Stability Condition of Parahoric Higgs Bundle

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Goal of this talk: To explain the stability condition of parahoric Higgs bundles defined in paper "Tame parahoric Higgs bundles for a complex reductive group".

1. Ramanathan Stability
2. Parabolic Higgs Bundle
3. Tame Parahoric Higgs Bundle

Stability Condition

The notion of stability comes from geometric invariant theory for construction of moduli spaces. In the cases of vector bundles, it is initiated by the following:

1. Slope stability, the most common one for curves.
2. μ-stability, namely, the slope is calculated by $c_1(V) \cdot H^{n-1}/\mathrm{rank}$, for some polarization H, suitable for vector bundles in higher dimension.
3. Bridgeland stability, by giving a *central charge* and a subcategory called *slicing*, one of the most general cases.

The cases for vector bundles can be generalized to "decorated" bundles: sheaves, principal bundles, Higgs bundles, where we will have many ways to do that. These are essential in the construction of moduli spaces.

Slope Stability Condition

A slope of a holomorphic vector bundle W over a nonsingular algebraic curve is a rational number $\mu(W) = \frac{\deg W}{\mathrm{rk}(W)}$.

Stability Condition for curves

A bundle W is stable if and only if for any all proper non-zero subbundles V of W we have

$$\mu(V) < \mu(W).$$

The generalization of this idea to G-principal bundle is done by Ramanathan, where when we set $G = GL_n$ we will recover this definition.

Parabolic subgroup

Let G be a connected reductive algebraic group. The parabolic subgroup P is the subgroup between Borel subgroup and G. It's also characterized by G/P is compact variety called "flag variety". Example: $G = \mathrm{GL}_4$, then its Borel subgroup B is upper triangular matrices,

$$\left\{ A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a_{22} & a_{23} & a_{24} \\ 0 & 0 & a_{33} & a_{34} \\ 0 & 0 & 0 & a_{44} \end{bmatrix} : \det(A) \neq 0 \right\}$$

and we have three maximal parabolic subgroups containing B:

$$\left\{ \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a_{22} & a_{23} & a_{24} \\ 0 & a_{32} & a_{33} & a_{34} \\ 0 & a_{42} & a_{43} & a_{44} \end{bmatrix} \right\}, \left\{ \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ 0 & 0 & a_{33} & a_{34} \\ 0 & 0 & a_{43} & a_{44} \end{bmatrix} \right\}, \left\{ \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ 0 & 0 & 0 & a_{44} \end{bmatrix} \right\}$$

Parabolic Reduction

Let G be the same as before, and let E be a G-principal bundle. Let P be a parabolic subgroup of G, then we define a *parabolic reduction* σ to be a holomorphic section of E/P.

Anti-dominant characters

A character $\chi : P \to \mathbb{C}^*$ is called *anti-dominant* if the line bundle $G \times_\chi \mathbb{C} \to G/P$ is ample.

Given a parabolic reduction, one has the P-principle bundle $\sigma^* E$ by pulling back E from the natural projection $E \to E/P$. Also, for any anti-dominant character $\chi : P \to \mathbb{C}^*$, we can defined the line bundle $E(\sigma, \chi) = \sigma^*(E \times_\chi \mathbb{C})$ if we see E as a P-bundle over E/P.

Parabolic Reduction, degree

We may want to calculate the degree of $E(\sigma, \chi)$ defined by a reduction and character.

Let's consider the example $G = \mathrm{GL}_4$ and P the matrix group $\left\{ \begin{bmatrix} A & B \\ 0 & C \end{bmatrix} \right\}$ where each of A, B, C is 2×2 matrix. Then we consider the character $\chi(p) = \det(A)$ for any $p \in P$ and it's anti-dominant.

For any $G-$bundle E, we have the associated rank 4 vector bundle $E(V)$. For any parabolic reduction $\sigma : X \to E/P$, one can associate a rank 2 subbundle W of $E(\mathbb{C}^4)$. And the degree

$$\deg E(\sigma, \chi) = \deg W.$$

Parabolic Reduction, GL_n

In the case of $G = \mathrm{GL}_n$, any parabolic reduction will correspond to a choice of flag of subbundles $V_1 \subset V_2 \subset \ldots \subset V_r = E(\mathbb{C}^n)$, an anti-dominant character will correspond to a choice of $\lambda_1 < \lambda_2 < \ldots < \lambda_r$ where it corresponds to the power of determinant of the matrix. Then

$$\deg E(\sigma, \chi) = \lambda_n \deg V_r + \sum_{i=1}^{r-1} (\lambda_i - \lambda_{i+1}) \deg V_i$$

Ramanathan Stability Condition

Ramanathan defined a stability condition in his paper "Stable principal bundles on a compact Riemann surface":

Ramanathan Stability

A principal G-bundle E with on X is called *R-stable* if for any parabolic reduction $\sigma : X \to E/P$. P is a **maximal** parabolic subgroup of G, we have

$$\deg \sigma^* T(G/P) > 0$$

where $T(G/P)$ denotes the tangent bundle along fiber of E/P.

