Finite Group Actions in Four-Manifolds

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I. Finite Group Actions in SU(2)-Gauge Theory

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I. Finite Group Actions in SU(2)-Gauge Theory

I.1 Review of Donaldson Invariant

- M : a simply connected, smooth, closed 4-manifold with positive definite intersection form.

- $E \longrightarrow M$: SU(2)-bundle with $c_2(E) = -1$.

- A connection $\nabla \in \mathfrak{C}(E)$: self-dual if $*R^{\nabla} = R^{\nabla}$.

$$\Longrightarrow \quad R^{\nabla} \quad \text{: harmonic}$$

∇ minimizes the Yang-Mills action

$$\mathcal{Y}_m(\nabla) = \frac{1}{2}\int_M \left\| R^{\nabla} \right\|^2 \text{dvol}.$$

- The gauge group $\mathfrak{g}$ of bundle automorphisms of E acts on the space $\&\mathcal{D}$ of self-dual connections.

$\mathfrak{M}(E) = \&\mathcal{D}/\mathfrak{g}$: the moduli space of self-dual connections.

By perturbation or generic metric and compactipication.

Theorem I.1.1. *$\overline{\mathfrak{M}}(E)$: a smooth, orientable, 5-manifold with λ singular points each of which has a cone nbd on $\mathbb{CP}^2$, where $\lambda = \text{rank } H^2(M;\, \mathbb{Z})$.*

Theorem I.1.2. *If $H^2(M;\mathbb{Z}) = < a_1, \cdots, a_\lambda >$, then*

$$M \simeq \coprod_1^\lambda \mathbb{CP}^2 : \textit{cobordant.}$$

Theorem I.1.3.

(1) The positive definite intersection form (p.d.i.f)

$$\omega(M) \simeq (1) \oplus \cdots \oplus (1).$$

(2) $\exists$ Non smoothable topological 4-manifold.
Ex. nE_8.

(3) $\exists$ Exotic smooth structures on $\mathbb{R}^4$.

(4) Nondecompasability, $M \neq M_1 \sharp M_2$ $(b_2^+(M_i) > 0, i = 1, 2)$ if M has nontrivial Donaldson invariants.

I.2. Obstruction for G-moduli space

- Let G be a finite group.
  Suppose that G acts smoothly, semi-freely, isometrically on a simply connected., closed, smooth 4-manifold M with a p.d.i.f.

  Let $\phi : E \longrightarrow M$: a G-bundle with $c_2(E) = -1$,
  $M^G = F := \{P_i\}_{i=1}^{n_1} \coprod \{T^{\lambda_i}\}_{i=1}^{n_2}$, G acts trivially on $E|_F$.

- $\overline{\mathfrak{M}}(E)$ is a G-space, but may not be a G-manifold. Transform the G-space $\overline{\mathfrak{M}}(E)$ into a smooth G-manifold with some singularities.

Theorem I.2.1. *There is a Baire set in the G-invariant metrics such that $\hat{\mathfrak{M}}^G$ is a smooth manifold in the $\hat{\mathfrak{M}}^G$ of irreducible self-dual connections.*

For each $\nabla \in \mathfrak{M}^G$, there is G-invariant fundamental elliptic complex

$$o \longrightarrow \Omega^0(\mathfrak{g}_E) \underset{\delta^\nabla}{\overset{d^\nabla}{\rightleftarrows}} \Omega^1(\mathfrak{g}_E) \xrightarrow{d^\nabla_-} \Omega^2_-(\mathfrak{g}_E) \longrightarrow o \quad .$$

Associate a Dirac operator

$$\begin{array}{ccc} \Gamma(V_+ \otimes V_- \otimes \mathfrak{g}_{\mathbb{C}}) & \xrightarrow{\qquad D \qquad} & \Gamma(V_- \otimes V_- \otimes \mathfrak{g}_{\mathbb{C}}) \\ & \searrow^{\nabla} \qquad \nearrow^{C} & \\ & \Gamma(\Lambda' \otimes V_+ \otimes V_- \otimes \mathfrak{g}_{\mathbb{C}}). & \end{array}$$

