轨道空间上的Frobenius流形结构

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Main References

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3. Dafeng Zuo, International Mathematics Research Notices 8(2007)rnm020-25

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   Extended affine Weyl groups and Frobenius manifolds-II

   The first draft: math.DG/0502365

5.—, Geometric structures related to new extended affine Weyl groups, in preparation.

Outline of this talk

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§1. Physical background

2-dimensional Toplogical field theory (2D TFT)

QFT on Σ consists of

- Local fields $\phi_\alpha(x)$, $x \in \Sigma$, eg. gravity field: the metric $g_{ij}(x)$
- Classical action

$$S[\phi] = \int_\Sigma L(\phi, \phi_x, \cdots)$$

Remark. Classical field theory is determined by the Euler-Lagrangian equations $\frac{\delta S}{\delta \phi_\alpha(x)} = 0$.

- Partition function

$$Z_\Sigma = \int [d\phi] e^{-S[\phi]}$$

- Correlators

$$\langle \phi_\alpha(x)\phi_\beta(y)\cdots\rangle = \int [d\phi]\phi_\alpha(x)\phi_\beta(y)\cdots e^{-S[\phi]}$$

Topological invariance

$$\frac{\delta S}{\delta g_{ij}(x)} = 0 \quad (i.e., \delta g_{ij}(x) = \text{arbitrary},\ \delta S = 0)$$

Remark. conformal field theory: $\delta g_{ij}(x) = \epsilon g_{ij}(x),\ \delta S = 0.$

$\Rightarrow$ correlators are numbers depending only on the genus $g = g(\Sigma)$

$$\langle \phi_\alpha(x)\phi_\beta(y)\cdots\rangle = \langle \phi_\alpha\phi_\beta\cdots\rangle_g$$

Example. 2D gravity with Hilbert-Einstein action

$$S = \frac{1}{2\pi}\int R\sqrt{g}\,d^2x = \chi(\Sigma)$$

There are two ways of quantization of this functional to obtain 2D quantum gravity.

1. [Matrix gravity] Base on an approximate discrete version of the model ($\Sigma \to$ Polyhedron) $\rightsquigarrow$ Matrix integrals

$$Z_N(t) = \int_{X=X^*} e^{-tr(X^2+t_1X^4+t_2X^6+\cdots)}dX$$

$N \to \infty \rightsquigarrow \tau$-function of KdV hierarchy $\rightsquigarrow$ a solution of 2D gravity

2.[Topological 2D gravity] Base on an approximate supersymmetric extension of Hilbert-Einstein Lagrangian $\rightsquigarrow$

$$\sigma_p \leftrightarrow c_p \in H_*(\mathcal{M}_{g,n})$$

and the genus g correlators of the topological gravity are expressed as

$$\langle \sigma_{p_1} \cdots \sigma_{p_n} \rangle = \#(c_{p_1} \cap \cdots c_{p_n}) = \prod_{i=1}^{n} (2p_i + 1)!! \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{p_1} \cdots \psi_n^{p_n},$$

where $\psi_i = c_1(L_i) \in H^*(\overline{\mathcal{M}}_{g,n})$ (the first Chern classes).

Witten conjecture [Proved by Kontsevich]

$\int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{p_1} \cdots \psi_n^{p_n}$ is governed by the τ-function of KdV hierarchy.

Problem. To find a rigorous mathematical foundation of 2D topological field theory.

M.F.Atiyah, Publ.Math. IHES. 68(1988)175–186. (inspired by G.Segal's axiomatization of CFT), for arbitrary dimension

Matter sector of a 2D TFT is specified by

1. $\mathcal{A}$=the space of local physical states, $\dim \mathcal{A} < \infty$
   basis $\{\phi_1 = 1, \cdots, \phi_n\}$ (primary observables)
2. an assignment

$$(\Sigma, \partial\Sigma) \mapsto v_{(\Sigma,\partial\Sigma)} \in A_{(\Sigma,\partial\Sigma)}$$

for any oriented 2-surface Σ with an oriented boundary $\partial\Sigma$ depends only on the topology of the the pair $(\Sigma, \partial\Sigma)$

$$A_{(\Sigma,\partial\Sigma)} := \begin{cases} \mathbb{C}, & \text{if} \quad \partial\Sigma = \emptyset; \\ A_1 \otimes \cdots \otimes A_k, & \text{if} \quad \partial\Sigma = C_1 \cup \cdots \cup C_k \end{cases}$$

$$A_i := \begin{cases} \mathcal{A}, & \text{oriention of } C_i \text{ is coherent to that of } \Sigma; \\ \mathcal{A}^*, & \text{otherwise} \end{cases}$$

The assignment satisfies three axioms: (see the attached files)

1. Normalization; 2. Multiplicativity; 3. Factorization.

Denote a symmetric polylinear function on the space of the states by

$$v_{g,s} := v_{(\Sigma,\partial\Sigma)} \in \underbrace{\mathcal{A}^* \otimes \cdots \otimes \mathcal{A}^*}_{s}, \quad g = g(\Sigma)$$

The genus g correlators of the fields $\phi_{\alpha_1}, \cdots, \phi_{\alpha_s}$ are defined by

$$\langle \phi_{\alpha_1} \cdots \phi_{\alpha_s} \rangle_g := v_{g,s}(\phi_{\alpha_1} \otimes \cdots \otimes \phi_{\alpha_s}).$$

Theorem[Dijgraff etc.] $\mathcal{A}$ carries a natural structure of a Frobenius algebra $(\mathcal{A}, \bullet, \langle\ ,\ \rangle, \phi_1)$. All correlators can be expressed in a pure algebraic way in terms of this algebra, i.e.,

$$\langle \phi_{\alpha_1} \cdots \phi_{\alpha_s} \rangle_g = \langle \phi_{\alpha_1} \bullet \ldots \bullet \phi_{\alpha_s}, H^g \rangle$$

where $H = \eta^{\alpha\beta} \phi_\alpha \bullet \phi_\beta$ and $\eta_{\alpha\beta} = \langle \phi_\alpha, \phi_\beta \rangle$.