Ramanathan Stability Condition, Second Definition

Ramanathan Stability

A principal G-bundle E with on X is called *R-stable* if for any parabolic reduction $\sigma : X \to E/P$ for any parabolic subgroup and any anti-dominant character $\chi : P \to \mathbb{C}^*$ **that is trivial on the center of** P, we have

$$\deg E(\sigma, \chi) > 0.$$

Quick Proof of Equivalence:
"$\leftarrow$:" $\sigma^* T(G/P) = E(\sigma, \mu)$, where μ is the determinant of action of P on $\mathfrak{g}/\mathfrak{p}$.
"$\rightarrow$:" Any antidominant character on maximal parabolic subgroup is a multiple of determinant action. And any non-maximal parabolic subgroup are determined by a combination of simple roots, which will goes a combination of degrees associated to maximal parabolic subgroup.

Ramanathan Stability Condition, applied to example

Let's consider the example $G = \mathrm{GL}_n$ and P the matrix group $\left\{ \begin{bmatrix} A & B \\ 0 & C \end{bmatrix} \right\}$ where A is $m \times m$ matrix, C is $(n-m) \times (n-m)$-matrix.
Then we consider the character $\chi(p) = \det(A)^{(n-m)} \det(C)^{-m}$ so that it's trivial on the center $Z = \left\{ \begin{bmatrix} \lambda I & 0 \\ 0 & \lambda I \end{bmatrix} \right\}$. Then the degree is by our general calculation

$$\deg E(\sigma, \chi) = (n-m) \cdot \deg E(V) + (-m - (n-m)) \cdot \deg W > 0$$

The condition implies

$$\deg E(V)/n > \deg W/(n-m)$$

for any subbundle W of rank $n - m$.

Ramanathan Stability Condition, Third Definition

Ramanathan Stability

A principal G-bundle E with on X is called *R-stable* if for any parabolic reduction $\sigma : X \to E/P$ for any parabolic subgroup and any anti-dominant character $\chi : P \to \mathbb{C}^*$, we have

$$\deg E(\sigma, \chi) - \langle \alpha, \chi \rangle > 0.$$

Here α is an element in $\mathfrak{g}$ determined by the topology of E, where we abuse the notation name χas its derivative and the $\langle -, - \rangle$ is the Killing form.

Remark: if $G = \mathrm{GL}_n$ we have $\alpha = \frac{\deg E(V)}{n} \cdot Id_{n\times n}$, where this is exactly the slope.

Parabolic Vector Bundles

All the previous ones are defined for compact Riemann surface, we would like to analyze open Riemann surface with points inside. Then the bundle structure will goes to the parabolic bundle structure, if we only want logarithmic singularity.

Parabolic Vector Bundles

A *parabolic vector bundle* with parabolic divisor D is a rank n bundle F together with a filtration of $F|_p$ for each $p \in D$ by

$$F|_p = F_1(p) \supset F_2(p) \supset \ldots \supset F_n(p) \supset F_{n+1}(p) = 0$$

together with a series of numbers called *weight* of the filtration

$$\alpha_1(p) \leq \alpha_2(p) \leq \ldots \leq \alpha_n(p).$$

Parabolic Degree

The *parabolic degree* of a parabolic vector bundle F is given by

$$\mathrm{pardeg}(F) = \deg(F) + \sum_{p \in D} \sum_{i=1}^{n} \alpha_i(p).$$

We call a parabolic vector bundle F stable if for all parabolic subbundle F', we have

$$\frac{\mathrm{pardeg}(F')}{\mathrm{rk}(F')} < \frac{\mathrm{pardeg}(F)}{\mathrm{rk}(F)}.$$

We would like to see its counterpart in parabolic principal bundles.

Parabolic Principal Bundles

Now we consider a G-principal bundle E and a collections of point $(x_i)_{i=1}^n$ on the curve X. Then we have the following:

Parabolic structure

A *parabolic structure* of weight α_i on E over a point x_i is defined as the choice of a parabolic subgroup $Q_i \subset E\left(H^{\mathbb{C}}\right)_{x_i}$ with an antidominant character α_i for Q_i.