For each $g \in G$,

$$\mathrm{ind}_g(D) = (-1)^m \frac{\mathrm{ch}_g(j^*\sigma(D))\,\mathrm{td}(T^g \otimes \mathbb{C})}{\mathrm{ch}_g(\Lambda_{-1} N^g \otimes \mathbb{C})}[TX^g]\,,$$

$m = \dim M^g$,

$j : M^g \longrightarrow M$,

$N^g = N(M, M^g)$: normal bundle.

- Perturb the Fredholm G-map $\Psi : \mathcal{C}/\mathfrak{g} \longrightarrow [\mathcal{C} \times \Omega^2_-(\mathfrak{g}_E)]/\mathfrak{g}$ given by $\Psi([\nabla]) = [\nabla, R^\nabla_-]$ to transverse the zero section.

  The Kuranishi map Ψ is locally equavalent to the sum of a G-equivariant linear map and a nonlinear G-equivariant map in finite dimensional range.

- Now we assume tat $G = \mathbb{Z}_2 =< h >$ is the group of order 2.

Proposition I.2.2. *Let* $A = \chi(F), \nabla \in \mathfrak{M}(E)^G$ *and* $h(\nabla) = g(\nabla)$ *for some* $g \in \mathfrak{g}$.

(i) *If* $(hg)^2 = I, \nabla \in \widehat{\mathfrak{M}}$, *then*

$$\begin{cases} \dim\ H^1_{\nabla_+} - \dim\ H^2_{\nabla_+} = \frac{1}{4}(10 + 3A) \\ \dim\ H^1_{\nabla_-} - \dim\ H^2_{\nabla_-} = \frac{1}{4}(10 - 3A) \end{cases},$$

where $H^*_{\nabla_\pm}$ *is* (± 1)*-eigen sp. of* (hg),

(ii) *If* $(hg)^2 = -I, \quad \nabla \in \widehat{\mathfrak{M}}(E)^G$, *then*

$$\begin{cases} \dim\ H^1_{\nabla_+} - \dim\ H^2_{\nabla_+} = \frac{1}{4}(10 + A) \\ \dim\ H^1_{\nabla_-} - \dim\ H^2_{\nabla_-} = \frac{1}{4}(10 - A) \end{cases},$$

where $H^*_{\nabla\pm}$ *is* (± 1)*-eigen space of* $(hg)^2$,

(iii) *If* ∇ *is reducible, then*

$$\begin{cases} \dim\ H^1_{\nabla_+} - \dim\ H^2_{\nabla_+} = \frac{1}{4}(14 + A) \\ \dim\ H^1_{\nabla_-} - \dim\ H^2_{\nabla_-} = \frac{1}{4}(10 - A) \end{cases},$$

where $H^*_{\nabla_\pm}$ *is* (± 1)*-eigen space of* $g_1 h g g_2$ *for some* $g_i \in \Gamma_\nabla$*.*

- Apply a G-transversality technique of T. Petrie.
  $X = \mathfrak{M}(E)^G$,
  $X_0 = (\mathrm{end}(\mathfrak{M}(E)) \cup$ nbd of reducible con. in $\mathfrak{M}(E)) \cap X$,

  For $\nabla \in X$, there is a fiber bundle $V \longrightarrow X$ with fiber
  $V_\nabla = \mathrm{Hom}^S_G(H^1_{\nabla_-}, H^2_{\nabla_-})$
  : the space of surjective G-homomorphisms
  : a Stiefel manifold with homotopy groups

$$\pi_i(V_\nabla) = \begin{cases} \mathbb{Z} & \text{if } (hg)^2 = -\mathrm{I} \text{ and } i = 2, \\ 0 & \text{if } (hg)^2 = \mathrm{I} \text{ and } i \leq 3. \end{cases}$$

Theorem I.2.3.