Definition. A Frobenius algebra is a pair $(\mathcal{A}, \bullet, \langle\ ,\ \rangle, e)$ satisfying

1. $\mathcal{A}$ is a commutative and associative algebra over $\mathcal{K}$ with a unit e;
2. $\langle\ ,\ \rangle : \mathcal{A} \times \mathcal{A} \to \mathcal{K}$ is a symmetric non-degenerate inner product;
3. $\langle\ ,\ \rangle$ is invariant, i.e., $\langle a \bullet b, c\rangle = \langle a, b \bullet c\rangle$.

Example A[Topological sigma model]. X a smooth projective variety,

$$\dim_{\mathbb{C}} X = d, H^{odd}(X) = 0, \mathcal{A} = H^*(X), \dim \mathcal{A} = n$$

primary observables $\leftrightarrow$ cohomologies $\phi_1 = 1, \cdots, \phi_n \in H^*(X)$

$$\langle \phi_i, \phi_j \rangle := \int_X \phi_i \cup \phi_j$$

Claim: $(\mathcal{A}, \cup, \langle\ ,\ \rangle, \phi_1)$ is a Frobenius algebra.

Example B[Topological Landau-Ginsburg model]. Let $f(x)$ be an analytic function, $x = (x_1, \cdots, x_N) \in \mathbb{C}^N$ with an isolated singularity at $x = 0$ of the multiplicity n, i.e., $df|_{x=0} = 0$.

$$\mathcal{A} := \mathbb{C}[x_1, \cdots, x_N]/(\frac{\partial f}{\partial x_1}, \cdots, \frac{\partial f}{\partial x_N}), \quad \dim \mathcal{A} = n$$

primary observables $\leftrightarrow \phi_1 = 1, \phi_2(x), \cdots, \phi_n(x) \in \mathcal{A}$

$$\langle \phi(x), \psi(x) \rangle := \frac{1}{(2\pi i)^N} \int_{\cap |\frac{\partial f}{\partial x_i}|=\epsilon} \frac{\phi(x)\psi(x)}{\frac{\partial f}{\partial x_1} \cdots \frac{\partial f}{\partial x_N}} d^N x$$

Claim: $(\mathcal{A}, \langle\ ,\ \rangle, \phi_1, \cdot)$ is a Frobenius algebra.

Next, they consider a particular topological perturbation, which preserves the topological invariance:

$$S \mapsto \tilde{S}(t) := S - \sum_{\alpha=1}^{n} t^{\alpha} \int_{\Sigma} \Omega$$

$$\langle \phi_\alpha(x)\phi_\beta(y)\cdots\rangle(t) \equiv \int [d\phi]\phi_\alpha(x)\phi_\beta(y)\cdots e^{-\tilde{S}(t)}$$

Theorem. [WDVV,1991] The perturbed Frobenius algebra $\mathcal{A}_t$ satisfies WDVV equations of associativity

$$\frac{\partial^3 F}{\partial t^\alpha \partial t^\beta \partial t^\lambda}\eta^{\lambda\mu}\frac{\partial^3 F}{\partial t^\mu \partial t^\delta \partial t^\gamma} = \frac{\partial^3 F}{\partial t^\delta \partial t^\beta \partial t^\lambda}\eta^{\lambda\mu}\frac{\partial^3 F}{\partial t^\mu \partial t^\alpha \partial t^\gamma},$$

with a quasihomogeneity condition

$$\mathcal{L}_E F = (3-d)F + \text{quadratic polynomial in t}$$

where

$$E = t^1\partial_1 + \text{linear in } t^2, \cdots, t^n$$

is Euler vector field and ϕ_1 is unit,

$$\eta_{\alpha\beta} = \langle\phi_\alpha\phi_\beta\rangle_0(t) = \text{const.}, \quad \langle\phi_\alpha\phi_\beta\phi_\gamma\rangle_0(t) = \frac{\partial^3 F}{\partial t^\alpha \partial t^\beta \partial t^\gamma}$$

for some function $F(t)$, called primary free energy.

[B.Dubrovin's idea,1992]: Add the above statement [WDVV] as a new axiom of 2D TFT. That is to say, to reconstruct the building of 2D TFT on the base of WDVV equations.

Example A'. Frobenius algebra $\mathcal{A}_t$: quantum cohomology of X

$$t = (t', t''), t' \in H^2(X)/2\pi i H^2(X, \mathbb{Z}), t'' \in H^{*\neq 2}(X)$$

The primary energy $F(t)$ is the generating function of the genus 0 Gromov-Witten invariants.

Particularly, $X = \mathbb{CP}^1$,

Quantum cohomology of $\mathbb{CP}^1 = \mathbb{C}[\phi]/(\phi_2 = e^{t^2})$

$$F(t) = \frac{1}{2}(t^1)^2(t^2) + e^{t^2}, E = t^1\partial_1 + 2\partial_2, e = \partial_1.$$

Example B'. Set $f_t(x) = f(x) + \sum_{\alpha=1}^{n} s^\alpha(t)\phi_\alpha(x)$, then the deformed Frobenius algebra $\mathcal{A}_t$ is

$$\mathcal{A}_t := \mathbb{C}[x_1, \cdots, x_N]/(\frac{\partial f_t}{\partial x_1}, \cdots, \frac{\partial f_t}{\partial x_N})$$

primary observables $\leftrightarrow$ elements of the Jacobi ring $\mathcal{A}$

$$\langle \phi(x), \psi(x) \rangle \equiv \eta_{ij}(t) := \frac{1}{(2\pi i)^N} \int_{\cap |\frac{\partial f_t}{\partial x_i}|=\epsilon} \frac{\phi(x)\psi(x)}{\frac{\partial f_t}{\partial x_1} \cdots \frac{\partial f_t}{\partial x_N}} d^N x$$

Particularly, $f(x) = x^{n+1}$

The simple singularity of type A_n, $\mathcal{A} = \mathbb{C}[x]/(x^n)$.

§2. Definitions and Examples

Back to the main problem: we have a family of Frobenius algebras $\mathcal{A}_t$ depending on the parameters $t = (t^1, \cdots, t^n)$. Write

$$M = \text{the space of parameters}$$

and we have a fiber bundle

$$\begin{array}{r l} & \downarrow \mathcal{A}_t \\ t \in & M \end{array}$$

The basic idea is to identify this fiber bundle with the tangent bundle TM of the manifold M.