We call the pair $(E, (Q_i, \alpha_i)_{i=1}^n)$ a **parabolic G-principal bundle**

Parabolic degree associated to Parabolic Reduction

For any parabolic reduction $\sigma : X \to E/P$, one can also define a parabolic degree:

$$\mathrm{pardeg}\, E(\sigma,\chi) = \deg E(\sigma,\chi) + \deg\left((P,\chi),(Q_i,\alpha_i)\right),$$

where $\deg\left((P,\chi),(Q_i,\alpha_i)\right)$ is the **relative degree** defined by Biquard-Garcia-Prada-Mundet i Riera which is really complicated.

The case of GL_n

A decreasing sequence of weights

$$\alpha_i^r > \alpha_i^{r-1} > \ldots > \alpha_i^1,$$

so that the increasing filtration is then

$$E_r \subset E_{r-1} \subset \ldots E_1 = E_{x_i}.$$

determines the parabolic structure (Q_i, α_i). Also for the parabolic subgroup P and $\chi : P \to \mathbb{C}^*$ we have a choice of increasing filtration $\lambda_1 < \lambda_2 < \ldots < \lambda_s$ and $V_1 \subset \ldots \subset V_s$, then the relative degree is calculated to be

$$\deg((P,\chi),(Q_i,\alpha_i)) = \sum_{i=1}^{s}\sum_{j=1}^{r}(\lambda_i - \lambda_{i+1})(\alpha_i^j - \alpha_i^{j-1})\dim(V_i \cap E_j)$$

Ramanathan Stability

Ramanathan Stability

A parabolic principal G-bundle E with on X is called *R-stable* if for any parabolic reduction $\sigma : X \to E/P$ for any parabolic subgroup and any anti-dominant character $\chi : P \to \mathbb{C}^*$ **that is trivial on the center of** P, we have

$$\text{pardeg}\, E(\sigma, \chi) > 0.$$

In the case of GL_n, this reduces to the case

$$\lambda_r(\text{pardeg}\, \mathcal{W} - \mu \mathrm{rk}(\mathcal{W})) + \sum_{k=1}^{r-1} (\lambda_k - \lambda_{k+1})(\text{pardeg}\, \mathcal{W}_k - \mu \mathrm{rk}(\mathcal{W}_k)) \geq 0.$$

where $\mathcal{W}_1 \subset \ldots \subset \mathcal{W}_r = E(V)$ the associated vector bundles and $\mu = \frac{\text{pardeg}\, \mathcal{W}}{n}$ the slope for parabolic vector bundle.

Incoporating with Higgs Field

A Higgs bundle on curve X is a pair (E, ϕ) with a vector bundle E and ϕ a section of $End(E) \otimes \Omega^1_X$.
It is called *stable* if for any sub-vector bundle $F \subset E$ that is preserved by ϕ, namely, $\phi(F) \subset F \otimes \Omega^1_X$, one has $\deg F/\mathrm{rk}(F) < \deg E/\mathrm{rk}(E)$.

Natural Question

How to define Higgs field for principal bundle, and what does the stability look like.

Incoporating with Higgs Field

A G-parabolic Higgs bundle is a parabolic principal bundle $(E, (Q_i, \alpha_i))$ and a section of $PE(\mathfrak{g}) \otimes K(D)$, where PE is the *sheaf of parabolic endomorphism* determined by the parabolic structure at the points.

$G = \mathrm{GL}_n$

$\mathbb{C}^n = F_1 \supset F_2 \supset \ldots \supset F_n$ the parabolic Higgs field is such that $\Phi(F_i) \subset F_i \otimes K(D)$, when we restrict to the point.

More details in the paper of Biquard-Garcia-Prada-Mundet i Riera. The stability condition now has an extra condition: the parabolic reduction σ is not arbitrary: Higgs field ϕ should lie in a specific sheaf $E(\mathfrak{g})^{-}_{\sigma,\chi} \otimes K(D)$.

Parahoric Groups

We would like to generalize all these above settings to parahoric Higgs bundles, when higher singularity is allowed instead of the logarithmic singularity for the parabolic case.
Let G be a connected complex reductive group. We fix a maximal torus T in G. For every root $r \in R$ there is a root homomorphism $u_r : \mathbb{G}_a \to G$. We denote the image of this morphism U_r.
Let $A := \mathbb{C}[[z]]$ and $K := \mathbb{C}((z))$.
Fix a co-character with coefficients in $\mathbb{Q}$, $\theta \in Y(T) \otimes_{\mathbb{Z}} \mathbb{Q}$, which we shall call a *weight*. Under differentiation, we can consider θ as an element in $\mathfrak{t}$, which is the Lie algebra of T. Define the integer $m_r(\theta) := \lceil -r(\theta) \rceil$, where $\lceil \cdot \rceil$ is the ceiling function and $r(\theta) := \langle \theta, r \rangle$.