(i) *To perturb* Ψ *to be* G*-transversal throughout of* $\mathfrak{M}(E)^G$ *there are obstruction classes* $\Theta_3(\Psi) \in H^3(X, X_0; \mathbb{Z}(= \pi_2(V_\nabla)))$.

(ii) *If the obstruction cohomology classes* $\Theta_3(\Psi) = 0$*, then we may have a smooth* G*-manifold* $\mathfrak{M}(E)$ *of dimension 5 with* λ *singlular points each of which has a cone nbd of* $\mathbb{CP}^2$*, where* $\lambda = \mathrm{rank} H^2(M; \mathbb{Z})$.

I.3. Involution and Donaldson invariant

- X : a closed, simply connected, smooth 4-manifold with an orientation-preserving smooth involution σ.

  $X^\sigma = F$: a 2-dim submanif.
  $E \longrightarrow X$: SU(2) vector bundle, $-c_2(E)$: even.
  $p : X \to X/\sigma = X'$: projection, $F' = p(F)$.

- For a generic σ-invariant metric on X, let $\mathfrak{M}(E)^\sigma$ be the moduli space of σ-invariant self-dual connections.

Theorem I.3.1. *There is 1-1 correspondence between* $\mathfrak{M}(E)^\sigma$ *and* $\mathfrak{M}(E')$*, where* $\mathfrak{M}(E')$ *is the moduli space of self-dual connections on* $E' \longrightarrow X'$.

Let $\dim \mathfrak{M}^\sigma(E) = \dim \mathfrak{M}(E') = 2d$.

For the coupled SU(2)-bundle $E^* \longrightarrow \overline{\mathfrak{M}}^\sigma(E) \times X$,

$$\text{the Donaldson } \mu : H_2(X) \xrightarrow{c_2(E^*)/} H^2(\overline{\mathfrak{M}}^\sigma(E)) \xrightarrow{PD} H_{2d-2}(\overline{\mathfrak{M}}^\sigma(E)).$$

The Donaldson invariant

$$D^\sigma : H_2(X)^\sigma \times \cdots \times h_2(X)^\sigma \longrightarrow \mathbb{Z},$$
$$D^\sigma(\alpha_1, \cdots, \alpha_d) = \#[\overline{\mathfrak{M}}^\sigma(E) \cap \mu(\alpha_1) \cap \cdots \cap \mu(\alpha_d)].$$

Theorem I.3.2. (Wang) *Let* $\alpha_1, \cdots, \alpha_d \in H_2(X;\mathbb{Z})^\sigma$ *and* $p_*(\alpha_i) = 2\beta_i \in H_2(X';\mathbb{Z})$, $i = 1, \cdots, d$. *Then* $D_\sigma(\alpha_1, \cdots, \alpha_d) = D'(\beta_1, \cdots, \beta_d)$, *where* $D' : H_2(X') \times \cdots \times H_2(X') \longrightarrow \mathbb{Z}$ *is the Donaldson invariant on the quotient* $E' \longrightarrow X'$.

II. Finite Group Actions in U(1)-Gauge Theory

II.1. Review of Seiberg-Witten Invariant

- X : a closed, oriented, smooth 4-manifold, $b_2^+(x) > 1$.
  L$\longrightarrow X : a$ U(1)$-$bundle with $c_1(L) = c_1(X)$ mod 2.
  $W^{\pm}$: twisted spinor bundles associated with L.
  Clifford multi. $W^+ \bigotimes T^*X \longrightarrow W^-$
  $\tau : W^+ \times W^+ \longrightarrow \text{End}(W^+)_\circ$
  given by $\tau(\phi, \phi) = (\phi \bigotimes \bar{\phi}^t)_\circ$: traceless endomorphism of W^+.