Definition. A Frobenius structure of charge d on M is the data $(M, \bullet, \langle , \rangle, e, E)$ satisfying

(i) $\eta := \langle\ ,\ \rangle$ is a flat pseudo-Riemannian metric and $\nabla e = 0$;

(ii) $(T_m M, \bullet, \eta, e)$ is a Frobenius algebra which depends smoothly on $m \in M$;

(iii) $(\nabla_w c)(x, y, z)$ is symmetric, where $c(x, y, z) := \langle x \bullet y, z \rangle$;

(iv) A linear vector field $E \in Vect(M)$ must be fixed on M, i.e. $\nabla\nabla E = 0$ such that

$$\mathcal{L}_E \langle\ ,\ \rangle = (2 - d)\langle\ ,\ \rangle, \quad \mathcal{L}_E \bullet = \bullet, \quad \mathcal{L}_E\ e = -e.$$

Theorem. [B.Dubrovin 1992] There is a one to one correspondence between a Frobenius manifold and the solution $F(\mathbf{t})$ of WDVV equations of associativity

$$\frac{\partial^3 F}{\partial t^\alpha \partial t^\beta \partial t^\lambda}\eta^{\lambda\mu}\frac{\partial^3 F}{\partial t^\mu \partial t^\delta \partial t^\gamma} = \frac{\partial^3 F}{\partial t^\delta \partial t^\beta \partial t^\lambda}\eta^{\lambda\mu}\frac{\partial^3 F}{\partial t^\mu \partial t^\alpha \partial t^\gamma},$$

with a quasihomogeneity condition

$$\mathcal{L}_E F = (3-d)F + \text{quadratic polynomial in t.}$$

Definition. A Frobenius manifold is called semisimple if the algebra $(T_m M, \bullet)$ are semisimple at generic m.

Main mathematical applications of Frobenius manifolds

★ The theory of Gromov - Witten invariants,

★ Singularity theory,

★ Hamiltonian theory of integrable hierarchies,

★ Differential geometry of the orbit spaces of reflection groups and of their extensions $\rightsquigarrow$ semisimple Frobenius manifolds.

Definition. An intersection form of Frobenius manifold is a symmetric bilinear form on the cotangent bundle T^*M defined by

$$(\omega_1, \omega_2)^* = i_E(\omega_1 \cdot \omega_2), \quad \omega_1, \omega_2 \in T^*M.$$

Here the multiplication law on the cotangent planes is defined using the isomorphism

$$\langle\ ,\ \rangle : TM \to T^*M.$$

The discriminant Σ is defined by

$$\Sigma = \{t|\det(\ ,\ )|_{T_t^*M} = 0\} \subset M.$$

Theorem. [B.Dubrovin 1992]

The metrics $\eta := \langle \ , \ \rangle$ *and* $g := (\ ,\ )^*$ *form a flat pencil on* $M \diagdown \Sigma$, *i.e.,*

1. *The metric* $h^{\alpha\beta} = \eta^{\alpha\beta} + \lambda g^{\alpha\beta}$ *is flat for arbitrary* λ *and*
2. *The Levi-Civita connection for the metric* $h^{\alpha\beta}$ *has the form*

$$\Gamma^{\alpha\beta}_{\delta_{(h)}} = \Gamma^{\alpha\beta}_{k_{(\eta)}} + \lambda \Gamma^{\alpha\beta}_{k_{(g)}},$$

where $\Gamma^{\alpha\beta}_{\delta_{(h)}} = -h^{\alpha\gamma}\Gamma^{\beta}_{\delta\gamma_{(h)}}$, $\Gamma^{\alpha\beta}_{\delta_{(g)}} = -g^{\alpha\gamma}\Gamma^{\beta}_{\delta\gamma_{(g)}}$, $\Gamma^{\alpha\beta}_{\delta_{(\eta)}} = -\eta^{\alpha\gamma}\Gamma^{\beta}_{\delta\gamma_{(\eta)}}$.

The holonomy of the local Euclidean structure defined on $M \setminus \Sigma$ by the intersection form $(\ ,\ )^*$ gives a representation

$$\mu : \pi_1(M \diagdown \Sigma) \to \mathit{Isometries}(\mathbb{C}^n).$$

Definition. The group

$$W(M) := \mu(\pi_1(M \diagdown \Sigma)) \subset \mathit{Isometries}(\mathbb{C}^n)$$

is called a monodromy group of Frobenius manifold.

$$\Longrightarrow \qquad M \diagdown \Sigma = \Omega / W(M), \qquad \Omega \subset \mathbb{C}^n.$$

[B.Dubrovin's conjecture] The monodromy group is a discrete group for a solution of WDVV equations with good properties.

Example. $[W(M)$=Coxeter group $A_1]$ $n=1$, $M=\mathbb{R}$, $t=t^1$,

$$F(t)=\frac{1}{6}t^3, \quad E=t\partial_t, \quad e=\partial_t, \quad \eta^{11}=<\partial_t,\partial_t>=1.$$

$\rightsquigarrow$ dispersionless KdV hierarchy $\rightsquigarrow$ Witten Conjecture.

Example. $[W(M)$=extended affine Weyl group $\widetilde{W}(A_1)]$

Quantum cohomology of $\mathbb{CP}^1$:

$$F=\frac{1}{2}(t^1)^2t^2+e^{t^2}, E=t^1\partial_1+2\partial_2, e=\partial_1.$$

$\rightsquigarrow$ dispersionless extended Toda hierarchy $\rightsquigarrow$ Toda Conjecture.

Question 1. Given a Frobenius manifold, how to find the monodromy group? (Some cases can be computed).

Question 2. Which kind of groups can be served as the monodromy groups of some Frobenius manifolds?

- ♣ Coxeter groups [B.Dubrovin1996]
- ♣ Extended affine Weyl groups [B.Dubrovin Youjin Zhang 1996]

  [Dubrovin-Zhang-Zuo 2005,general], [2007,new cases]

  For the general case of type E, still open?

