$ of κ in $\mathcal{K}$ consisting solely of standard rational curves we define a distribution $\mathcal{D} \subset T_{\mathcal{K}}|_{\mathbb{U}}$ by setting $\mathcal{D}_\kappa := H^0(\mathbb{P}^1, Q) \bmod df(H^0(\mathbb{P}^1, T_{\mathbb{P}^1})) \cong \mathbb{C}^{2p}$. Then, for $\xi \in T_{[\alpha]}$ we have $d\widehat{\rho}(\xi) \in \mathcal{D}_\kappa$ if and only if $d\pi(\xi) \in P_x$, hence $\mathcal{P}_{[\alpha]} = (d\widehat{\rho})^{-1}(\mathcal{D}_\kappa)$. Finally there is a 1-dimensional distribution underlying the tautological foliation $\mathcal{F}$ on Ω which will be denoted by the same symbol $\mathcal{F}$. Thus, $\mathcal{F}_{[\alpha]} := T_{[\alpha]}(\widehat{\rho}^{\,-1}(\kappa))$.

To relate the distributions $\mathcal{F}, \mathcal{P}$ on Ω and the distribution $\mathcal{D}$ on $\mathcal{G}$ we recall the notion of the Cauchy characteristic of a distribution. Given a complex manifold M and a holomorphic distribution $E \subset T_M$ and denoting by $\mathcal{E}$ the corresponding locally free sheaf of germs of holomorphic sections of E, then $Ch(\mathcal{E}) \subset \mathcal{E}$ is the subsheaf consisting of germs of holomorphic sections ζ such that $[\zeta, \mathcal{E}] \subset \mathcal{E}$. Thus, the Cauchy characteristic $Ch(\mathcal{E}) = \mathcal{E}$ if and only if $E \subset T_M$ is integrable. Outside an analytic subvariety of codimension ≥ 2 the Cauchy characteristic is locally free, and from now on we will make no distinction between a distribution and its associated locally free sheaf, and think of the Cauchy characteristic as a distribution defined outside an analytic subvariety of codimension ≥ 2. To proceed we note

Lemma 7. — *Let $U \subset \mathbb{C}^n$, $V \subset \mathbb{C}^m$ be Euclidean domains, and $\lambda : U \times V \to V$ be the canonical projection. Let $S \subset T_V$ be a holomorphic distribution and $G := (d\lambda)^{-1}(S)$. Write $H \subset T_{U\times V}$ for the distribution corresponding to the foliation by fibers of λ, i.e., $H = (d\lambda)^{-1}(0)$. Then, $H \subset Ch(G)$.*

At a general point of the fibered space $\pi : \mathcal{C} \to X$ of VMRTs, *a priori* there can be more than one tautological foliation coming from different sets of families of local holomorphic curves. The question whether the tangent map τ_x is birational at a general point $x \in X$ has to do with uniqueness of the tautological foliation. Such a uniqueness result would follow if the tautological foliation $\mathcal{F}$ can be characterized as in fact the Cauchy characteristic of $\mathcal{P}$ at a general point of $\mathcal{C}$. We have proven that $\mathcal{F} \subset Ch(\mathcal{P})$. For the inverse inclusion we impose an additional assumption on the Gauss map on the VMRT $\mathcal{C}_x \subset \mathbb{P}T_x(X)$ at a general point, a condition that is always satisfied whenever the $\mathcal{C}_x$ is nonsingular and non-linear.

Proposition 5. — *Let $(X, \mathcal{K})$ be a uniruled projective manifold equipped with a minimal rational component, and $\pi : \mathcal{C} \to \mathcal{X}$ be the associated fibered space of* VMRTs. *Let $\Omega \subset \mathcal{C}$ be a connected nonempty open subset consisting of nonsingular points on which both a tautological foliation $\mathcal{F}$ by standard $\mathcal{K}$-curves and hence the corresponding distribution $\mathcal{P}$ are defined. Suppose at a general point $[\alpha] \in \Omega$, $\pi([\alpha]) := x$, the Gauss map of $\mathcal{C}_x \subset \mathbb{P}T_x(X)$ is a holomorphic immersion at $[\alpha]$. Then, $\mathcal{F} = Ch(\mathcal{P})$.*

Proof. — In what follows we denote by $\widetilde{\Omega} = \pi^{-1}(\Omega) \subset \widetilde{\mathcal{C}}$, $\widetilde{\mathcal{P}} = (d\pi)^{-1}(\mathcal{P})$, etc., by lifting to homogenizations. At a general point $\alpha \in \widetilde{\Omega}$ choose local holomorphic coordinates $(z_1, \cdots, z_n)$ at $x = \widetilde{\pi}(\alpha)$ and corresponding fiber coordinates $(w_1, \cdots, w_n)$ on T_X in a neighborhood of u. Suppose $s := \sum g^i \frac{\partial}{\partial z_i} + \sum h^j \frac{\partial}{\partial w_j}$ is a germ of holomorphic section of $\widetilde{\mathcal{P}}$ at u such that $[s, \eta]$ is a germ of $\widetilde{\mathcal{P}}$ at α. Denote by $\mathcal{V} \subset \mathcal{P}$ the subbundle of vertical vectors, i.e., of vectors tangent to the fibers $\mathcal{C}_y$ of $\pi|_\Omega$. Now for η an arbitrary germ of vertical holomorphic vector field at α we have

$$\left[\sum g^i \frac{\partial}{\partial z_i} + \sum h^j \frac{\partial}{\partial w_j}, \sum \eta^k \frac{\partial}{\partial w_k}\right] = \sum \eta^k \frac{\partial g^i}{\partial w_k} \frac{\partial}{\partial z_i} \text{ mod } \widetilde{\mathcal{V}} \,. \qquad (*)$$

The condition that $[s, \eta]$ takes values in $\widetilde{\mathcal{P}}$ implies that $\sum \eta^k \frac{\partial g^i}{\partial w_k} \frac{\partial}{\partial w_i} \in P_\alpha$. Since the germ of vertical vector field η is arbitrary, it follows that $\sum g^i \frac{\partial}{\partial w_i}(\alpha) \in \mathrm{Ker}(\widetilde{\sigma}_\alpha) = \mathbb{C}\alpha$. Thus, $s = \lambda \sum w^i \frac{\partial}{\partial z_i} + \sum h^j \frac{\partial}{\partial w_j}$ for some λ holomorphic. Suppose the holomorphic vector field $\sum w^i \frac{\partial}{\partial z_i} + \sum r^j \frac{\partial}{\partial w_j}$ takes values in $\widetilde{\mathcal{F}}$. Since $\widetilde{\mathcal{F}} \subset Ch(\widetilde{\mathcal{P}})$, comparing s with $\widetilde{\mathcal{F}}$ we conclude that for $\xi := \sum (h^j - \lambda r^j) \frac{\partial}{\partial w_j} \in Ch(\widetilde{\mathcal{P}})$, and to prove Proposition 5 it remains to show that ξ is pointwise a multiple of the Euler vector field $\sum w^j \frac{\partial}{\partial w_j}$ (which descends to 0 when we project from $\widetilde{\mathcal{C}}$ to $\mathcal{C}$). Write $\xi^j := h^j - \lambda r^j$. By the same formula $(*)$ above for Lie bracket, replacing η^k by ξ^k and letting $\sum g^i \frac{\partial}{\partial z_i} + \sum h^j \frac{\partial}{\partial w_j}$ now stand for an arbitrary germ of $\widetilde{\mathcal{P}}$-valued holomorphic vector field at α we conclude that $\sum \xi^k \frac{\partial g^i}{\partial w_k} \frac{\partial}{\partial w_i} \in P_\alpha$ for any choice of (g^i) such that $\sum g^i \frac{\partial}{\partial w_i}$ is a $\widetilde{\mathcal{V}}$-valued germ of holomorphic vector field at α. Hence $\xi \in \mathrm{Ker}(\sigma_\alpha) = \mathbb{C}\alpha$, as desired. □

5.2. Birationality of the tangent map and Cartan-Fubini extension. — The characterization of the tautological foliation under the Gauss map condition (†) in [(4.1), Propoition 5] implies the birationality of the tangent map $\tau_x : \mathcal{K}_x \to \mathcal{C}_x$ under the same condition (Hwang-Mok [17], 1999). Kebekus [25] (2002) proved that any $\mathcal{K}$-curve marked at a general point x is immersed at the marking, and deduced

***Theorem 7* (Kebekus [25]).** — *Let $(X, \mathcal{K})$ be a uniruled projective manifold equipped with a minimal rational component. Then, at a general point $x \in X$, the tangent map $\tau_x : \mathcal{K}_x \to \mathbb{P}T_x(X)$ is a finite holomorphic map.*

Together with Theorem 7 one obtains a proof of [(2.4), Theorem 1], the structure theorem on the tangent map and VMRTs stating that the tangent map is a birational finite holomorphic map at a general point, under the additional Gauss map condition (†). To remove (†) the first question is to characterize the case where $\mathcal{C}_x = \mathbb{P}T_x(X)$. This was obtained by Cho-Miyaoka-Shepherd-Barron ([2], 2002) by a method involving the holomorphicity of the tangent map made possible by Kebekus [25].

***Theorem 8* (Cho-Miyaoka-Shepherd-Barron [3]).** — *Let $(X, \mathcal{K})$ be a uniruled projective manifold equipped with a minimal rational component,* $\dim(X) := n$. *Suppose at a general point $x \in X$ the associated variety of minimal rational tangents $\mathcal{C}_x$ is the same as $\mathbb{P}T_x(X)$. Then, X is biholomorphic to $\mathbb{P}^n$.*

To prove [(2.4), Theorem 1] in its full generality, we considered in Hwang-Mok [22] (2004) the integrable distribution $Ch(\mathcal{P})$ for the distribution $\mathcal{P}$ defined in (4.1). We showed using [25] and [3] that a local leaf of $Ch(\mathcal{P})$ is the projectivized tangent bundle of a locally closed complex submanifold on X which extends to an immersed projective space, and deduce from there the birationality of the tangent map at a general point, leading to a proof of Theorem 1.