- Levi-Civita connection. on X and a connection.
  A on L induce a Dirac operator $D_A : \Gamma(W^+) \longrightarrow \Gamma(W^-)$.
  Seiberg-Witten (SW) equations :

(✠) $$\begin{cases} D_A\phi & = 0 \\ F_A^+ & = -\tau(\phi,\, \phi) \end{cases} .$$

- Gauge group $c^{\infty}(X,\mathrm{U}(1))$ of L acts on $\mathcal{SW}(L) = \{(A,\phi)|(\maltese)\}$. $\mathfrak{M}(L) = \mathcal{SW}(L)/c^{\infty}(X,\mathrm{U}(1))$: the moduli space associated with the spin^c structure L on X.

- For a generic metric on X, $\mathfrak{M}(L)$: a compact orientable manifold with dimeusion $d = \frac{1}{4}[c_1(L)^2 - (2\chi + 3\sigma)]$.

- Fix a point $x_\circ \in X$, $\varrho : c^{\infty}(X,\mathrm{U}(1)) \longrightarrow \mathrm{U}(1)$: evaluation induces a $U(1)$-bundle
  $E = (\mathrm{SW}(L) \times \mathbb{C})/c^{\infty}(X,\mathrm{U}(1)) \longrightarrow \mathfrak{M}(L)$.

  $\mathcal{SW}(L) =< c_1(E)^s, \mathfrak{M}(L) >$
  : the Seiberg-Witten invariant of L, where $d = 2s$.

Theorem II.1.1.

(1) $\exists$ *finitely many spin*^{c} *str.* L *on* X *for which* SW$(L) \neq 0$.

(2) *If* $X = X_1 \sharp X_2$, $b_2^+(X_i > 0), i = 1, 2$,
then SW$(L) = 0$ *for spin*^{c} *str.* L *on* X.

(3) SW(L) *: indep. on the metrics on* X, *dep. only on* $c_1(L)$.

(4) f *: a self-diffeo. of* X, SW$(L) = \pm$SW(f^*L).

(5) *If* X *admits a metric of positive scalar curvature,*
then SW $\equiv 0$ *on* X.

(6) *If* X *: symplectic, then* SW$(K_\times) = \pm 1$.

(7) *Thom conjecture is true.*

II.2. Finite Group Action and spin^c structure

Let $\pi_1(X)$ be finite, and G be a finite group.

Theorem II.2.1. *If G acts smoothly, freely on X,* $\mathrm{SW}_X \not\equiv 0$.
Then X/G *cannot decomposed as a smooth* $X_1 \sharp X_2$
with $b_2^+(X_i) > 0, i = 1, 2$.

Theorem II.2.2. *Let X be symplectic,* $c_1(X)^2 > 0$, $b_2^+(X) > 3$.
If σ *: free anti-sym. involution on X, then* $\mathrm{SW} \equiv 0$ *on* X/σ *and* X/σ *can not be symplectic.*

Theorem II.2.3. *Let* $\mathrm{SW}_X \not\equiv 0$ *on X,* $b_2^+(X) > 1$.
If Y has negative definite intersection form and $n_1, \cdots, n_k$ *: even integers such that* $4b_1(Y) = 2n_1 + \cdots + 2n_k + n_1^2 + \cdots + n_k^2$*, and* $\pi_1(Y)$ *has a nontrivial finite quotient. Then* $X \sharp Y$ *has a nontrivial* SW*-invariant, but does not admit a symplectic str.*

Theorem II.2.4. *Let X : symplectic, and σ : anti-sympl. involution on X,*

$$X^\sigma = \coprod_\lambda \Sigma_\lambda$$

: lagrangian surfaces, genus$(\Sigma_\lambda) > 1$ *for some* λ, $b_2^+(X/\sigma) > 1$.
Then X/σ *has vanishing* SW*-invariants.*