- ♣ Jacobi forms $J(A_n)$,$J(B_n)$, $J(G_2)$ [n=1, B.Dubrovin 1996, general n, M.Bertola 2000], $J(E_6)$, $J(D_4)$ [Satake.I 1993, 1998] Open for the rest?
- ♣ Elliptic Weyl groups [Satake.I 2006, math.AG/0611553]

§3 Frobenius manifolds and Coxeter groups

Let W be a finite irreducible *Coxeter group*.

$$W \curvearrowright V \qquad \rightsquigarrow \qquad W \curvearrowright S(V)$$

[Chevalley Theorem]. *The ring $S(V)^W$ of W-invariant polynomial functions on V*

$$\mathbb{C}[x_1, \cdots, x_n]^W \simeq \mathbb{C}[y^1, \cdots, y^n],$$

where $y^i = y^i(x_1, \cdots, x_n)$ are certain homogeneous W-invariant polynomials of degree $\deg y^i = d_i,\ i = 1, \cdots, n$.

The maximal degree h is called the Coxeter number. We use the ordering of the invariant polynomials

$$\deg y^n = d_n = h > d_{n-1} > \cdots > d_1 = 2.$$

The degrees satisfy the *duality condition*

$$d_i + d_{n-i+1} = h + 2, \quad i = 1, \cdots, n.$$

$$W \curvearrowright V \qquad \rightsquigarrow \qquad W \curvearrowright V \otimes \mathbb{C}$$

$$\mathcal{M} = V \otimes \mathbb{C}/W \qquad \text{affine algebraic variety}$$

$$S(V)^W \qquad \text{the coordinate ring of} \quad \mathcal{M}$$

$V \rightsquigarrow$ flat manifold $\quad (V, \{x_1, \cdots, x_n\}, (dx_a, dx_b)^* = \delta_{ab})$

$\rightsquigarrow (\mathcal{M} \setminus \Sigma, g^{ij}(y))$

$$g^{ij}(y) := (dy^i, dy^j)^* = \sum_{a,b=1}^{n} \frac{\partial y^i}{\partial x_a} \frac{\partial y^j}{\partial x_b} \delta_{ab}$$

Lemma.[K.Saito etc 1980]

1. The metric $(g^{ij}(y))$ is flat on $\mathcal{M} \setminus \Sigma$.

2. These $g^{ij}(y)$ are at most linear w.r.t y^n.

Write

$$e := \frac{\partial}{\partial y^n}.$$

Introduce a new metric,

$$\eta^{ij}(y) := \langle dy^i, dy^j \rangle = \mathcal{L}_e g^{ij}(y) = \frac{\partial g^{ij}(y)}{\partial y^n}.$$

Theorem. [K.Saito etc. 1980, B.Dubrovin 1992]

The metrics $\langle\ ,\ \rangle$ *and* $(\ ,\ )^*$ *form a flat pencil of metrics. Moreover, there exist homogeneous polynomials*

$$t^1(x), \cdots, t^n(x)$$

of degrees $d_1, \cdots, d_n$ *respectively such that the matrix*

$$\langle dt^i,\ dt^j\rangle := \eta^{ij} = \frac{\partial g^{ij}(t)}{\partial t^n}$$

is a constant nondegenerate matrix.

Theorem.[B.Dubrovin, 1992] *There exists a unique Frobenius structure of charge* $d = 1 - \frac{2}{h}$ *on the orbit space* $\mathcal{M}$ *polynomial in* $t^1, t^2, \cdots, t^n$ *such that*

1. *The unity vector field e coincides with* $\frac{\partial}{\partial y^n} = \frac{\partial}{\partial t^n}$;
2. *The Euler vector field has the form*

$$E = \sum_{\alpha=1}^{n} d_\alpha t^\alpha \frac{\partial}{\partial t^\alpha}.$$

Theorem. [B.Dubrovin's conjecture, 1996. C.Hertling, 1999]

Any irreducible semisimple polynomial Frobenius manifold with positive invariant degrees is isomorphic to the orbit space of a finite Coxeter group.

Our question and result I

Lemma. [M.Bertola, 1998] *For* B_n *and* $1 \leq k \leq n$,

1. *These* $g^{ij}(y)$ *are at most linear w.r.t* y^k
2. *The space* $\mathcal{M}$ *carries a flat pencil of metrics*

$$g^{ij}(y) \;\; and \;\; \eta^{ij}(y) = \frac{\partial g^{ij}(y)}{\partial y^k}. \tag{0.1}$$

Question: If $k \neq n$, how to construct the flat coordinates of $\eta^{ij}(y)$ and the corresponding Frobenius manifolds?

M.Bertola's results (unpublished 1999)

M.Bertola started from the superpotential

$$\lambda(p) = p^{-2(n-k)}(\sum_{a=1}^{n} p^{2(n-a)} y_a + p^{2n})$$

to compute the corresponding potential $F(t)$ and obtained n different Frobenius structures related to B_n. For example,

$$\eta(\partial', \partial'') = -\sum_{|\lambda|<\infty} res_{d\lambda=0} \frac{\partial'(\lambda(p)dp)\partial''(\lambda(p)dp)}{d\lambda(p)},$$

Our construction is different.

We started from the flat pencil of metrics.

The first step is to construct the flat coordinate $t^1, \cdots, t^n$.

The second step is to show that $g^{ij}(t)$ and the $\Gamma^{ij}_m(t)$ are weighted homogeneous polynomials of $t^1, \ldots, t^n, \frac{1}{t^n}$.

The last step is to get the Frobenius structure.