The statement of birationality of the tangent map leads to a method of analytic continuation, which we call Cartan-Fubini extension, for local VMRT-preserving biholomorphic maps. In 2004 we proved

***Theorem 9* (Hwang-Mok [22]).** — *Let X and X' be Fano manifolds of Picard number 1 with minimal rational components. Assume that at a general point $x \in X$ the variety of minimal rational tangents $\mathcal{C}_x(X)$ of X is non-linear and of positive dimension. Let $f : U \to U'$ be a biholomorphic map from an open connected subset $U \subset X$ onto $U' \subset X'$. Suppose the differential df sends each irreducible component of $\mathcal{C}(X)|_U$ to an irreducible component of $\mathcal{C}(X')|_{U'}$ biholomorphically. Then, f extends to a biholomorphic map $F : X \to X'$.*

Sketch of proof. — In the case of an irreducible Hermitian symmetric space S of the compact type and of rank ≥ 2, Cartan-Fubini extension is equivalent to Ochiai's Theorem, and in (4.2) we sketched a proof using VMRTs. The analogue of [(4.2), Lemma 4] for Theorem 9 under the additional Gauss map condition (†) is given by [(5.1), Proposition 5]. In Hwang-Mok [18] we proved Theorem 9 under the condition (†), and in [22] the latter condition was removed starting with an extension of the birationality result for non-linear VMRTs. To explain the special case in [18], along the line of argument of (4.2) for a proof of Ochiai's Theorem we can likewise pass to the moduli space $\mathcal{K}$ resp. $\mathcal{K}'$ of minimal rational curves on X resp. X'. Picking a base point $x \in X$, and denoting by $\mathcal{Q}_x \subset \mathcal{K}$ the subspace of minimal rational curves passing through x, $f : U \cong U'$ extends by Proposition 5 to some holomorphic map $f^\sharp$ on some neighborhood $\mathbb{U}$ of $\mathcal{Q}_x$ in $\mathcal{K}$ as in (4.2). In the general case we do not however have the Hartogs-type extension theorem as used in Mok-Tsai [MT] to extend $f^\sharp$ meromorphically to $\mathcal{K}$. Instead, we developed in [18] a method of parametrized analytic continuation along minimal rational curves. Let $\rho : \mathcal{U} \to \mathcal{K}$, $\mu : \mathcal{U} \to X$ be the universal family of $(X, \mathcal{K})$. Fix a standard $\mathcal{K}$-curve $\ell \in \mathcal{K}$ passing through $x \in U$. We have a map $\lambda := f^\sharp \circ \rho \circ \tau^{-1}$ which is defined on some arbitrarily small neighborhood Ω of the tautological lifting $\widehat{\ell}$ of ℓ in $\mathcal{C}$. To extend f meromorphically on a neighborhood of $\ell \in X$ by the argument in (4.2) in which a point y is regarded as the intersection of minimal rational curves passing through y, it is not necessary to have λ defined on all of $\mathcal{C}|_\ell$. It suffices to have λ defined on the arbitrarily small neighborhood Ω of $\widehat{\ell}$, and the upshot is that we can do meromorphic extension of f and $f^\sharp$ simultaneously along a standard $\mathcal{K}$-curve issuing from U. Each general point of X is accessible from U by a finite chain of standard $\mathcal{K}$-curves. Since X is of Picard number 1, the inaccessible points can be cut down to codimension ≥ 2. A major difficulty in completing the proof after meromorphic extension along standard $\mathcal{K}$-curves lies in the lack of univalence, and, after proving univalence, there remains the difficulty due to singularities of the extended map. Overcoming these difficulties necessitates the use of the deformation theory of rational curves, and for the latter difficulty we need to further use the Fano property of both X and X', which gives rise to projective embeddings using positive powers of the anti-canonical line bundle. The proof of Theorem 9 in the general case requires a combination of [18] and the use of integral manifolds of $Ch(\mathcal{P})$ as mentioned in relation to Theorem 8. □

The method of analytic continuation on VMRT-preserving maps makes explicit use of the geometry arising from minimal rational curves. From the perspective of Several Complex Variables, it would be of interest to prove an extension result solely basing on the neighborhood structure of the cycles $\mathcal{Q}_x \subset \mathcal{K}$. Examination of the Hermitian symmetric case suggests that in general one can hope for constructing a fundamental system of pseudoconcave neighborhoods $\mathcal{Q}_x$, thereby guaranteeing meromorphic extension of $f^\sharp$ and hence of f from methods in Several Complex Variables. In this direction the following formulation in a special case is of independent interest.

Conjecture 5. — *Let $(X, \mathcal{K})$ be a Fano manifold of Picard number 1 equipped with a minimal rational component. Assume that at a general point $x \in X$ the moduli space $\mathcal{K}_x$ of $\mathcal{K}$-curves marked at x is irreducible and non-linear, and that the tangent map $\tau_x : \mathcal{K}_x \to \mathcal{C}_x$ is a biholomorphism onto $\mathcal{C}_x$, so that, denoting by $p : \mathcal{K}_x \to \mathcal{K}$ the canonical map, the image $\mathcal{Q}_x = p(\mathcal{K}_x)$ is nonsingular. Let $\mathbb{U} \supset \mathcal{Q}_x$ be any connected open neighborhood of $\mathcal{Q}_x$ in $\mathcal{K}$. Then, any meromorphic function on $\mathbb{U}$ extends to a meromorphic function on $\mathcal{K}$.*

5.3. The Lazarsfeld Problem and other applications of Cartan-Fubini extension. — As an application of the Cartan-Fubini extension on uniruled projective manifolds with non-linear VMRTs ([(5.2), Theorem 9]) we have the following result on the local rigidity of generically finite surjective holomorphic maps of a fixed projective manifold X' onto a Fano manifold $(X, \mathcal{K})$ of Picard number 1 equipped with a minimal rational component with non-linear VMRTs. We have

Theorem 10 **(Hwang-Mok [22])**. — *Let $\pi : \mathcal{X} \to \Delta := \{t \in \mathbb{C}, |t| < 1\}$ be a regular family of Fano manifolds of Picard number 1 so that $X_0 = \pi^{-1}(0)$ has a minimal rational component with non-linear varieties of minimal rational tangents. For a given projective manifold Y, suppose there exists a surjective holomorphic map $f : \mathcal{Y} = Y \times \Delta \to \mathcal{X}$ respecting the projections to Δ so that $f_t : Y \to X_t$ is a generically finite for each $t \in \Delta$. Then, there exists $\epsilon > 0$ and a holomorphic family of biholomorphic maps $\Phi_t : X_0 \to X_t$ for $|t| < \epsilon$, satisfying $\Phi_0 = \mathrm{id}$ and $f_t = \Phi_t \circ f_0$.*

Sketch of proof. — Fix a minimal rational component $\mathcal{K}_0$ on X_0 with non-linear VMRTs. To simplify notations we assume minimal rational curves to be embedded. Let $\ell_0 \subset X_0$ be a $\mathcal{K}_0$-curve. ℓ_0 is also free on $\mathcal{X}$ since $T_{\mathcal{X}}|_{\ell_0} = T_{X_0}|_{\ell_0} \oplus \mathcal{O}$. Consider the space $\mathcal{K}$ of free rational curves on $\mathcal{X}$ obtained by deforming some ℓ_0 in $\mathcal{X}$. Any $\ell \in \mathcal{K}$ must lie on some $X_t, t \in \Delta$. We may think of $(\mathcal{X}, \mathcal{K})$ as a holomorphic family of $(X_t, \mathcal{K}_t)$ fibered over Δ. To simplify the discussion we assume that the VMRTs are irreducible at a general point of $X = X_0$. Shrinking Δ around 0 if necessary we may assume that the VMRT at a general point of X_t remains irreducible.

In Hwang-Mok [16] we introduced the notion of varieties of distinguished tangents on a projective manifold Y (cf. Hwang-Mok [17], §5) which generalizes the notion of VMRTs. Let $y \in Y$ be a very general point, i.e., a point outside some countable union of proper subvarieties. Consider an irreducible component $\mathcal{M}$ of the Chow space of curves on Y, and denote by $\mathcal{M}_y \subset \mathcal{M}$ the subvariety corresponding to curves through y. For curves belonging to $\mathcal{M}_y$ and smooth at y we have the notion of the tangent map. The rank on the tangent map leads to stratifications of $\mathcal{M}_y$ such that the tangent map is of constant rank on each stratum. Fix a uniruled projective manifold $(X, \mathcal{K})$ equipped with a minimal rational component and denote by $\mathcal{C}_x \subset \mathbb{P}T_x(X)$ the VMRT at a general point $x \in X$. For a generically finite surjective holomorphic map $h : Y \to X$ and for a very general point $y \in Y$ such that $dh(y)$ is of maximal rank at y, $[dh(y)]^{-1}(\mathcal{C}_{h(y)})$ is a variety of distinguished tangents at y.