Parahoric Group Scheme

We define the *parahoric subgroup* G_θ of $G(K)$ as

$$G_\theta := \langle T(A), U_r(z^{m_r(\theta)}A), r \in R \rangle.$$

Denote by $\mathcal{G}_\theta$ the corresponding group scheme of G_θ, which is called the *parahoric group scheme*.
Remark: An equivalent analytic definition of the parahoric group G_θ as

$$G_\theta := \{g \in G(K) \mid z^\theta g z^{-\theta} \text{ has a limit as } z \to 0 \text{ along any ray}\},$$

Parahoric Torsor

We define a group scheme $\mathcal{G}_{\boldsymbol{\theta}}$ over X by gluing the following local data

$$\mathcal{G}_{\boldsymbol{\theta}}|_{X \backslash D} \cong G \times (X \backslash D), \quad \mathcal{G}_{\boldsymbol{\theta}}|_{\mathbb{D}_x} \cong \mathcal{G}_{\theta_x}, x \in D,$$

Parahoric Subgroup

Recall that a parabolic subgroup P of G can be determined by a subset of roots $R_P \subseteq R$. We define the following parahoric group as a subgroup of $P(K)$

$$P_\theta := \langle T(A), U_r(z^{m_r(\theta)}A), r \in R_P \rangle.$$

Denote by $\mathcal{P}_\theta$ the corresponding group scheme over $\mathbb{D} = \mathrm{Spec}(A)$. Now we consider the global picture. Let X be a smooth algebraic curve with reduced effective divisor D. Let $\boldsymbol{\theta} = \{\theta_x, x \in D\}$ be a collection of rational weights and define the group scheme $\mathcal{P}_{\boldsymbol{\theta}}$ over X by gluing the local data

$$\mathcal{P}_{\boldsymbol{\theta}}|_{\mathbb{D}_x} \cong P \times X \backslash D, \quad \mathcal{P}_{\boldsymbol{\theta}}|_{\mathbb{D}_x} \cong \mathcal{P}_{\theta_x}, x \in D.$$

Parahoric Reduction

let E be a $\mathcal{G}_{\boldsymbol{\theta}}$-torsor. Given a character κ, we can define a line bundle over $E/\mathcal{P}_{\boldsymbol{\theta}}$, and we use the same notation $L^{\kappa}_{\boldsymbol{\theta}}$ for this line bundle. Let $\varsigma : X \to E/\mathcal{P}_{\boldsymbol{\theta}}$ be a reduction of the structure group. A line bundle $L^{\kappa}_{\boldsymbol{\theta}}(\varsigma) := \varsigma^* L^{\kappa}_{\boldsymbol{\theta}}$ over X can be then defined by pullback in the following diagram

$$\begin{tikzcd} \varsigma^* L^{\kappa}_{\boldsymbol{\theta}} \arrow[r, dotted] \arrow[d, dotted] & L^{\kappa}_{\boldsymbol{\theta}} \arrow[d] \\ X \arrow[r, "\varsigma"] & E/\mathcal{P}_{\boldsymbol{\theta}} \end{tikzcd}$$

We use the notation $L^{\kappa}_{\boldsymbol{\theta}}$ for the line bundle over X.

Parahoric Degree, Stability

We define the *parahoric degree* of a $\mathcal{G}_{\boldsymbol{\theta}}$-torsor E with respect to a given reduction ς and a character κ as follows

$$\mathit{parh}\deg E(\varsigma, \kappa) = \deg(L^{\kappa}_{\boldsymbol{\theta}}) + \langle \boldsymbol{\theta}, \kappa \rangle,$$

where $\langle \boldsymbol{\theta}, \kappa \rangle := \sum_{x \in D} \langle \theta_x, \kappa \rangle$.

R-stablity of Parahoric torsor

A parahoric $\mathcal{G}_{\boldsymbol{\theta}}$-torsor E is called *R-stable* (resp. *R-semistable*), if for

- any proper parabolic group $P \subseteq G$,
- any reduction of structure group $\varsigma : X \to E/\mathcal{P}_{\boldsymbol{\theta}}$,
- any nontrivial anti-dominant character $\chi : \mathcal{P}_{\boldsymbol{\theta}} \to \mathbb{G}_m$ trivial on the center of $\mathcal{P}_{\boldsymbol{\theta}}$, we have

$$\mathit{parh}\deg E(\varsigma, \chi) > 0.$$