Write

$$\tilde{d}_j = \frac{j}{k}, \quad j \leq k, \quad \tilde{d}_m = \frac{2k(n-m)+l}{2k(n-k)}, \quad m > k.$$

Main Theorem.[Zuo IMRN-2007] *For any fixed integer* $1 \leq k \leq n$*, there exists a unique Frobenius structure of charge* $d = 1 - \frac{1}{k}$ *on the orbit space* $\mathcal{M} \setminus \{t^n = 0\}$ *of* B_n *(or* D_n*) polynomial in* $t^1, t^2, \cdots, t^n, \frac{1}{t^n}$ *such that*

1. *The unity vector field e coincides with* $\frac{\partial}{\partial y^k} = \frac{\partial}{\partial t^k}$*;*
2. *The Euler vector field has the form*

$$E = \sum_{\alpha=1}^{n} \tilde{d}_\alpha t^\alpha \frac{\partial}{\partial t^\alpha}.$$

Theorem. [Zuo IMRN-2007] *There is an isomorphism between them.*

Motived by James T.Ferguson and I.A.B. Strachan' work,

Logarithmic deformations of the rational superpotential/Landau-Ginzburg constructions of solutions of the WDVV equations, arXiv:Math-ph/0605078

we consider a water-bag reduction as follows

$$\lambda(p) = p^{-2(n-k)}(\sum_{a=1}^{n} p^{2(n-a)} y_a + p^{2n}) + \sum_{i=1}^{M} k_i \log(p^2 - b_i^2).$$

Remark. Don't determine a full Frobenius manifold because of the nonexistence of E.

Theorem. [Zuo IMRN-2007] The prepotential F is at most quadratic in the parameters k_α, that is, up to quadratic terms in the flat coordinates

$$F(\mathbf{t},\mathbf{b}) = F^{(0)}(\mathbf{t}) + \sum_{\alpha=1}^{M} k_\alpha F^{(1)}(\mathbf{t}, b_\alpha) + \sum_{\alpha\neq\beta}^{M} k_\alpha k_\beta F^{(2)}(b_\alpha, b_\beta),$$

where $\mathbf{t} = (t_1, \cdots, t_l)$ and $\mathbf{b} = (b_1, \cdots, b_M)$. Here $F^{(0)}$ is the potential associated to B_n (D_n) and

$$F^{(2)}(b_\alpha, b_\beta) = \frac{1}{2}(b_\alpha - b_\beta)^2 \log(b_\alpha - b_\beta)^2 + \frac{1}{2}(b_\alpha + b_\beta)^2 \log(b_\alpha + b_\beta)^2,$$

$$\deg F = \deg F^{(0)} = 4K + 2,\ \deg F^{(1)} = 2K + 2,\ \deg F^{(2)} = 2.$$

§4. Frobenius manifolds and Extended affine Weyl groups

Motivation. Quantum cohomology of $\mathbb{P}^1$:

$$F = \frac{1}{2} t_1^2 t_2 + e^{t_2}, E = t_1 \partial_1 + 2\partial_2, e = \partial_1, W(M) = \widetilde{W}(A_1)$$

Question: How to construct this kind of Frobenius manifolds? That is,

$$F = F(t_1, \cdots, t_n, t_{n+1}, e^{t_{n+1}})$$

$$E = \sum_{\alpha=1}^{n} d_\alpha t_\alpha \partial_\alpha + d_{n+1} \partial_{n+1}.$$

Notations

Let R be an irreducible reduced root system defined on $(V, (\ ,\ ))$.

$\{\alpha_j\}$: a basis of simple roots, $\{\alpha_j^\vee\}$: the corresponding coroots.

W Weyl group, $W_a(R)$ affine Weyl group (the semi-direct product of W by the lattice of coroots)

$W_a(R) \curvearrowright V$: affine transformations

$$\mathbf{x} \mapsto w(\mathbf{x}) + \sum_{j=1}^{l} m_j \alpha_j^\vee, \quad w \in W,\ m_j \in \mathbb{Z}.$$

ω_j: the fundamental weights, $(\omega_i, \alpha_j^\vee) = \delta_{ij}$

Definition.[B.Dubrovin, Y.Zhang 1998]

The extended affine Weyl group $\widetilde{W} = \widetilde{W}^{(k)}(R)$ acts on the extended space

$$\widetilde{V} = V \oplus \mathbb{R}$$

and is generated by the transformations

$$x = (\mathbf{x}, x_{l+1}) \mapsto (w(\mathbf{x}) + \sum_{j=1}^{l} m_j \alpha_j^\vee,\ x_{l+1}), \quad w \in W,\ m_j \in \mathbb{Z},$$

and

$$x = (\mathbf{x}, x_{l+1}) \mapsto (\mathbf{x} + \gamma\,\omega_k,\ x_{l+1} - \gamma).$$

Here $\gamma = 1$ except for the cases when $R = B_l, k = l$ and $R = F_4, k = 3$ or $k = 4$, in these three cases $\gamma = 2$.

Definition.[B.Dubrovin, Y.Zhang 1998]

$\mathcal{A} = \mathcal{A}^{(k)}(R)$ is the ring of all $\widetilde{W}$-invariant Fourier polynomials of the form

$$\sum_{m_1,\ldots,m_{l+1}\in\mathbb{Z}} a_{m_1,\ldots,m_{l+1}} e^{2\pi i(m_1x_1+\cdots+m_lx_l+\frac{1}{f}\, m_{l+1}x_{l+1})}$$

that are bounded in the limit

$$\mathbf{x} = \mathbf{x}^0 - i\,\omega_k\tau, \quad x_{l+1} = x_{l+1}^0 + i\,\tau, \quad \tau \to +\infty$$

for any $x^0 = (\mathbf{x}^0, x_{l+1}^0)$, where f is the determinant of the Cartan matrix of the root system R.