If we take $y \in Y$ to be a very general point of Y, a variety of distinguished tangents at y is the closure under the tangent map of a stratum of $\mathcal{M}_y$. Since there are only countably many irreducible components of the Chow space of curves on Y, from the construction by stratification there are at most countably many varieties of distinguished tangents passing through y. In the context of Theorem 10, choose a connected open subset $U \subset Y$ such that f_t is a biholomorphism of U onto $V_t \subset X_t$. Let $y \in Y$ be a very general point lying on U. We have a holomorphic family of VMRTs $\mathcal{C}_{f_t(y)}(X_t)$. Then, for each $t \in \Delta$, $f_t^{-1}(\mathcal{C}_{f_t(y))}(X_t)) := \mathcal{D}_t \subset \mathbb{P}T_y(Y)$ is a variety of distinguished tangent at y. By the countability of the space of varieties of distinguished tangents at y it follows that $\mathcal{D}_t$ is actually *independent* of t. There is an obvious identification $\varphi_t : V_t \cong V_0$ given by $\varphi_t = f_t \circ f_0^{-1}$, and we have $[d\varphi_t](\mathcal{C}_{f(t)}(X_t)) = \mathcal{C}_{f(0)}(X_0)$. Thus φ_t is VMRT-preserving, and by Cartan-Fubini extension as given in Theorem 9, φ_t extends to a biholomorphism $\Phi_t : X_0 \cong X_t$ such that $f_0 = \Phi_t \circ f_t$. □

In relation to finite holomorphic maps on rational homogeneous manifolds $S = G/P$, Lazarsfeld [32] proved that for any finite holomorphic map $f : \mathbb{P}^n \to X$ from the complex projective space onto a projective manifold X, X must itself be biholomorphic to $\mathbb{P}^n$. He raised the question of characterizing finite holomorphic maps $f : S \to X$ from a rational homogeneous manifold S of Picard number 1 onto a projective manifold. Hwang-Mok [16] solved the problem in 1999, and obtained [22] (2004) a new proof using Cartan-Fubini extension as given in Theorem 10.

Theorem 11 **(Hwang-Mok [16], [22]).** — *Let $S = G/P$ be a rational homogeneous manifold of Picard number 1. Let $f : S \to X$ be a nonconstant surjective holomorphic map onto a projective manifold X. Then either $X \cong \mathbb{P}^n$, where $n = \dim(S)$; or f is a biholomorphism.*

In the first proof in [16] we considered intertwining maps of $f : S \to X$, as follows. Suppose $f : S \to X$ is not a biholomorphism and $X \ncong \mathbb{P}^n$, and write s for the sheeting number of the map. The image manifold X is necessarily Fano. Equip X with a minimal rational component $\mathcal{K}$. Denote by $\mathcal{C}_x$ the variety of $\mathcal{K}$-tangents at x and assume known in the ensuing discussion that $\mathcal{C}_x \neq \mathbb{P}T_x(X)$ at a general point. Let $x \in X$ be outside the branch locus of f, and let V be a sufficiently small connected open neighborhood of x in X such that $f^{-1}(V)$ decomposes into a union of s open subsets $U_i, 1 \leq i \leq s,$, where $f_i := f|_{U_i} : U_i \to V$ is a biholomorphism for each i. For $i \neq j$ let $\varphi : U_i \to U_j$ be defined by $\varphi(z) = f_j^{-1} \circ f_i$. Consider the pull-back $\mathcal{D} := [df]^{-1}(\mathcal{C}|_V)$. We have tautologically $[d\varphi] : \mathcal{D}|_{U_i} \cong \mathcal{D}|_{U_j}$. For a general point $s \in S$, $\mathcal{D}_s$ is a variety of distinguished tangents. At any such point $\mathcal{D}_s$ is shown to be invariant under the isotropy subgroup $P_s \subset G$ at s. For instance, in the Hermitian symmetric case this implies that $\mathcal{D}_s$ must be one of the finitely many proper P_s-invariant subsets defined in terms of ranks of tangent vectors, $\mathcal{D}$ is actually G-invariant, and the condition $[d\varphi] : \mathcal{D}|_{U_i} \cong \mathcal{D}|_{U_j}$ forces $[d\varphi]$ to be VMRT-preserving, since at $s \in S$ the variety of minimal rational tangents $\mathcal{C}_s(S)$ is the most singular P_s-invariant stratum of $\mathcal{D}_s$. In this case by Ochiai's Theorem [47] the intertwining map must extend to an automorphism of S, and that is enough to force a contradiction. In the general case there may be continuous families of P_s-orbits, but using the fact that there are at most countably many distinct varieties of distinguished tangents at $s \in S$, it remains true that $\mathcal{D}$ is G-invariant. This leads to the conclusion that either $\mathcal{D}_s \subset \mathbb{P}T_s(S)$ is linearly non-degenerate, in which case we proved using Hwang-Mok [14] that S must be Hermitian symmetric, or $\mathcal{D}_s \subset \mathbb{P}T_s(S)$ is linearly degenerate, and the intertwining map φ must preserve some proper G-invariant distribution, after which we can work with results of Yamaguchi [51] to show that φ extends to $\Phi \in \mathrm{Aut}(S)$ to reach a contradiction. This line of argument has been recently generalized to the case of rational homogeneous spaces of Picard number ≥ 2, leading to a solution to a generalized Lazarsfeld Problem.

Theorem 12 **(Lau [31])**. — *Let G be a simple complex Lie group and $Q \subset G$ be a parabolic subgroup. Denote by $S = G/Q$ the corresponding rational homogeneous manifold,* $\dim(S) = n$. *Let $f : S \to X$ be a surjective holomorphic map from S onto a projective manifold X. Then one of the following holds:* (1) *f is a biholomorphism;* (2) *$f : S \to X$ is a finite map and X is the projective space $\mathbb{P}^n$;* (3) *there exists a parabolic subgroup Q' of G containing Q as a proper subgroup such that f factors through a finite map $g : G/Q' \to X$.*

The generalized Lazarsfeld Problem for $S = G/Q$ of Picard number ≥ 2 leads to a Fano manifold $(X, \mathcal{K})$ equipped with a minimal rational component and admitting the structure of a holomorphically fibered space $\lambda : X \to B$ such that the $\mathcal{K}$-curves lie on the fibers of λ. The principal algebro-geometric difficulty, solved in [31], is to produce a minimal rational component $\mathcal{K}'$ such that the $\mathcal{K}'$-curves are transversal to the fibration λ. After that Lau made use of multi-graded differential systems using Yamaguchi [51]. As in [16] the proof involves a substantial amount of Lie theory.

As far as the original Lazarsfeld Problem is concerned, Hwang-Mok [22] gave a new proof which frees the solution from Lie theory, deriving Theorem 11 as a consequence of Theorem 10, as follows. Let $S = G/P$ be an n-dimensional rational homogeneous manifold of Picard number 1 and $f : S \to X$ be a generically finite surjective holomorphic map onto a projective manifold X, which is necessarily Fano, such that $X \not\cong \mathbb{P}^n$ and f is not a biholomorphism. Equip X with a minimal rational component $\mathcal{K}$ and suppose that the associated VMRT at a general point is non-linear. Let θ be a holomorphic vector field on S and $\Theta_t = \exp(t\theta)$ be a holomorphic 1-parameter group of automorphism of S. Write $f_t = f \circ \Theta_t$. Then, applying the local rigidity result Theorem 11 we have $f_t = \Phi_t \circ f$. Thus $df_t(\eta) = 0$ whenever $df(\eta) = 0$. Thus the non-empty ramification divisor R of $f = f_0$ remains the ramification divisor of f_t for $t \neq 0$. On the other hand from the definition $f_t = f \circ \Theta_t$ it follows that the ramification divisor of f_t is $\Theta_{-t}(R)$, and a contradiction is obtained when we choose the vector field θ not to vanish identically on R. Finally, it remains to rule out the possibility that the VMRT of $(X, \mathcal{K})$ is linear at a general point $x \in X$. Choose a general point $x \in X$

lying outside the branching locus of f, $s \in S$ such that $f(s) = x$. An irreducible component of $[df]^{-1}(\mathcal{C}_x)$ then gives a P_s-invariant projective linear subspace of $\mathbb{P}T_s(S)$, giving rise to one of the finitely many G-invariant holomorphic distributions on S. D is non-integrable since S is of Picard number 1. On the other hand in the case of linear VMRTs on X an irreducible component of $\mathcal{C}$ over a sufficiently small open subset corresponds to an integrable distribution, a contradiction.

It would be interesting to give a proof of Theorem 12 along the line of Cartan-Fubini extension for special classes of Fano manifolds of Picard number ≥ 2.

6. Parallel transport of the second fundamental form

6.1. VMRTs in a differential-geometric context-parallel transport in the solution of the Generalized Frankel Conjecture. — In Algebraic Geometry Hartshorne conjectured that over an algebraically closed field a projective manifold with ample tangent bundle is isomorphic to the projective space. The conjecture was solved by Mori [45] (1979) by proving an existence theorem on rational curves using methods of characteristic $p > 0$, and the deformation theory of rational curves. In the context of Kähler Geometry, Frankel conjectured that a compact Kähler manifold of positive holomorphic bisectional curvature is biholomorphic to the complex projective space. The conjecture was resolved in the affirmative by the method of stable harmonic maps by Siu-Yau [50] (1980) who further formulated the conjecture that a compact Kähler manifold of nonnegative holomorphic bisectional curvature is locally symmetric. The latter conjecture, commonly called the Generalized Frankel Conjecture, was resolved in the affirmative by Mok [39] (1988).