Example II.2.5. Let $X = (\Sigma_g \times \Sigma_g, w \bigoplus w)$, and $f : \Sigma_g \longrightarrow \Sigma_g$: an involution such that $f^* w = -w$.
Let $\sigma_f : \Sigma_g \times \Sigma_g \longrightarrow \Sigma_g \times \Sigma_g,$ be given by $\sigma_f(x, y) = (f^{-1}(y), f(x))$.
Then σ_f is an anti-symplectic involution,
the fixed point set is $(\Sigma_g \times \Sigma_g)^{\sigma_f} \simeq \Sigma_g$.
By Hirzebruch signature thm.,

$$b_2^+(X/\sigma_f) = \frac{1}{2}(b_2^+(X) - 1) = g^2 > 1 \text{ if } g > 1.$$

Theorem II.2.6. *Let* X *: Kähler surf.,* $b_2^+(X) > 3$,
$H_2(X;\mathbb{Z})$ *has no 2-torsion.*
$\sigma : X \longrightarrow X$ *: anti-holom. involution,*
$X^\sigma = \Sigma$ *has genus* > 0, $[\Sigma] \in 2H_2(X;\mathbb{Z})$.
If $K_X^2 > 0$ *or* $K_X^2 = 0$ *and* $g(\Sigma) > 1$, *then* $\mathrm{SW} \equiv 0$ *on* X/σ.

- Let $\mathbb{Z}_p$ act on X, p : prime, $H_1(X,\mathbb{R}) = 0, b_2^+(X) > 1$.

Theorem II.2.7. (Fang) *Supp. $\mathbb{Z}_p$ acts trivially on $H^{2+}(X,\mathbb{R})$. L : $\mathbb{Z}_p$-equivariant spin^c str. on X, the equivariant Dirac operator $D_A : \Gamma(W^+) \longrightarrow \Gamma(W^-)$ has the form*

$$\operatorname{ind} \mathbb{Z}_p(D_A) = \sum_{j=0}^{p-1} k_j t^j \in R(\mathbb{Z}_p) = \frac{\mathbb{Z}[t]}{(t^p = 1)}.$$

$$Then\ \mathrm{SW}(L) = 0 \bmod p$$

if $k_j \leq \frac{1}{2}(b_2^+(X) - 1), j = 0, 1, \cdots, p-1$.

III. Finite Group Actions in Gromov-Witten Theory.

III.1. Finite Group Action in Symplectic 4-Manifolds

- Let (X, ω) be a closed symplectic 4-manifold.
  Let a finite group G act pseudo-holomorhpically, semifreely on X with a codim.2 fixed pt. set F.

- Let $p : X \to X/G = X'$ be the projection.

$$\begin{array}{ccc} H^2(X;\mathbb{Z})^G & \xleftarrow{p^*} & H^2(X';\mathbb{Z}) \\ {\scriptstyle PD}\downarrow & & \downarrow{\scriptstyle |G|\cdot PD} \\ H_2(X;\mathbb{Z})^G & \xrightarrow{p_*} & H_2(X';\mathbb{Z}) \end{array}$$

  commutes, where $|G|$: the order of G, and PD : Poincaré dual.

- Let $A' \in H_2(X';\mathbb{Z})$, $A' = PD(\alpha') \in H_2(X';\mathbb{Z})$, and

$$L' \to X' : U(1) \text{ vector bundle } c_1(L') = \alpha'.$$

$$L \equiv p^* L' \to X,\ A = PD(c_1(L)).$$

Then $A \in H_2(X;\mathbb{Z})^G$ and $p_*(A) = |G|A'$.

- Let J be G-invariant, ω-comptible almost complex st. on X.
ω', J' : push downs of ω, J, resp., on X'.

$$p_* : H = \text{im}(PD \circ P^*) \subset H_2(X;\mathbb{Z})^G \longrightarrow H' = \text{im}(|G|PD) \subset H_2(X';\mathbb{Z})$$

is an isomorphism.