We introduce a set of numbers

$$d_j = (\omega_j, \omega_k), \quad j = 1, \dots, l$$

and define the following Fourier polynomials

$$\tilde{y}_j(x) = e^{2\pi i d_j x_{l+1}}\, y_j(\mathbf{x}), \quad j = 1, \dots, l,$$

$$\tilde{y}_{l+1}(x) = e^{\frac{2\pi i}{\gamma} x_{l+1}}.$$

Here

$$y_j(\mathbf{x}) = \frac{1}{n_j} \sum_{w \in W} e^{2\pi i(\omega_j, w(\mathbf{x}))},$$

$$n_j = \#\{w \in W | e^{2\pi i(\omega_j, w(\mathbf{x}))} = e^{2\pi i(\omega_j, \mathbf{x})}\}.$$

B.Dubrovin and Y.Zhang considered a particular choice of α_k based on the following observations

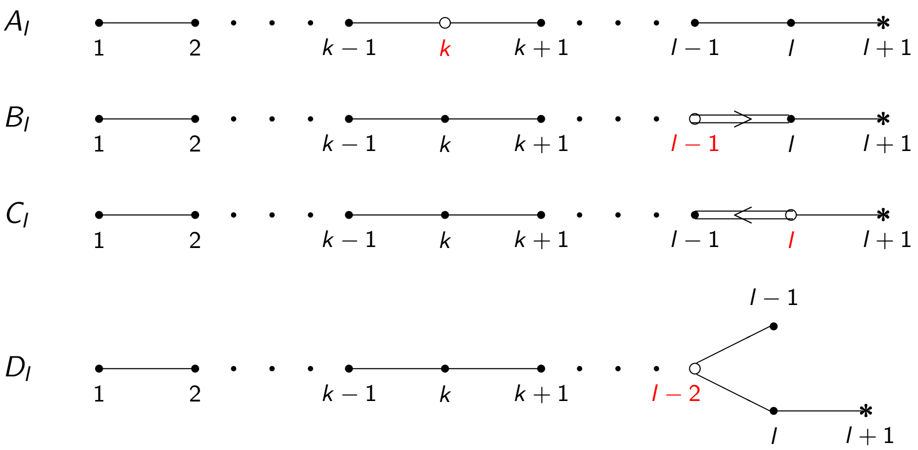
1. The Dynkin graph of $R_k := \{\alpha_1, \cdots, \hat{\alpha_k}, \cdot, \alpha_l\}$ (α_k is omitted) consists of 1, 2 or 3 branches of A_r type for some r.

2. $d_k > d_s, s \neq k$.

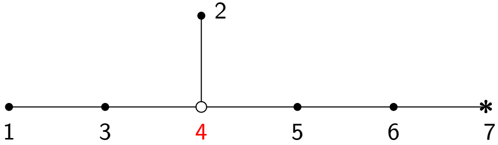
Chevalley-Type Theorem [B.Dubrovin, Y.Zhang 1998]

For the above particluar choice of α_k*,*

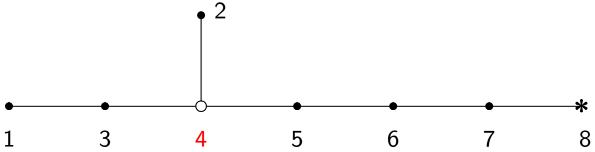
$$\mathcal{A}^{(k)}(R) \simeq \mathbb{C}[\tilde{y}_1, \cdots, \tilde{y}_{l+1}].$$


A_l
1 2 $k-1$ k $k+1$ $l-1$ l $l+1$
B_l
1 2 $k-1$ k $k+1$ $l-1$ l $l+1$
C_l
1 2 $k-1$ k $k+1$ $l-1$ l $l+1$
D_l
1 2 $k-1$ k $k+1$ $l-2$ $l-1$ l $l+1$

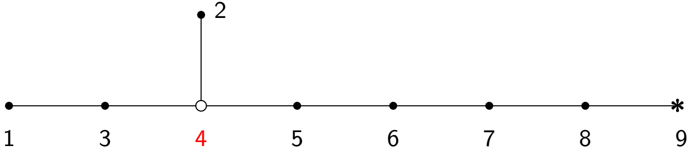
E_6



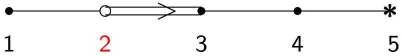
E_7



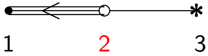
E_8



F_4



G_2



$\mathcal{M} = \mathrm{Spec}\mathcal{A}$: the orbit space of $\widetilde{W}^{(k)}(R)$

global coordinates on $\mathcal{M}$: $\{\tilde{y}_1(x), \cdots, \tilde{y}_{l+1}(x)\}$

local coordinates on $\mathcal{M}$:

$$y^1 = \tilde{y}_1, \ldots, y^l = \tilde{y}_l, \;\; y^{l+1} = \log \tilde{y}_{l+1} = 2\pi i \, x_{l+1}.$$

the metric $(\ ,\ )^{\sim}$ on $\widetilde{V} = V \oplus \mathbb{C}$

$$(dx_a, dx_b)^{\sim} = \frac{1}{4\pi^2}(\omega_a, \omega_b),$$

$$(dx_{l+1}, dx_a)^{\sim} = 0, \qquad 1 \le a, \;\; b \le l,$$

$$(dx_{l+1}, dx_{l+1})^{\sim} = -\frac{1}{4\pi^2(\omega_k, \omega_k)} = -\frac{1}{4\pi^2 d_k}$$

$\rightsquigarrow (\mathcal{M} \setminus \Sigma, g^{ij}(y)),$

$$g^{ij}(y) := (dy^i, dy^j)^{\sim} = \sum_{a,b=1}^{l+1} \frac{\partial y^i}{\partial x^a} \frac{\partial y^j}{\partial x^b} (dx^a, dx^b)^{\sim}. \tag{0.2}$$

Claim: $g^{ij}(y)$ *is flat. Moreover for the particular choice,* $g^{ij}(y)$ *are at most linear w.r.t* y^k.

$$\rightsquigarrow \eta^{ij}(y) = \mathcal{L}_e g^{ij}(y) = \frac{\partial g^{ij}(y)}{\partial y^k}, \quad e := \frac{\partial}{\partial y^k}.$$

Theorem. [B.Dubrovin, Y.Zhang 1998]

For the particular choice of α_k, $\eta^{ij}(y)$ and $g^{ij}(y)$ form a flat pencil. Moreover there exists a unique Frobenius structure on the orbit space $\mathcal{M} = \mathcal{M}(R, k)$ polynomial in $t^1, \dots, t^l, e^{t^{l+1}}$ such that

1. *the unity vector field coincides with $\frac{\partial}{\partial y^k} = \frac{\partial}{\partial t^k}$;*

2. *the Euler vector field has the form*

$$E = \frac{1}{2\pi i d_k} \frac{\partial}{\partial x_{l+1}} = \sum_{\alpha=1}^{l} \frac{d_\alpha}{d_k} \, t^\alpha \frac{\partial}{\partial t^\alpha} + \frac{1}{d_k} \frac{\partial}{\partial t^{l+1}}.$$

3. *The intersection form of the Frobenius structure coincides with the metric $(\, , \,)^{\sim}$ on $\mathcal{M}$.*

Theorem.[P.Slodowy 1998,Preprint but unpublished]

The ring $\mathcal{A}^{(k)}(R)$ is isomorphic to the ring of polynomials of $\tilde{y}_1(x), \cdots, \tilde{y}_{l+1}(x)$ for arbitrary choice.