Mok [39] made use of the Kähler Ricci flow, proving that nonnegativity of holomorphic bisectional curvature is preserved under the flow for the evolved metric $g_t, t > 0$. From earlier reduction of the problem, to confirm the Generalized Frankel Conjecture it suffices to consider the case where we have a compact Kähler manifold (X, g) of nonnegative holomorphic bisectional curvature and of positive Ricci curvature at some point such that furthermore $b_2(X) = 1$. For the latter class of (X, g), the evolved Kähler metric (X, g_t) is shown to be of positive Ricci curvature. Thus, X is Fano and hence uniruled by Miyaoka-Mori [38]. Since (X, g) is of nonnegative holomorphic bisectional curvature, the pull-back of its tangent bundle by any $f : \mathbb{P}^1 \to X$ is nonnegative, hence every rational curve on X is free. In [39] we studied minimal rational curves on X and the associated varieties of minimal rational tangents $\mathcal{C}_x \subset \mathbb{P}T_x(X)$ (although the terminology was not used there). We proved that there are the following alternatives on the evolved metrics g_t defined for $t > 0$ sufficiently small. For such $t > 0$, either (X, g_t) is of positive holomorphic bisectional curvature, or (X, g_t) admits non-trivial zeros of holomorphic bisectional curvature at any point of X. Write n for $\dim(X)$. If the VMRT $\mathcal{C}_x \subset \mathbb{P}T_x(X)$ is of dimension $p < n - 1$ at a general point, we showed that $\mathcal{C}$ is invariant under parallel transport of (X, g_t). If however $\mathcal{C}_x$ agrees with $\mathbb{P}T_x(X)$, we showed that there exists a hypersurface $\mathcal{S} \subset \mathbb{P}T_X$ such that $\mathcal{S}$ is invariant under parallel transport of (X, g_t). In either case we applied Berger's Theorem which characterizes Riemannian locally symmetric spaces by the fact that at any point there exists some proper subset of the unit sphere invariant under parallel transport. Thus (X, g_t) is an irreducible Hermitian symmetric space of the compact type for $t > 0$ and hence for $t = 0; g_0 = g$. More precisely we have proved

Theorem 13 **(Mok [39]).** — *Let (X, g) be a compact Kähler manifold of nonnegative holomorphic bisectional curvature and of positive Ricci curvature at some point. Assume that X is of Picard number 1. Then, either X is biholomorphically equivalent to the complex projective space, or (X, g) is biholomorphically isometric to an irreducible Hermitian symmetric space S of rank ≥ 2.*

On an irreducible Hermitian symmetric space of the compact type and of rank ≥ 2, the fibered space $\pi : \mathcal{C} \to S$ is invariant under parallel transport with respect to *any* choice of a canonical Kähler-Einstein metric, an elementary fact that follows from the parallelism of the Riemannian curvature tensor. Theorem 13 says in particular that on S this basic fact can be derived from curvature properties. In the negative direction, Berger's Theorem implies that for a rational homogeneous manifold $S = G/P$ of Picard number 1 which is not isomorphic to a Hermitian symmetric space, the VMRTs are not invariant under parallel transport. In an algebro-geometric context it remains interesting to introduce some algebraic notion of parallel transport applicable to any uniruled projective manifold $(X, \mathcal{K})$ equipped with a minimal rational component. A related problem is the Campana-Peternell Conjecture, which is a form of Generalized Hartshorne Conjecture (cf. (6.4)). Here the principal geometric problem is whether the notion of invariance of VMRTs under some restricted form of parallel transport is sufficient to characterize rational homogeneous manifolds $S = G/P$ of Picard number 1 by means of some algebro-geometric condition of nonnegativity on the tangent bundle. Such an approach in a very special situation has been established for Fano manifolds of Picard number 1 with nef tangent bundle and 1-dimensional VMRTs by Mok [41] (2001) and Hwang [13] (2007).

6.2. Propagation of the second fundamental form along a standard rational curve. — Let $(X, \mathcal{K})$ be a uniruled projective manifold equipped with a minimal rational component, $\rho : \mathcal{U} \to \mathcal{K}$, $\mu : \mathcal{U} \to X$ be the universal family of $\mathcal{K}$-curves and $\pi : \mathcal{C} \to X$ be the fibered space of varieties of minimal rational tangents. Let $B \subset X$ be the largest subvariety, necessarily of codimension ≥ 2, such that $\pi|_{X-B} : \mathcal{C}|_{X-B} \to X - B$ is flat. Let $f : \mathbb{P}^1 \to X$ be a parametrized standard rational curve, $f(\mathbb{P}^1) := C$, such that $C \subset X - B$. C lifts canonically to $\widetilde{C} \subset \mathcal{U}$, whose image under the tangent map gives the tautological lifting $\widehat{C} \subset \mathcal{C}$. At each of the finitely many points $\widetilde{x}_k$ of $\widetilde{C} \cap \mu^{-1}(x)$ there is an open neighborhood U_k such that τ_x embeds U_k holomorphically onto a smooth submanifold $\mathcal{C}^k_x$, which is the germ of some irreducible component of $\mathcal{C}_x$ at $[\alpha_k] = \tau_x(\widetilde{x}_k)$. In what follows $\widehat{C}$ will mean the pull-back of the tautological lifting of C to $f^*\mathcal{C}$, so that $\widehat{C}$ is smooth. For $t \in \mathbb{P}^1$ we write $\mathcal{C}_t$ for $(f^*\mathcal{C})_t$, $[\alpha(t)]$ for $\widehat{C} \cap \mathcal{C}_t$, and V_t for $f^*T_{f(t)}(X)$. We have $\mathcal{C}_t \subset \mathbb{P}V_t$. For every $t \in \mathbb{P}^1$ we have a germ of smooth projective submanifold $\mathcal{C}^o_t \subset \mathcal{C}_t \subset \mathbb{P}V_t$ at $[\alpha(t)]$ corresponding to one of the germs $\mathcal{C}^k_x, x = f(t)$, chosen in such a way that the union of $\mathcal{C}^o_t$ is a germ of complex submanifold along the smooth curve $\widetilde{C} \subset \mathbb{P}V$. Write $T_{[\alpha(t)]}$ for $T_{[\alpha(t)]}(\mathcal{C}^o_t)$. In Mok [14] (§3.2, p.2651ff.) we introduced implicitly the notion of parallel transport of the second fundamental form along the tautological lifting $\widehat{C}$ of a standard rational curve C. By this we mean that the second fundamental form can be interpreted in a natural way as a holomorphic section of a vector bundle which is trivial over $\widehat{C}$. We formulate the notion of isomorphisms of second fundamental forms and the result on parallel transport, as follows.

Definition 4. — *Let V and V' be two complex Euclidean spaces of the same dimension, and $A \subset \mathbb{P}V$, $A' \subset \mathbb{P}V'$ be two local complex submanifolds of the same dimension. Let $a \in A$, $a = [\alpha]$; $a' \in A'$, $a' = [\alpha']$. Write $T_a(A) = \mathbb{P}E/\mathbb{C}\alpha$ (resp. $T_a(A') = \mathbb{P}E'/\mathbb{C}\alpha'$ where $E \subset V$ (resp. $E' \subset V'$) is a vector subspace containing α (resp. α'). We say that the second fundamental form σ_a of $A \subset \mathbb{P}V$ at $a \in A$ is isomorphic to the second fundamental form $\sigma_{a'}$ of $A' \subset \mathbb{P}V'$ at $a' \in A'$ if and only if there exists a linear isomorphism $\varphi : V \cong V'$ such that $\varphi(\alpha) = \alpha'$, $\varphi(E) = E'$, and such that φ satisfies the following additional property ($\sharp$)*

($\sharp$) *Let $\overline{\varphi} : V/E \to V'/E'$ be the linear map induced by φ, $\varphi(E) = E'$, and denote by $\widetilde{\sigma}_\alpha$ (resp. $\widetilde{\sigma}_{\alpha'}$) the second fundamental form of $\widetilde{A}$ at α (resp. of $\widetilde{A}'$ at α'). Then, for any $\xi, \eta \in E$ we have $\widetilde{\sigma}_{\alpha'}\big(\varphi(\xi), \varphi(\eta)\big) = \overline{\varphi}\big(\widetilde{\sigma}_\alpha(\xi,\eta)\big)$.*

Proposition 6. — *For every $t \in \mathbb{P}^1$, denote by $\sigma_{[\alpha(t)]} : S^2T_{[\alpha(t)]} \to N_{\mathcal{C}_t^o|\mathbb{P}V_t,[\alpha(t)]}$ the second fundamental form of $\mathcal{C}_t^o \subset \mathbb{P}V_t$ at $[\alpha(t)]$. Then, for $t_1, t_2 \in \mathbb{P}^1$, $\sigma_{[\alpha(t_1)]}$ is isomorphic to $\sigma_{[\alpha(t_2)]}$.*

Proof. — Write $\nu : \mathbb{P}V \to \mathbb{P}^1$ for the canonical projection, where $V = f^*T_X$, and T_ν for its relative tangent bundle. Write $\lambda = \nu|_{f^*\mathcal{C}}$, and recall that $T_{[\alpha(t)]} = T_{[\alpha(t)]}(\mathcal{C}_t^o)$. Write $N_{[\alpha(t)]} = T_{\nu,[\alpha(t)]}/T_{[\alpha(t)]}$. Putting together $T_{[\alpha(t)]}$, $t \in \mathbb{P}^1$, we obtain a holomorphic vector bundle $T_\lambda|_{\widehat{C}}$ on $\widehat{C}$. Likewise, putting together $N_{[\alpha(t)]}$, $t \in \mathbb{P}^1$, we obtain a holomorphic vector bundle $N_\lambda|_{\widehat{C}}$ on $\widehat{C}$. For a nonzero vector $\alpha(t) \in V_t$ we have the canonical isomorphism $T_{[\alpha(t)]}(\mathbb{P}V_t) \otimes L_{[\alpha(t)]} \cong \nu^*V_t/L_{[\alpha(t)]}$, where $L_{[\alpha(t)]} = \mathbb{C}\alpha(t)$ is the tautological line at $[\alpha(t)]$. Varying over $\widehat{C}$ we obtain a canonical isomorphism $T_\nu \otimes L \cong \nu^*V_t/L$ over $\widehat{C}$. Since $L|_{\widehat{C}} \cong T_{\widehat{C}}$ canonically, and C is a standard rational curve, we have $\nu^*V|_{\widehat{C}} \cong \mathcal{O}(2) \oplus \big(\mathcal{O}(1)\big)^p \oplus \mathcal{O}^q$, so that

$$T_\nu|_{\widehat{C}} \cong \nu^*V|_{\widehat{C}}/T_{\widehat{C}} \otimes T^*_{\widehat{C}} \cong \big((\mathcal{O}(1))^p \oplus \mathcal{O}^q\big) \otimes \mathcal{O}(-2) \cong \big(\mathcal{O}(-1)\big)^p \oplus \big(\mathcal{O}(-2)\big)^q.$$