For $A \in H, p_*(A) = |G|A', F = X^G, F' = p(F) \subset X'$,

$$\begin{aligned}
& \overline{\mathfrak{M}}_{0,k}(X, F : A, J)^G \\
= \ & [\{u : (C, x_1, \cdots, x_k) \to (X, F; F)| \\
& u : J - \text{holo. stable map representing } A \text{ relative to } F, \\
& \text{C is a curve with arithmatic genus 0 and } k \text{ marked points}, \\
& u(C) : G - \text{invariant.}\}/\mathrm{Aut}(u)] \\
& \overline{\mathfrak{M}}_{0,k}(x', F'; A', J') \\
= \ & [\{u' : (C, x_1, \cdots x_k) \to (X', F'; F')|u' : J' - \text{holo.} \\
& \text{stable map representing } A' \text{ relative to } F', \text{ C is a curve with} \\
& \text{arithmatic genus 0 and } k \text{ marked points,}\}/\mathrm{Aut}(u')].
\end{aligned}$$

Theorem III.1.1.

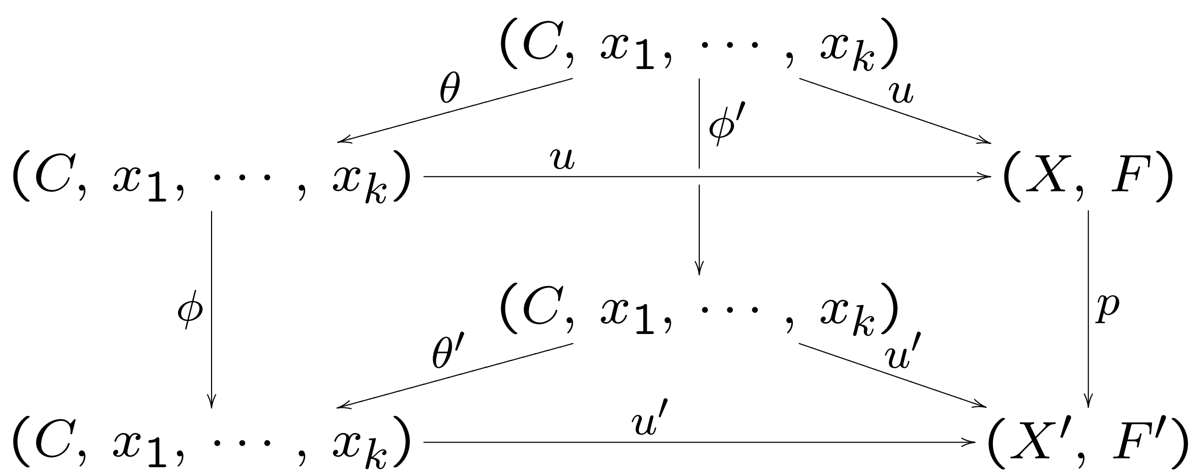
(i) *There is a homeomorphism*

$$\Psi : \overline{\mathfrak{M}}_{0,k}(X, F; A, J)^G \to \overline{\mathfrak{M}}_{0,k}(X', F'; A', J')$$

which is an orientation preserving diffeomorphism on each strata.

(ii) *And we have*

$$\begin{aligned} &= \text{dim } (\overline{\mathfrak{M}}_{0,k}(X, F; A, J)^G) \\ &= \text{dim } (\overline{\mathfrak{M}}_{0,k}(X', F'; A', J')) \\ &= 2c_1(X')A' + 2k - 2 - 2A' \cdot F' \\ &= d'. \end{aligned}$$



III.2. Relative Gromov-Witten Invariant

The evaluation map

$$ev_k : \overline{\mathfrak{M}}_{0,\, k}(X,\ F;\ A,\ J)^G \to X^k$$

$$ev_k([C, x_1, \cdots x_k; u]) = (u(x_1), \cdots, u(x_k)).$$

im$(ev_k) \subset X^k$: a d'-dim. pseudo-cycle whose boundary has at most dim. $d' - 2$.