Another proof [B.Dubrovin, Y.Zhang and D.Zuo 2006]

We give an alternative proof of Chevelly-Type theorem associated to the root system $B_l, C_l, D_l, (F_4, G_2)$.

Our question and result—II

An natural question:[P.Slodowy, B.Dubrovin and Y.Zhang 1998]

Is whether the geometric structures that were revealed in the above for particular choice also exist on the orbit spaces of the extended affine Weyl groups for an arbitrary choice of α_k?

Difficulty: d_k will be not the maximal number except the particular choice.

1. Note that the $g^{ij}(y)$ may be not linear with respect to y^k. Thus if we define $\eta^{ij}(y) = \dfrac{\partial g^{ij}(y)}{\partial y^k}$ as before, we can not obtain the flat pencil.

2. If we can obtain a flat pencil, how to find flat coordinates and construct Frobenius manifolds?

For the question 1, our strategy is to change the unity vector field.

Main theorem 1. *For any fixed integer $0 \le m \le l - k$ there is a flat pencil of metrics $(g^{ij}(y)), (\eta^{ij}(y))$ (bilinear forms on T^*M) with $(g^{ij}(y))$ given by* (**??**) *and $\eta^{ij}(y) = \mathcal{L}_e g^{ij}(y)$ on the orbit space $\mathcal{M}$ of $\widetilde{W}^{(k)}(C_l)$. Here the unity vector field*

$$e := \sum_{j=k}^{l} a_j \frac{\partial}{\partial y^j}$$

is defined by the generating function

$$\sum_{j=k}^{l} a_j u^{l-j} = (u+2)^m (u-2)^{l-k-m}$$

for the constants $a_k, \dots, a_l$.

For the question 2, it is very technical.

Main theorem 2. *In the flat coordinates $t^1, \dots, t^{l+1}$, the nonzero entries of the matrix (η^{ij}) are given by*

$$\eta^{ij} = \begin{cases} k, & j = k - i, & 1 \leq i \leq k-1, \\ 1, & i = l+1, j = k & \text{or } i = k,\ j = l+1, \\ C, & j = l - m + k - i + 1, & k+2 \leq i \leq l-m-1, \\ 2, & i = l-m, j = k+1 & \text{or } i = k+1,\ j = l-m, \\ 4m, & j = 2l - m - i + 1, & l-m+2 \leq i \leq l-1, \\ 2, & i = l, j = l-m+1 & \text{or } i = l-m+1,\ j = l, \end{cases}$$

where $C = 4(l-m-k)$. The entries of the matrix $(g^{ij}(t))$ and the Christoffel symbols $\Gamma^{ij}_m(t)$ are weighted homogeneous polynomials in $t^1, \dots, t^l, \frac{1}{t^{l-m}}, \frac{1}{t^l}, e^{t^{l+1}}$.

Main theorem 3. *For any fixed integer* $0 \leq m \leq l - k$, *there exists a unique Frobenius structure of charge* $d = 1$ *on the orbit space* $\mathcal{M} \setminus \{t^{l-m} = 0\} \cup \{t^l = 0\}$ *of* $\widetilde{W}^{(k)}(C_l)$ *weighted homogeneous polynomial in* $t^1, t^2, \cdots, t^l, \frac{1}{t^{l-m}}, \frac{1}{t^l}, e^{t^{l+1}}$ *such that*

1. *The unity vector field e coincides with* $\sum_{j=k}^{l} a_j \frac{\partial}{\partial y^j} = \frac{\partial}{\partial t^k}$;
2. *The Euler vector field has the form*

$$E = \sum_{\alpha=1}^{l} \tilde{d}_\alpha t^\alpha \frac{\partial}{\partial t^\alpha} + \frac{\partial}{\partial t^{l+1}}$$

3. *The intersection form of the Frobenius structure coincides with the metric* $(g^{ij}(t))$.

Main theorem 4. *The Frobenius manifold structures that we obtain in this way from B_l and D_l, by fixing the k-th vertex of the corresponding Dynkin diagram, are isomorphic to the one that we obtain from C_l by choosing the k-th vertex of the Dynkin diagram of C_l.*

Example. $[C_5, k=1, m=2]$Let R be the root system of type C_5, take $k=1, m=2$, then

$$
\begin{aligned}
F = \ & \frac{1}{2}\, t_6 {t_1}^2 + \frac{1}{2}\, t_1 t_2 t_3 + \frac{1}{2}\, t_1 t_4 t_5 - \frac{1}{72}\, {t_3}^4 {t_5}^4 - \frac{1}{8}\, t_2 t_3 t_4 t_5 \\
& - \frac{1}{2268}\, {t_5}^8 - \frac{1}{36288}\, {t_3}^8 - \frac{1}{48}\, {t_3}^2 {t_2}^2 - \frac{1}{48}\, {t_4}^2 {t_5}^2 + \frac{1}{24}\, {t_5}^4 t_2 t_3 \\
& + \frac{1}{96}\, {t_3}^4 t_4 t_5 + \frac{1}{1440}\, {t_3}^5 t_2 + \frac{1}{360}\, t_4 {t_5}^5 + t_2 t_3 e^{t_6} - t_4 t_5 e^{t_6} \\
& - \frac{2}{3}\, {t_5}^4 e^{t_6} + \frac{1}{6}\, {t_3}^4 e^{t_6} + \frac{1}{2}\, e^{2t_6} + \frac{1}{48}\, \frac{{t_2}^3}{t_3} + \frac{1}{192}\, \frac{{t_4}^3}{t_5}.
\end{aligned}
$$

