Volume Conjecture of Reshetikhin-Turaev invariants

<u>Quantum invariants</u> $\quad K$ knot, L link

M 3-mfd

- '85 Jones polynomial $\quad J(K,t) \in \mathbb{Z}[t^{\pm 1}]$
- '89 Witten "invariants" $\quad W_r(M) \in \mathbb{C}$, $r \in \mathbb{N}$
- '90, '91 Reshetikhin – Turaev
  - colored Jones polynomial

    $J_n(K,t) \in \mathbb{Z}[t^{\pm}]$, $\quad n \in \mathbb{Z}$.
  - RT - invariants : M closed, ori.

    $RT_r(M,q) \in \mathbb{C}$, $\quad r \in \mathbb{N}$
- '92 Turaev – Viro invariants

  $TV_r(M,q) \in \mathbb{R}$, $\quad r \in \mathbb{N}$

Volume Conj:

Conj 1: (Kashaev '97, Murakami × 2 '01).

K hyperbolic knot in S^3.

$$\lim_{n\to\infty} \frac{2\pi}{n} \ln \left| J_n(K, e^{\frac{2\pi i}{n}}) \right| = \mathrm{Vol}(S^3 \setminus K),$$

Conj 2. (Chen - Y. '15). M closed hyperbolic.

$$\lim_{\substack{r\to\infty \\ r \text{ odd}}} \frac{4\pi}{r} \ln \left| RT_r(M, e^{\frac{2\pi i}{r}}) \right| = \mathrm{Vol}(M^3).$$

- VC for TV - invariants. for M w/ $\partial M \neq \emptyset$.

- Relative RT - invariants.

$$RT_r(M, L, \vec{n}) \in \mathbb{C},$$

$r \geq 3$, $\vec{n} \in \{0, \cdots, r-2\}^{|L|}$

M closed, ori.

$L \subset M$ framed link.

$$\boxed{q = e^{\frac{2\pi i}{r}}}$$

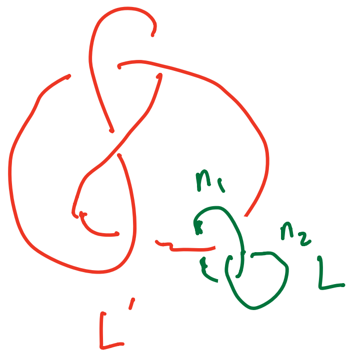
- If $M = S^3$, then

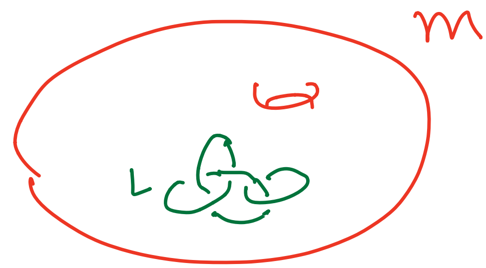
$$RT_r(M, L, \vec{n}) = J_{\vec{n}}(L, e^{\frac{4\pi i}{r}})$$

- If $L = \emptyset$ or $\vec{n} = \vec{0}$, then

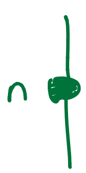
$$RT_r(M, L, \vec{n}) = RT_r(M).$$

Suppose $M = S^3_{L'}$, L' framed link





$$RT_r(M, L, \vec{n}) = c \cdot \Big\langle$$  $$\Big\rangle$$

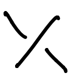


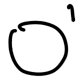
n-th Jones-Wenzl projector

Kirby colouring.

$$\omega = \sum_{n=0}^{r-2} (-1)^n [n+1] \cdot$$  n

Kauffman bracket

(1) $\langle$  $\rangle = q^{\frac{1}{2}} \langle\,)(\,\rangle + q^{-\frac{1}{2}} \langle$  $\rangle$

(2) $\langle$  $' \sqcup L \rangle = (-q - q^{-1}) \cdot \langle L \rangle$

$$\langle \bigcirc^{n} \rangle = (-1)^n [n+1] = (-1)^n \frac{q^n - q^{-n}}{q - q^{-1}}$$

Conj 3 (Wong – Y. '20) (M, L) $r \geq 3$.

$\vec{n}^{(r)} = (n_1^{(r)}, \dots, n_{|L|}^{(r)})$, $n_i^{(r)} \in \{0, \dots, r-2\}$.

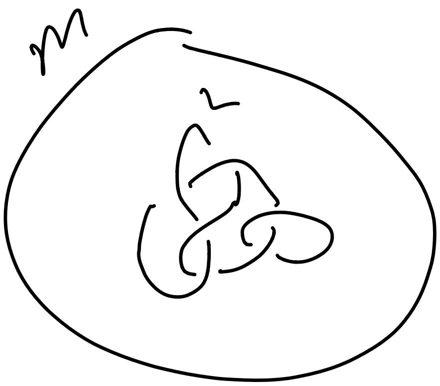
- $\vec{\theta} = (\theta_1, \dots, \theta_{|L|})$, $\theta_i = \left| \lim_{r\to\infty} \frac{4\pi n_i^{(r)}}{r} - 2\pi \right|$
- $M_{L\vec{\theta}}$ = M w/ a <u>hyperbolic cone metric</u> w/ signular locus L & cone angles $\vec{\theta}$

Then

$$\lim_{\substack{n\to\infty \\ \text{odd}}} \frac{4\pi}{r} \ln \left| RT_r(M, L, \vec{n}^{(r)}) \right| = \mathrm{Vol}(M_{L\vec{\theta}})$$

Conj 3 $\Rightarrow$ Conj 1 & 2.

Hyperbolic cone metric.



- metric on M.
- hyperbolic on $M \setminus L$.
- L is a geodesic w/ a cone angle at each comp.

- cone angles $= 2\pi \iff$ hyperbolic metric on M.
- cone angle $= 0 \iff$ hyperbolic metric on $M \setminus L$.

hyp Dehn

- (Hodgson - Kerckhoff '98).

  local rigidit for $\vec{\theta} \leq 2\pi$

- (Kojima '98)

  Global rigidity for $\vec{\theta} \leq \pi$ $\left(\leq 2\pi \right)$ believe.

Thm (Wong – Y. '20). Conj 3 is true for (M, L) s.t $M \setminus L$ is homeomorphic to a fundamental shadow link complement w/ sufficiently small $\vec{\theta}$. (Asym. Exp. Conj).

Rm. By Costantino – Thurston, M in Thm 1 covers all 3-mfds.

- If can push $\vec{\theta}$ from small to $\vec{2\pi}$, then solve Conj 2!! π

Thm 2 (Wong – Y. '20). Conj 3 is true for (M, L) s.t $M \setminus L \cong S^3 \setminus K_{4_1}$, for all $\vec{\theta}$ in $[0, 2\pi]^{|L|}$.

Conj 3

$$\lim_{\substack{r\to\infty \\ \text{odd}}} \frac{4\pi}{r} \ln RT_r(M,L,\vec{n}) = Vol(M_{L_{\vec{\theta}}}) + i\, CS(M_{L_{\vec{\theta}}})$$

$$RT_r(M,L,\vec{n}) = c\cdot \frac{e^{\frac{1}{2}\sum_{i=1}^{|L|} H(\gamma_i)}}{\sqrt{\mathbb{T}(M,m)}}\, e^{\frac{r}{4\pi}\left(Vol(M_{L_{\vec{\theta}}}) + i\, CS(M_{L_{\vec{\theta}}})\right)} \left(1 + O\left(\frac{1}{r}\right)\right)$$