Since at $[\alpha(t)], T_{[\alpha(t)]} \otimes L_{[\alpha(t)]} \cong P_{\alpha(t)}/\mathbb{C}\alpha(t)$, where $P_{\alpha(t)} \subset V_t$ is the positive part of V_t at $[\alpha(t)]$, over $\widehat{C}$ we have $T_\lambda|_{\widehat{C}} \cong \big(\mathcal{O}(1)\big)^p \otimes \mathcal{O}(-2) \cong \big(\mathcal{O}(-1)\big)^p$ and $N_\lambda|_{\widehat{C}} \cong \mathcal{O}^q \otimes \mathcal{O}(-2) \cong \big(\mathcal{O}(-2)\big)^q$. Thus, over $\widehat{C}$

$$\mathrm{Hom}\big(S^2T_\lambda|_{\widehat{C}}, N_\lambda|_{\widehat{C}}\big) \cong \mathrm{Hom}\Big(\big(\mathcal{O}(-2)\big)^{\frac{p(p+1)}{2}}, \big(\mathcal{O}(-2)\big)^q\Big) \cong \mathcal{O}^{\frac{qp(p+1)}{2}}$$

is holomorphically trivial. Hence, at $t_1, t_2 \in \mathbb{P}^1$ the second fundamental forms $\sigma_{[\alpha(t_i)]} : S^2T_{[\alpha(t)]} \to N_{[\alpha(t)]}$; $i = 1, 2$; must be isomorphic to each other, as desired. □

Taking $\sigma_{[\alpha(t)]}$ as defining a holomorphic section of a holomorphically trivial vector bundle $E := S^2T_\lambda|_{\widehat{C}} \otimes N_\lambda|_{\widehat{C}}$ over $\mathbb{P}^1$, parallel transport of the second fundamental form from $t_1 \in \mathbb{P}^1$ to $t_2 \in \mathbb{P}^1$ can be understood as sending an element of $\epsilon_{t_1} \in$

E_{t_1} to the unique element $\epsilon_{t_2} \in E_{t_2}$ for which there exists $\epsilon \in \Gamma(\mathbb{P}^1, E)$ such that $\epsilon(t_1) = \epsilon_{t_1}$, $\epsilon(t_2) = \epsilon_{t_2}$. Fixing a decomposition of $V = f^*T_X$ over $\mathbb{P}^1$ given by $V = \mathcal{O}(2) \oplus (\mathcal{O}(1))^p \oplus \mathcal{O}^q$. there is a linear isomorphism $\varphi : V_{t_1} \to V_{t_2}$ which respects the decomposition of V and which induces parallel transport from $\sigma_{[\alpha(t_1)]}$ and $\sigma_{[\alpha(t_2)]}$.

6.3. Recognition of certain rational homogeneous manifolds from VMRTs at a general point. — We consider the question of characterizing certain rational homogeneous manifolds of Picard number 1 by their VMRTs at general points. Let S be an irreducible Hermitian symmetric space of Picard number 1, and denote its VMRT at $0 \in S$ by $\mathcal{C}_0 \subset \mathbb{P}T_0(S)$. Suppose $(X, \mathcal{K})$ is a uniruled projective manifold equipped with a minimal rational component such that at a general point $x \in X$ the VMRT $\mathcal{C}_x \subset \mathbb{P}T_x(X)$ is congruent to $\mathcal{C}_0 \subset \mathbb{P}T_0(S)$ of the model space. Let $B \subset X$ be a proper subvariety such that $\pi|_{X-B} : \mathcal{C} \to X$ is a locally trivial holomorphic fiber bundle with fibers $\mathcal{C}_x \subset \mathbb{P}T_x(X)$ being congruent to $\mathcal{C}_0 \subset \mathbb{P}T_0(S)$ as a projective submanifold. By [(4.3), Theorem 4], which in particular characterizes irreducible Hermitian symmetric spaces S of rank ≥ 2 by means of S-structures, to prove $X \cong S$ it suffices to show that B can be reduced to the empty set by methods of holomorphic extension. By Hartogs extension of S-structures (cf. (3.4)) it is enough to show that for every irreducible component $E_i \subset B$ of codimension 1 in X and any general point $y \in E_i$, there exists a neighborhood U_y of y such that $\pi|_{U_y - E_i} : \mathcal{C}|_{U_y - E_i} \to U_y - E_i$ extends holomorphically across $U_y \cap E_i$ as a holomorphic fiber subbundle of $\pi : \mathbb{P}T_{U_y} \to U_y$. Since X is of Picard number 1, for $y \in E_i$ sufficiently general there exists a standard parametrized rational curve $f : \mathbb{P}^1 \to X$ such that $f(0) \notin B$ and $f(\infty) = y$. The idea is to consider the tautological lifting of $C = f(\mathbb{P}^1)$ to $\widehat{C} \subset f^*\mathcal{C}$, and to recapture $\mathcal{C}_\infty$ which corresponds to $\mathcal{C}_y$ by knowing its second fundamental form at the point $[\alpha(\infty)] \in \mathcal{C}_\infty$ corresponding to $[df(\infty)] \in \mathcal{C}_y$.

The simplest case for this to work is the case of the n-dimensional hyperquadric $Q^n, n \geq 3$. For the family $f^*\mathcal{C} \subset \mathbb{P}(f^*T_X)$, the general fiber is isomorphic to a hyperquadric in $\mathbb{P}^{n-1}$. Degeneration of the hyperquadrics can occur at $t = \infty$, to give a degenerate hyperquadric defined by a degenerate symmetric bilinear form. However, this is precisely the case if and only if the second fundamental form σ at a general point of $\mathcal{C}_\infty$ is degenerate. The method of parallel transport of second fundamental forms then rules out the latter possibility, showing that $\mathcal{C}_y \subset \mathbb{P}T_y(X)$ is congruent to the VMRT of the model space for a general point y of the hypersurface E_i. With this holomorphic extension result of VMRTs across general points of hypersurfaces and Hartogs extension for bad sets of codimension ≥ 2 we have shown that X is biholomorphically isomorphic to the hyperquadric whenever the VMRT at a general point is congruent to $Q^{n-2} \subset \mathbb{P}^{n-1}$.

As seen from the table in (2.4) in the general symmetric case the VMRT $\mathcal{C}_0 \subset \mathbb{P}T_0(S)$ is itself a Hermitian symmetric space, either of rank 2 and embedded by the minimal canonical embedding, or of rank 1 and embedded by the second canonical embedding. In some sense they are quadratic objects. In fact, $\mathcal{C}_0$ is the closure of the graph of a vector-valued quadratic function Q on the tangent space $T_{[\alpha]}(\mathcal{C}_0)$. Q is essentially the second fundamental form. To illustrate how the argument of parallel transport of second fundamental forms works in the other cases, we consider the cases where $\mathcal{C}_0 \subset \mathbb{P}T_0(S)$ is an irreducible Hermitian symmetric space of rank 2, so that it carries a canonical G-structure for some reductive Lie subgroup of the general linear group. In the notations analogous to those in the preceding discussion, $\mathcal{C}_\infty \subset \mathbb{P}(f^*T_X)$ has the same second fundamental form at $[\alpha(\infty)]$ as that of the model space. $\mathcal{C}_0 \subset \mathbb{P}T_0(S)$ is uniruled by lines. Denoting by $\mathcal{K}'$ the minimal rational component on $\mathcal{C}_0$ consisting of lines, the G-structure of $\mathcal{C}_0$ is completely determined by VMRTs $\mathcal{C}'_{[\alpha]}$ associated to $(\mathcal{C}_0, \mathcal{K}')$, where $\mathcal{C}'_{[\alpha]}$ is defined by the set of non-zero tangent vectors $\eta \in T_{[\alpha]}(\mathcal{C}_0)$ such that $\sigma_{[\alpha]}(\eta, \eta) = 0$. Parallel transport of second fundamental forms then implies that $\mathcal{C}_\infty$ inherits a G-structure. By making use of developing maps $\mathcal{C}_\infty \subset \mathbb{P}(f^*T_y(X)$ can be shown to be congruent to $\mathcal{C}_0 \subset \mathbb{P}T_0(S)$. Here one has to exclude the possibility of linear degeneration of $\mathcal{C}_\infty \subset \mathbb{P}(f^*T_y(X)$, a possibility that is ruled out by the surjectivity of the second fundamental $\sigma_{[\alpha]}$ on the model space, and hence of $\sigma_{[\alpha(\infty)]}$ at $y = f(\infty)$ on X by parallel transport.

The preceding line of argumentation can be strengthened to yield

***Theorem 14* (Mok [42], Hong-Hwang [8]).** — *Let G be a simple complex Lie group, $P \subset G$ be a maximal parabolic subgroup corresponding to a long simple root, and by $S := G/P$ be the corresponding rational homogeneous manifold of Picard number 1. Denote by $\mathcal{C}_0 \subset \mathbb{P}T_0(S)$ the variety of minimal rational tangents at a reference point $0 \in S$ associated to the minimal rational component of lines on S. Let X be a Fano manifold of Picard number 1 and $\mathcal{K}$ be a minimal rational component on X. Suppose the variety of $\mathcal{K}$-tangents $\mathcal{C}_x \subset \mathbb{P}T_x(X)$ at a general point $x \in X$ is congruent to $\mathcal{C}_0 \subset \mathbb{P}T_0(S)$ as a projective submanifold. Then, X is biholomorphic to S.*

For the case where S is the projective space Theorem 14 follows from [3]. A sketch of the proof for S Hermitian symmetric and of rank ≥ 2 has been given in the above. When $P \subset G$ corresponds to a long simple root, the VMRT $\mathcal{C}_0 \subset \mathbb{P}D_0$ for the minimal nontrivial G-invariant distribution D on S. $\mathcal{C}_0$ is the highest weight orbit in $\mathbb{P}D_0$, and it is itself a Hermitian symmetric space. $D \neq T_S$ unless S is Hermitian symmetric. When S is non-symmetric and $\mathcal{C}_0$ is irreducible as a Hermitian symmetric space, it is of rank 3, embedded by the minimal canonical embedding. In general $\mathcal{C}_0 \subset \mathbb{P}D_0$ is of rank 3 as an embedded Hermitian symmetric space, when the degree for the embedding on each irreducible factor of $\mathcal{C}_0$ is taken into account in the obvious way. In fact, $\mathcal{C}_0 \subset \mathbb{P}D_0$ is a cubic object, being the closure of the graph of a vector-valued cubic polynomial on the tangent space $T_{[\alpha]}(\mathcal{C}_0)$ (cf. Hwang-Mok [17], p.377). The cubic nature of the VMRT is reflected in the table for Fano contact homogeneous manifolds of Picard number 1 in (3.1), and applies in general to the long-root case.