The Euler vector field is given by

$$E = t_1\partial_1 + \frac{3}{4} t_2\partial_2 + \frac{1}{4} t_3\partial_3 + \frac{3}{4} t_4\partial_4 + \frac{1}{4} t_5\partial_5 + \partial_6.$$

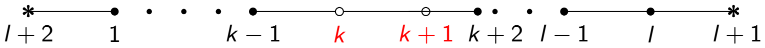
§5. Recent developments

Theorem.[2007] For any fixed integer $1 \le k < l$, there exists a unique Frobenius structure of charge $d = 1$ on the orbit space $\mathcal{M}^{k,2}$ of $\widetilde{W}^{k,2}(A_l)$ such that the potential $F(t) = \tilde{F}(t) + \frac{1}{2}(t^{k+1})^2 \log(t^{k+1})$, where $\tilde{F}(t)$ is a weighted homogeneous polynomial in $t^1, t^2, \cdots, t^l, e^{t^{l+1}}, e^{t^{l+2}-t^{l+1}}$, satisfying

1. The unity vector field e coincides with $\frac{\partial}{\partial y^{k+1}} + e^{ky^{l+1}} \frac{\partial}{\partial y^k} = \frac{\partial}{\partial t^k}$;
2. The Euler vector field has the form

$$E = \sum_{\alpha=1}^{l} \tilde{d}_\alpha t^\alpha \frac{\partial}{\partial t^\alpha} + \frac{1}{k} \frac{\partial}{\partial t^{l+1}} + \frac{l}{k(l-k)} \frac{\partial}{\partial t^{l+2}} .$$

A_l



Thanks

Appendix. Main techniques to obtain flat coordinates

The first step: $y \to \tau$

$$\sum_{j=0}^{l} \theta^j u^{l-j} = \sum_{j=0}^{l-m} \varpi^j (u+2)^m (u-2)^{l-m-j} - \sum_{j=l-m+1}^{l} \varpi^j (u+2)^{l-j} (u-2)^{j-k-1}.$$

where

$$\theta^j = \begin{cases} e^{k\, y^{l+1}}, \\ y^j e^{(k-j) y^{l+1}}, \\ y^j, \end{cases} \quad \varpi^j = \begin{cases} e^{k\, \tau^{l+1}}, & j = 0, \\ \tau^j e^{(k-j)\tau^{l+1}}, & j = 1, \cdots, k-1, \\ \tau^j, & j = k, \cdots, l. \end{cases}$$

The second step: $\tau \to z$

$$z^{l+1} = \tau^{l+1},\ z^j = \tau^j + p_j(\tau^1, \ldots, \tau^{j-1}, e^{\tau^{l+1}}),\ 1 \le j \le k,$$

$$z^j = \tau^j + \sum_{s=j+1}^{l-m} c_s^j \tau^s, \quad k+1 \le j \le l-k-m,$$

$$z^j = \tau^j + \sum_{s=j+1}^{l} h_s^j \tau^s, \quad l-k-m+1 \le j \le l,$$

where p_j are some weighted homegeoneous polynomials and c_s^j and h_s^j are determined by the following function respectively

$$\cosh\left(\frac{\sqrt{t}}{2}\right)\left(\frac{2\sinh\left(\frac{\sqrt{t}}{2}\right)}{\sqrt{t}}\right)^{2i-1}, \quad \left(\frac{\tanh(\sqrt{t})}{\sqrt{t}}\right)^{2i-1}.$$

The third step: $z \to w$

$$
\begin{aligned}
w^i &= z^i, \quad i = 1, \ldots, k,\ l+1,\\
w^{k+1} &= z^{k+1}(z^{l-m})^{-\frac{1}{2(l-m-k)}},\\
w^s &= z^s(z^{l-m})^{-\frac{s-k}{l-m-k}},\ s = k+2, \cdots, l-m-1,\\
w^{l-m} &= (z^{l-m})^{\frac{1}{2(l-m-k)}},\\
w^{l-m+1} &= z^{l-m+1}(z^l)^{-\frac{1}{2m}},\\
w^r &= z^r(z^l)^{-\frac{r+m-l}{m}},\ r = l-m+2, \cdots, l-1,\\
w^l &= (z^l)^{\frac{1}{2m}}.
\end{aligned}
$$

The last step: $w \to t$

$$
\begin{aligned}
& t^1 = w^1, \ldots, t^k = w^k, \; t^{l+1} = w^{l+1}, \\
& t^{k+1} = w^{k+1} + w^{l-m}\, h_{k+1}(w^{k+2}, \ldots, w^{l-m-1}), \\
& t^j = w^{l-m}(w^j + h_j(w^{j+1}, \ldots, w^{l-m-1})), \; k+2 \le j \le l-m-1, \\
& t^{l-m+1} = w^{l-m+1} + w^l\, h_{l-m+1}(w^{l-m+2}, \ldots, w^{l-1}), \\
& t^s = w^l(w^s + h_s(w^{s+1}, \ldots, w^{l-1})), \; l-m+2 \le s \le l-1 \\
& t^{l-m} = w^{l-m}, \quad t^l = w^l.
\end{aligned}
$$

Here $h_{l-m-1} = h_{l-1} = 0$, h_j are weighted homogeneous polynomials of degree $\frac{k\,(l-m-j)}{l-m-k}$ for $j = k+1, \ldots, l-m-2$ and h_s are weighted homogeneous polynomials of degree $\frac{k\,(l-s)}{m}$ for $s = l-m+2, \ldots, l-1$.

Due to the above construction, we can associate the following natural degrees to the flat coordinates

$$\tilde{d}_j = \deg t^j := \frac{j}{k}, \quad 1 \le j \le k,$$

$$\tilde{d}_s = \deg t^s := \frac{2l - 2m - 2s + 1}{2(l - m - k)}, \quad k + 1 \le s \le l - m,$$

$$\tilde{d}_\alpha = \deg t^\alpha := \frac{2l - 2\alpha + 1}{2m}, \quad l - m + 1 \le \alpha \le l,$$

$$\tilde{d}_{l+1} = \deg t^{l+1} := 0.$$