The relative G-invariant Gromov-Witten invariant

$$\Phi^{F,\, G}_{A,\, k} : H_d(X^k) \to Q$$

is given by the integral

$$\Phi^{F,\, G}_{A,\, k}(D) = \int_{\overline{\mathfrak{M}}_{0,k}(X,\, F;\, A,\, J)^G} {ev_k}^* PD(D) = (\mathrm{im}\, ev_k) \cdot D$$

: intersection number in X^k, where $d' + d = 4k$.

Similarly, the evaluation map

$$ev_k{}' : \overline{\mathfrak{M}}_{0,k}(X', F'; A', J') \to X'^k$$

is given by

$$ev_k{}'([C, x_1, \cdots x_k; u']) = (u'(x_1), \cdots, u'(x_k)).$$

The relative Gromov-Witten invariant on the quotient

$$\Phi^{F'}_{A',k} : H_d(X'^k) \to Q$$

is defined by the integral

$$\Phi^{F'}_{A',\,k}(D') = \int_{\overline{\mathfrak{M}}_{0,k}(X',\,F';\,A',\,J')} ev_k{}^* PD(D') = (\mathrm{im}\,(ev_k{}')) \cdot D$$

: intersection number in X^k, where $d' + d = 4k$.

- If $2c_1(X')A' = 2 + 2k + 2F' \cdot A'$, then the invariant is the degree of the $ev_k{}'$ which is the number of J'-holomorphic curves representing the homology A' meeting generic k distinct points in F' tangently.

Theorem III.2.1. *Let* $D \in H \subset H_d(X^k;\mathbb{Z})^G$ *and* $D' \in H' \subset H_d(X'^k;\mathbb{Z})$ *such that* $p_*(D) = |G|^k D'$, *then*

$$\Phi^{F,G}_{A,k}(D) = \Phi^{F'}_{A',K}(D').$$

[*i.e.,* $D \in H_d(X^k;\mathbb{Z})^G = \sum_{d_1+\cdots+d_k=d} H_{d_1}(X)^G \times \cdots \times H_{d_k}(X)^G$

$\exists D_{d_i} \in H_{d_i}(X)^G, i = 1, \cdots, k$ *such that*

$$D = \sum_{d_1+\cdots+d_k=d} D_{d_1} \times \cdots \times D_{d_k}.$$

$$p_*(D_{d_i}) = |G| {D_{d_i}}', i = 1, \cdots, k.$$

$$\begin{aligned} p_*(D) &= p_*(\sum_{d_1+\cdots+d_k=d} D_{d_1} \times \cdots \times D_{d_k}) \\ &= \sum_{d_1+\cdots d_k=d} p_*(D_{d_1}) \times \cdots \times p_*(D_{d_k}) \\ &= |G|^k p_*(D'). \end{aligned}$$

$\Phi^{F,G}_{A,k}(D) = (\mathrm{im}(ev_k)) \cdot D$: *intersection number in* X^k.]

III.3. Example

- Recall the number $R(d)$ of rational curves of the degree d in $\mathbb{CP}^2$.

$$\dim \overline{\mathfrak{M}}_{0,k}(\mathbb{CP}^2, d[S^2]) = 2(3d - 1 + k).$$

If $k = 3d - 1$, then $\dim \overline{\mathfrak{M}}_{0,k}(\mathbb{CP}^2, d[S^2]) = 4(3d - 1)$,

$$ev_k : \overline{\mathfrak{M}}_{0,k}(\mathbb{CP}^2, d[S^2]) \to (\mathbb{CP}^2)^k.$$

$$R(D) = \int_{\overline{\mathfrak{M}}_{0,k}(\mathbb{CP}^2, d[S^2])} \prod_{i=1}^{k} {ev_k}^*(c_1(\mathcal{O}(1)_i^2)) = \deg(ev_k)$$

$$= \begin{cases} \sum_{d_1+d_2=d} R(d_1)R(d_2)d_1^2 d_2 \left[k_2\binom{3d-4}{3k_1-2} - k_1\binom{3d-4}{3k_1-1}\right], & d \geq 2 \\ 1, \quad d = 1 \end{cases}$$

computed by the WDVV-equations for the potential.