For non-symmetric S there is an additional notion of the third fundamental form for $\mathcal{C}_0 \subset \mathbb{P}D_0$, defined as follows. The image of the second fundamental form $\sigma_{[\alpha]} : S^2T_{[\alpha]} \to T_0(S)/P_\alpha$ is not surjective. For $\alpha \in \widetilde{\mathcal{C}}_0$ one can define a filtration $\mathbb{C}\alpha \subset P_\alpha \subset Q_\alpha \subset T_0(S)$, where Q_α is obtained by adjoining the image of the second fundamental form at α. This filtration corresponds to the splitting $D|_\ell \cong \mathcal{O}(2) \oplus (\mathcal{O}(1))^p \oplus \mathcal{O}^q \oplus (\mathcal{O}(-1))^r$ for the minimal proper distribution $D \subset T_S$. At every point $[\alpha] \subset \mathcal{C}_0$ one can define the third fundamental form $\kappa_{[\alpha]} : S^3T_{[\alpha]} \to T_0(S)/Q_\alpha$. In the case of a Fano manifold X of Picard number 1 satisfying the hypothesis of Theorem

14 for a non-symmetric S defined by a long simple root, Proposition 6 generalizes to show that over a standard parametrized rational curve $f : \mathbb{P}^1 \to X$, the corresponding third fundamental form on its tautological lifting $\widehat{C}$ defines a holomorphic section of a holomorphically trivial vector bundle over $\mathbb{P}^1$. Using this we have a version of parallel transport of the third fundamental form κ, with which one can prove extension results of VMRTs across a general point of a hypersurface as in the Hermitian symmetric case. In the contact case Theorem 14 is proved in Mok [42] by resorting to Hong's characterization of Fano contact homogeneous manifolds of Picard number 1 in [Ho]. In the remaining cases Theorem 14 was established in Hong-Hwang [8].

In view of Theorem 14, one may raise the following conjecture.

Conjecture 6. — *Let $S = G/P$ be any Fano homogeneous contact manifold of Picard number 1 and denote by $\mathcal{C}_0(S) \subset \mathbb{P}T_0(S)$ its variety of minimal rational tangents at a reference point $0 \in S$. Let $(X, \mathcal{K})$ be a Fano manifold of Picard number 1 equipped with a minimal rational component such that the associated* VMRT *at a general point is congruent to $\mathcal{C}_0(S) \subset \mathbb{P}T_0(S)$. Then, X is biholomorphic to S.*

To resolve Conjecture 6 it remains to consider the short-root case. Confirmation of the conjecture would provide a unified proof of rigidity of Fano homogeneous manifolds of Picard number 1 under Kähler deformation [(3.4), Theorem 2].

6.4. Projective manifolds with nef tangent bundles and 1-dimensional VMRTs. — In analogy with the Generalized Frankel Conjecture in Kähler Geometry one can formulate a Generalized Hartshorne Conjecture in Algebraic Geometry. This is given by the Campana-Peternell Conjecture [2] (1991). In particular, restricting to Fano manifolds X of Picard number 1, the Campana-Peternell Conjecture asserts that X is biholomorphic to a rational homogeneous manifold $S = G/P$ whenever the tangent bundle of X is nef, i.e., numerically effective. The latter assumption implies that the deformation of any rational curve on X is unobstructed. As a consequence, for any choice of a minimal rational component $\mathcal{K}$ on X, the evaluation map $\mu : \mathcal{U} \to X$ associated to the universal family for $\mathcal{K}$ gives a regular family of projective manifolds. This imposes some restrictions on possible complex structures of moduli spaces $\mathcal{K}_x \cong \mathcal{U}_x$ of $\mathcal{K}$-curves marked at x by restricting $\mathcal{U}$ over minimal rational curves. While there is so far no strong evidence why the Campana-Peternell Conjecture should hold, with the latter fact in mind Mok [41] considered a special case of the conjecture, under the restrictive assumption that the VMRT at a general point is 1-dimensional. In [42] we considered Fano manifolds whose second and fourth Betti numbers are equal to 1. The condition on the fourth Betti number was removed recently by Hwang [12], and we have now

Theorem 15. — **(Mok [42], Hwang [12])** *Let X be a Fano manifold of Picard number 1 with nef tangent bundle. Suppose X is equipped with a minimal rational component for which the variety of minimal rational tangents at a general point $x \in X$ is 1-dimensional. Then, X is biholomorphic to the projective plane $\mathbb{P}^2$, the 3-dimensional hyperquadric Q^3, or the 5-dimensional Fano contact homogeneous manifold $K(G_2)$ of type G_2. In particular, X is a rational homogeneous manifold.*

We note that the only algebro-geometric property used which arises from the nefness of the tangent bundle is the fact that the restriction of the tangent bundle to any $\mathcal{K}$-curve is nonnegative. In particular, the nefness assumption in Theorem 15 can be replaced by the assumption that any rational curve on X is free. The approach of [41] was to reconstruct X under the given assumptions from its VMRTs by making use of the canonical double fibration $\rho : \mathcal{U} \to \mathcal{K}$, $\mu : \mathcal{U} \to X$ associated to $\mathcal{K}$.We note that no *a priori* assumption is placed on $\dim(X)$.

To start with, restricting $\mu : \mathcal{U} \to X$ to a minimal rational curve we obtain an algebraic surface Σ holomorphically fibered over $\mathbb{P}^1$ which admits a holomorphic section Γ corresponding to the tautological lifting of the minimal rational curve. Thus, $\Gamma \subset \Sigma$ is an exceptional curve. Since the base is $\mathbb{P}^1$, if the fibers are of genus ≥ 1 the family must be holomorphically trivial, and the existence of the exceptional curve $\Gamma \subset \Sigma$ forces a contradiction. Thus, any $\mathcal{U}_x$ is isomorphic to $\mathbb{P}^1$. At a general point $x \in X$ the tangent map $\tau_x : \mathcal{U}_x \to \mathbb{P}T_x(X)$ is a holomorphic map. To determine the VMRT at a general point the next step is to bound $d := \deg(\tau_x^*(\mathcal{O}(1))$. For this purpose we introduce the use of Chern class inequalities. First, the universal $\mathbb{P}^1$-bundle $\rho : \mathcal{U} \to \mathcal{K}$ gives rise to a holomorphic rank-2 vector bundle $\nu : V \to \mathcal{K}$ such that $\mathbb{P}V \cong \mathcal{U}$. We prove that V is stable and deduce that $d \leq 4$ from the Bogomolov inequality $c_1^2(V) \cdot [\omega]^{n-2} \leq 4c_2(V) \cdot [\omega]^{n-2}$ for stable rank-2 vector bundles V over an n-dimensional projective manifold, where ω stands for the first Chern form of a positive line bundle on X, and $[\omega]$ for its cohomology class. It is here that we make use of the assumption $b_4(X) = 1$ when applying Chern class inequalities. Using the existence of Hermitian-Einstein metrics due to Uhlenbeck-Yau the equality case in the Bogomolov inequality can be ruled out, and we end up with $d = 1, 2, 3$, which we eventually prove to correspond to the three examples in the statement of Theorem 15. To proceed we make use of results from (2.3) on the integrability of differential systems generated by VMRTs to show that in each of the three cases $d = 1, 2, 3$ the VMRT $\mathcal{C}_x \subset \mathbb{P}T_x(X)$ is congruent to $\mathcal{C}_0 \subset S$ of the model space, and the proof is completed by invoking special cases of Theorem 14. The condition $b_4(X) = 1$ is removed in Hwang [10] by resorting to the determination of a certain Chow group pertinent to the problem in the application of the Bogomolov inequality.

Finally, from Theorem 15, together with earlier works of Campana-Peternell [2] and Zheng [53], and Miyaoka's characterization of the hyperquadric [37], one confirms the Campana-Peternell Conjecture up to 4 dimensions. More precisely, we have

Theorem 16. — *Let X be a Fano manifold of dimension ≤ 4 on which all rational curves are free. Then, X is biholomorphic to a rational homogeneous manifold.*

7. Privileged subvarieties of uniruled projective manifolds

7.1. Subvarieties saturated with minimal rational curves. — In analogy to totally geodesic submanifolds in Riemannian geometry we introduce for uniruled projective manifolds $(X\mathcal{K})$ endowed with minimal rational components the notion of $\mathcal{K}$-saturated subvarieties, as follows.

Definition 5. — *Let $(X, \mathcal{K})$ be a uniruled projective manifold equipped with a minimal rational component, $\pi : \mathcal{C}_X \to X$ be the associated fibered space of varieties $\mathcal{K}$-tangents. Let $\Sigma \subset X$ be an irreducible analytic subvariety of some connected open subset $U \subset X$ and $\mathcal{E} \subset \mathcal{C}_X\big|_\Sigma$ be an analytic subvariety. For $y \in \Sigma$ denote by $\mathcal{E}_y$ the fiber of $\mathcal{E}$ over y. We say that $(\Sigma, \mathcal{E}) \hookrightarrow (X, \mathcal{C}_X)$ is $\mathcal{K}$-saturated if and only if*

(a) *$\mathcal{E}_y = \mathbb{P}T_y(\Sigma) \cap \mathcal{C}_X \neq \emptyset$ for a smooth point $y \in \Sigma$, and*

(b) *for a general smooth point y on Σ, and for the germ C of an irreducible branch of a standard $\mathcal{K}$-curve passing through y, C must lie on Σ whenever $\big[T_y(C)\big] \in \mathcal{E}_y$.*

When the choice of $\mathcal{K}$ is understood, we simply say that Σ is saturated with respect to minimal rational curves. If we take a minimal rational curve on $(X, \mathcal{K})$ to play the role of a geodesic, a $\mathcal{K}$-saturated subvariety is the analogue of a totally geodesic subspace in Riemannian geometry, except that the 'geodesics' are now only defined for tangent directions corresponding to varieties of minimal rational tangents.

7.2. A relative version of the Gauss map condition for linear sections of VMRTs. — In (5.1) we have introduced a non-degeneracy condition (†) on the Gauss map of the variety of minimal rational tangents $\mathcal{C}_x$ at a general point x of a uniruled projective manifold $(X, \mathcal{K})$ equipped with a minimal rational component, viz., we require that the Gauss map is generically finite on $\mathcal{C}_x$. Equivalently (†) is satisfied if and only if at a general smooth point $[\alpha]$ of $\mathcal{C}_x$, the kernel $\mathrm{Ker}\,\sigma_{[\alpha]} = 0$ for the second fundamental form $\sigma_{[\alpha]}$ at $[\alpha] \in \mathrm{Reg}(\mathcal{C}_x)$. We extend this to the situation of a linear section of $\mathcal{C}_x$ and define a non-degeneracy condition (††) which reduces to (†) when the linear section is $\mathcal{C}_x$ itself. Recall that a variety is said to be of pure dimension n if and only if each irreducible component is of the same dimension n.

Definition 6. — *Let $m \geq 2$, $\mathcal{A} \subset \mathbb{P}^m$ be a projective subvariety of pure dimension $a \geq 1$. Let $\Pi \subset \mathbb{P}^m$ be a projective linear subspace, and $\mathcal{B} := \Pi \cap \mathcal{A}$ be a non-empty projective subvariety of pure dimension $b \geq 1$. We say that the pair $(\mathcal{B}, \mathcal{A})$ satisfies the non-degeneracy condition (††) if and only if for every general smooth point $[\beta] \in \mathcal{B}$, $[\beta]$ is also a smooth point of $\mathcal{A}$ and* $\mathrm{Ker}\,\sigma_{[\beta]}\big(T_{[\beta]}(\mathcal{B}), \cdot\big) = 0$.

By an adaptation of the proof of Cartan-Fubini extension in the equidimensional case under the non-degeneracy assumption (†) as explained in (5.2) we have the following non-equidimensional analogue of Cartan-Fubini extension under some non-degeneracy assumption involving (††) on second fundamental forms. For the formulation a point $x \in X$ is said to be a good point if and only if every minimal rational curve passing through x is free, and a general element of every irreducible component of $\mathcal{K}_x$ represents a standard rational curve, otherwise x is called a bad point. The bad locus of $(X, \mathcal{K})$ is the set of bad points on X, which is a subvariety of X.

***Theorem 17* (Hong-Mok [8]).** — *Let $(Z, \mathcal{H})$ and $(X, \mathcal{K})$ be two uniruled projective manifolds equipped with minimal rational components. Assume that Z is of Picard number 1 and that $\mathcal{C}_z(Z)$ is of positive dimension at a general point $z \in Z$. Let $U \subset Z$ be a connected open subset and $f : U \to X$ be a holomorphic embedding onto a locally closed complex submanifold $S \subset X$ lying outside the bad locus of $(X, \mathcal{K})$. Suppose f respects varieties of minimal rational tangents in the sense* *that* $df(\widetilde{C}_z(Z)) = df(T_z(Z)) \cap \widetilde{C}_{f(z)}(X)$. *Assume furthermore that at a general point* $x \in S$, *the non-degeneracy condition* (††) *on second fundamental forms is satisfied for the pair* $(\mathcal{C}_x \cap \mathbb{P}T_x(S), \mathcal{C}_x)$. *Then,* f *extends to a rational map* $F : Z \to X$.

In terms of the holomorphic map f, the non-degeneracy condition on second fundamental forms translate into

$$\mathrm{Ker}\ \widetilde{\sigma}_{df(\alpha)}\big(T_{df(\alpha)}(df(\widetilde{\mathcal{C}}_z(Z))), \cdot\,\big) = \mathbb{C}df(\alpha).$$

As an important intermediate step in the proof of Theorem 17, Hong-Mok established under the assumption there the following result.

Proposition 7. — *Under the assumptions of Theorem 17 and in the notations there,* f *sends germs of standard* $\mathcal{H}$*-curves into germs of standard* $\mathcal{K}$*-curves. In particular,* $(S, \mathcal{C} \cap \mathbb{P}T_S) \subset (X, \mathcal{C})$ *is saturated with respect to* $\mathcal{K}$*-curves.*

7.3. Parallel transport of VMRTs along minimal rational curves. — As an application of non-equidimensional Cartan-Fubini extension, Mok [43] gave a characterization of standard embeddings between Grassmannians of rank ≥ 2. The result by itself had been known and proven by different methods by Neretin [46] and Hong [7]. Our proof started with non-equidimensional Cartan-Fubini extension in the Hermitian symmetric case with a proof relying on the use of Harish-Chandra coordinates. More recently, Hong-Mok [9] have established the general form of Proposition 7, obtaining at the same time a characterization of a general class of standard embeddings between rational homogeneous manifolds of Picard number 1. On a rational homogeneous manifold Y of Picard number 1 we consider the minimal rational component consisting of lines on Y and denote by $\mathcal{C}_y(Y)$ the associated VMRT at $y \in Y$.

***Theorem 18* (Hong-Mok [9]).** — *Let* $X = G/P$ *be a rational homogeneous manifold of Picard number 1 associated to a long simple root and let* $Z = G_0/P_0$ *be a rational homogeneous space associated to a subdiagram of the marked Dynkin diagram of* G/P. *Assume that* Z *is not linear. If* $f : U \to X$ *is a holomorphic embedding from a connected open subset* U *of* Z *into* X *satisfying* $df_z(\widetilde{\mathcal{C}}_z(Z)) = df_z(T_z(Z)) \cap \widetilde{\mathcal{C}}_{f(z)}(X)$ *for a general point* $z \in \mathcal{U}$, *then* f *extends to a standard embedding of* Z *into* X.

Sketch of proof. — A marked Dynkin subdiagram defines naturally an embedding λ from $Z = G_0/P_0$ into $X = G/P$. By a standard embedding from Z into X we mean $\varphi \circ \lambda$ for some $\varphi \in \mathrm{Aut}(X)$. For the proof of Theorem 18, first of all the method of non-equidimensional Cartan-Fubini extension as given in [(7.2), Theorem 17] can be implemented by checking the validity of the non-degeneracy condition (††) on the Gauss map yielding therefore a rational extension $F : Z \to X$. Write $S = F(Z)$ for the total transform of F. By Proposition 7, $S \subset X$ is $\mathcal{K}$-saturated. The condition $df_z(\widetilde{\mathcal{C}}_z(Z)) = df_z(T_z(Z)) \cap \widetilde{\mathcal{C}}_{f(z)}(X)$ says that S is tangent at a general point $s \in f(U)$ to a (unique) copy Z_s of a standard embedding of Z into X. Extending $f : U \to X$ to $F : Z \to X$ the same applies for a general point $s \in S$.

Start with a base point $0 \in Z$, $f(0) = 0$. Z_0 and S are tangent to each other at 0 and they share the same VMRTs at 0. Let $\mathcal{A}$ be the subvariety on Z_0 swept out by lines ℓ on Z_0 passing through 0. Since $S \subset X$ is $\mathcal{K}$-saturated, $\ell \in \mathcal{A} \subset Z_0 \cap S$.

At a general point $s \in \ell$, write $\mathcal{E}_s := \mathbb{P}T_s(S) \cap \mathcal{C}_s(X) = \mathbb{P}T_s(Z_s) \cap \mathcal{C}_s(X)$. We argue that Z_0 and S are tangent at $s \in \ell$, i.e., $\mathcal{C}_s(Z_0) = \mathcal{E}_s$. Write $T_s(\ell) = \mathbb{C}\alpha$. From deformation theory of rational curves $T_{[\alpha]}(\mathcal{C}_s(Z_0)) = T_s(\mathcal{A})/T_s(\ell)$ while also $T_{[\alpha]}(\mathcal{E}_s) = T_s(\mathcal{A})/T_s(\ell)$. This means that $\mathcal{E}_s$ and $\mathcal{C}_s(Z_0)$ are tangent to each other at $[\alpha]$. In general the tangency property does not imply identity of the two VMRTs, but we have found that this is the case for pairs (Z, X) of rational homogeneous manifolds of Picard number 1 as given in Theorem 18. We may think of this as a form of parallel transport of VMRTs for $\mathcal{K}$-saturated subvarieties along a minimal rational curve in special situations. Thus, $Z_s = Z_0$ for any line ℓ on Z_0 passing through 0 and for a general point $s \in \ell$. It follows that $Z_s = Z_0$ for a general point $s \in \mathcal{A}$, and s can now play the same role as the initial base point 0. Finally, $S = F(Z)$ can be recovered from the single point $0 \in Z$ in a finite number of steps by the procedure of adjoining minimal rational curves (cf. (3.1)), and we have proven that $S = Z_0$, as desired. □

References

[1] W.M. Boothby, *Homogeneous complex contact manifolds*, pp.144– 154 in Differential geometry, edited by C.B. Allendoerfer, Proc. Sympos. Pure Math. **3**, Amer. Math. Soc., Providence, RI, 1961.