- For $\mathbb{CP}^1 \subset \mathbb{CP}^2$,

$$\dim \overline{\mathfrak{M}}_{0,k}(\mathbb{CP}^2, \mathbb{CP}^1; d\mathbb{CP}^1) = 2(2d - 1 + k).$$

$$\text{If } k = 2d-1, \text{ then } = 4(2d-1).$$

The number of rational curves in $\mathbb{CP}^2$ of degree d relative to $\mathbb{CP}^1$ passing through generic $(2d-1)$ points in $\mathbb{CP}^1$ is

$$\begin{aligned} R'(d) &= \int_{\overline{\mathfrak{M}}_{0,k}(\mathbb{CP}^2,\mathbb{CP}^1;d\mathbb{CP}^1)} \prod_{i=1}^{k} {ev_k}'^*(c_1(Q(1)_i)^2) \\ &= \text{degree of } [{ev_k}' : \overline{\mathfrak{M}}_{0,k}(\mathbb{CP}^2, \mathbb{CP}^1; d\mathbb{CP}^1) \to (\mathbb{CP}^2)^k]. \end{aligned}$$

- Let $\sigma : \mathbb{CP}^1 \times \mathbb{CP}^1 \to \mathbb{CP}^1 \times \mathbb{CP}^1$ be the involution given by $\sigma(z_1, z_2) = (z_2, z_1)$.

  Then $(\mathbb{CP}^1 \times \mathbb{CP}^1)^\sigma = \Delta \simeq \mathbb{CP}^1$, and $\mathbb{CP}^1 \times \mathbb{CP}^1/\sigma \simeq \mathbb{CP}^2$.

Commutative diagram

$$\begin{array}{ccc} H^2(\mathbb{CP}^1 \times \mathbb{CP}^1)^\sigma & \xleftarrow{p^*} & H^2(\mathbb{CP}^2) \\ PD \downarrow & & \downarrow 2\cdot PD \\ H_2(\mathbb{CP}^1 \times \mathbb{CP}^1)^\sigma & \xrightarrow{p_*} & H_2(\mathbb{CP}^2) \end{array} .$$

Let $A, B \in H_2(\mathbb{CP}^1 \times \mathbb{CP}^1), C \in H_2(\mathbb{CP}^2)$: generators.

$$\begin{aligned} & p_*(\triangle) = p_*(A, B) = 2C, \\ & \dim \overline{\mathfrak{M}}_{0,k}(\mathbb{CP}^1 \times \mathbb{CP}^1, \triangle; d(\triangle))^\sigma \\ & \quad = 2C_1(\mathbb{CP}^1 \times \mathbb{CP}^1) \cdot d(\triangle) + 2k - 2 - 2\cdot \triangle \cdot d(\triangle) \\ & \quad = 2(2d - 1 + k). \end{aligned}$$

For $D \in H[(\mathbb{CP}^1 \times \mathbb{CP}^1)^k]^\sigma, p_*(D) = 2^k D' \in H(\mathbb{CP}^2)^k$,
the relative Gromov-Witten invariant

$$\begin{aligned} \Psi^{\triangle,\sigma}_{d(\triangle),k}(D) &= \Psi^C_{dC,k}(D') \\ &= \Psi^C_{dC,k}(pt, \cdots, pt) \text{ if } k = 2d - 1 \\ &= R'(d) \\ &= deg[{ev_k}' : \overline{\mathfrak{M}}_{0,k}(\mathbb{CP}^2, \mathbb{CP}^1, dC) \to (\mathbb{CP}^2)^k]. \end{aligned}$$