[2] F. Campana and T. Peternell, *Projective manifolds whose tangent bundles are numerically effective*, Math. Ann. **289** (1991), 169–187.

[3] K. Cho, Y. Miyaoka and N.I. Shepherd-Barron, *Characterizations of projective space and applications to complex symplectic manifolds*, in Higher dimensional birational geometry (Kyoto, 1997), pp.1–88, Adv. Stud. Pure Math. **35**, Math. Soc. Japan, Tokyo, 2002.

[4] A. Grothendieck, *Sur la classification des fibrés holomorphes sur la sphère de Riemann*, Amer. J. Math. **79** (1957), 121–138.

[5] V. Guillemin, *The integrability problem for G-structures*, Trans. A.M.S. **116** (1965), 544–560.

[6] J.-H. Hong, *Fano manifolds with geometric structures modeled after homogeneous contact manifolds*, Internat. J. Math. **11** (2000), 1203–1230.

[7] J.-H. Hong: *Rigidity of smooth Schubert varieties in Hermitian symmetric spaces*, Trans. Amer. Math. Soc. **359** (2007) 2361–2381 (electronic).

[8] J.-H. Hong and J.-M. Hwang, *Characterization of the rational homogeneous manifold associated to a long simple root by its variety of minimal rational tangents*. Preprint.

[9] J.-H. Hong and N. Mok, *Non-equidimensional Cartan–Fubini extension of holomorphic maps respecting varieties of minimal rational tangents*, in preparation.

[10] J.-M. Hwang, *Rigidity of homogeneous contact manifolds under Fano deformation*, J. Reine Angew. Math. **486** (1997), 153–163.

[11] J.-M. Hwang, *Geometry of minimal rational curves on Fano manifolds*, School on Vanishing Theorems and Effective Results in Algebraic Geometry (Trieste 2000), ICTP Lect. Notes, 6, Abdus Salam Int. Cent. Theoret. Phys., Trieste, 2001, pp.335–393.

[12] J.-M. Hwang, *Rigidity of rational homogeneous spaces*, Proceedings of the International Congress of Mathematicians, Madrid 2006, Volume II, European Math. Soc. Publishing House, Zürich 2007, pp.613–626.

[13] J.-M. Hwang, *Deformation of holomorphic maps onto Fano manifolds of second and fourth Betti numbers 1*, Ann. Inst. Fourier **57** (2007), 815–823.

[14] J.-M. Hwang and N. Mok, *Uniruled projective manifolds with irreducible reductive G-structures*, J. Reine Angew. Math. **490** (1997), 55–64.

[15] J.-M. Hwang and N. Mok, *Rigidity of irreducible Hermitian symmetric spaces of the compact type under Kähler deformation*, Invent. Math. **131** (1998), 393–418.

[16] J.-M. Hwang and N. Mok, *Holomorphic maps from rational homogeneous spaces of Picard number 1 onto projective manifolds*, Invent. Math. **136** (1999), 209–231.

[17] J.-M. Hwang and N. Mok, *Varieties of minimal rational tangents on uniruled manifolds*, in Several Complex Variables, edited by M. Schneider and Y.-T. Siu, MSRI publications **37**, Cambridge University Press, 1999, pp.351–389.

[18] J.-M. Hwang and N. Mok, *Cartan-Fubini type extension of holomorphic maps for Fano manifolds of Picard number 1*, J. Math. Pure Appl. **80** (2001), 563–575.

[19] J.-M. Hwang and N. Mok, *Deformation rigidity of the rational homogeneous space associated to a long root*, Ann. Scient. Ec. Norm. Sup. **35** (2002), 173–184.

[20] J.-M. Hwang and N. Mok, *Finite morphisms onto Fano manifolds of Picard number 1 which have curves with trivial normal bundle*, J. Alg. Geom. **12** (2003), 627–652.

[21] J.-M. Hwang and N. Mok, *Deformation rigidity of the 20-dimensional F_4-homogeneous space associated to a short root*, to appear in Algebraic Transformation Groups and Algebraic Varieties, ed. V.L. Popov, Encyclopaedia of Mathematical Sciences, Springer-Verlag, Berlin-Heidelberg-New York, 2004, pp.37-58.

[22] J.-M. Hwang and N. Mok, *Birationality of the tangent map for minimal rational curves*, Asian J. Math. **8**, Special Volume in honor of Yum-Tong Siu's sixtieth birthday, (2004), 51-64.

[23] J.-M. Hwang and N. Mok, *Prolongations of infinitesimal linear automorphisms of projective varieties and rigidity of rational homogeneous spaces of Picard number 1 under Kähler deformation*, Invent. Math. **160** (2005), 591–645.

[24] J.-M. Hwang and S. Ramanan, *Hecke curves and Hitchin discriminant*, Ann. Scient. Éc. Norm. Sup. **37** (2004), 801–817.

[25] S. Kebekus, *Families of singular rational curves*, J. Alg. Geom. **11** (2002), 245–256.

[26] S. Kebekus, *Lines on contact manifolds*, J. Reine Angew. Math. **539** (2001), 167–177.

[27] S. Kebekus, T. Peternell, A. Sommese and J. Wiśniewski, *Projective contact manifolds*, Invent. Math. **142** (2000), 1–15.

[28] S. Kobayashi and T. Ochiai, *Holomorphic structures modeled after compact hermitian symmetric spaces*, in Hano et al. (eds.), Manifolds and Lie groups, Papers in honor of Yozo Matsushima, Progr. Math. **14** (1981), 207–221.

[29] J. Kollár, *Rational curves on algebraic varieties*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3. Folge) **32**, Springer Verlag, Berlin, 1996.

[30] J. Kollár, Miyaoka and S. Mori, *Rational connectedness and boundedness of Fano manifolds*, J. Diff. Geom. **36** (1992), 765–779.

[31] C.-H. Lau, *Holomorphic maps from rational homogeneous manifolds onto projective manifolds*, to appear in J. Alg. Geom.

[32] R. Lazarsfeld, *Some applications of the theory of positive vector bundles*, in Complete intersections (Acireale, 1983), edited by S. Greco and R. Strano, Lecture Notes in Math. **1092**, Springer, Berlin, 1984, pp.29–61.

[33] C. LeBrun, *A rigidity theorem for quaternionic-Kähler manifolds*, Proc. Amer. Math. Soc. **103:4** (1988), 1205–1208.

[34] C. LeBrun, *Fano manifolds, contact structures and quaternionic geometry*, Int. J. Math. **3** (1995), 419–437.

[35] Y. Matsushima and A. Morimoto, *Sur certains espaces fibrés holomorphes sur une variété de Stein*, Bull. Soc. Math. France **88** (1960), 137–155.

[36] Y. Miyaoka, *Geometry of rational curves on varieties*, in Geometry of higher-dimensional algebraic varieties, Birkhäuser Verlag, Basel, 1997, pp.1–127.

[37] Y. Miyaoka, *Numerical characterisation of hyperquadrics*, in *Complex Analysis in Several Variables – Memorial Conference of Kiyoshi Oka's Centennial Birthday*, Adv. Stud. Pure Math. 42, Math. Soc. Japan, Tokyo 2004, pp.209–235.

[38] Y. Miyaoka and S. Mori, *A numerical criterion for uniruledness*, Ann. Math. **124** (1986), 65–69.

[39] N. Mok, *The uniformization theorem for compact Kähler manifolds of nonnegative holomorphic bisectional curvature*, J. Diff. Geom. **27** (1988), 179–214.

[40] N. Mok, *G-structures on irreducible Hermitian symmetric spaces of rank* ≥ 2 *and deformation rigidity*, Contemp. Math. **222** (1999), 81–107.

[41] N. Mok, *On Fano manifolds with nef tangent bundles admitting 1-dimensional varieties of minimal rational tangents*, Trans. A.M.S. **354** (2002), 2639–2658.

[42] N. Mok, *Recognizing certain rational homogeneous manifolds of Picard number 1 from their varieties of minimal rational tangents*, AMS/IP Studies in Advanced Mathematics **42** (2008), 41-61.

[43] N. Mok, *Characterization of standard embeddings between complex Grassmannians by means of varieties of minimal rational tangents*, Science in China Ser. A: Mathematics **51** (2008), 660-684.

[44] N. Mok and I.-H. Tsai, *Rigidity of convex realizations of irreducible bounded symmetric domains of rank 2*, J. Reine Angew. Math. **431**(1992), 91–122.

[45] S. Mori, *Projective manifolds with ample tangent bundles*, Ann. Math. **110** (1979), 593–606.

[46] Yu. A. Neretin, *Conformal geometry of symmetric spaces, and generalized linear-fractional Krein-Smul'yan mappings* (Russian), Mat. Sb. **190** (1999), 93–122; translation in Sb. Math. **190** (1999), 255–283.

[47] T. Ochiai, *Geometry associated with semisimple flat homogeneous spaces*, Trans. Amer. Math. Soc. **152** (1970), 159–193.

[48] J.-P. Serre, *Algèbres de Lie semi-simples complexes*, W.A. Benjamin, New York, 1966. Translated as *Complex semisimple Lie algebras*, Springer, New York, 1987.

[49] I. Singer and S. Sternberg, *The infinite groups of Lie and Cartan*, J. Anal. Math. **15** (1965), 1–114.

[50] Y.-T. Siu and S.-T. Yau. *Compact Kähler manifolds of positive bisectional curvature*, Invent. Math. **59** (1980), 189-204.

[51] K. Yamaguchi, *Differential systems associated with simple graded Lie algebras*, in Progress in differential geometry, edited by K. Shiohama, Adv. Stud. Pure Math. **22**, Math. Soc. Japan and Kinokuniya, Tokyo, 1993, pp.413–494.

[52] F.L. Zak *Tangents and secants of algebraic varieties*, Translations of Mathematical Monographs, Vol.127, Amer. Math. Soc., Providence 1993.

[53] F. Zheng, *On semi-positive threefolds*, Thesis, Harvard University, 1990.

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