Gromov-Hausdorff Limit of Manifolds and Some Applications

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April 30, 2021

Outline

Sectional Curvature and Topology

Let (M^n, g) be a closed manifold with $n \geq 2$

- $Sec_M > 0$: $\pi_1(M)$ is finite.
- $Sec_M \equiv 0$: $\pi_1(M)$ is infinite.
- $Sec_M < 0$: $\pi_1(M)$ is infinite.

No intersection of $\{M : Sec_M > 0\}$ and $\{M : Sec_M < 0\}$.

Negative Ricci Curvature and Scalar Curvature

No topological obstruction for negative Ricci and negative Scalar:

- Aubin 1970: For each closed manifold M^n with $n \geq 3$, $\exists$ a complete g such that $R_g \equiv -1$.
- Lohkamp 1994: For each closed manifold M^n with $n \geq 3$, $\exists$ a complete g such that $-a(n)g \leq \mathrm{Ric}_g \leq -b(n)g$.

Nonnegative Ricci Curvature

- Milnor 1968: Each finitely generated subgroup of $\pi_1(M)$ has polynomial growth. (Conjecturally, $\pi_1(M)$ is finitely generated.)
- Cheeger-Gromoll 1971: If (M^n, g) is compact then $b_1(M) \leq n$ and $b_1(M) = n$ iff (M^n, g) is a flat torus.
- Cheeger-Gromoll 1971: Let (M^n, g) be complete then M^n splits isometrically as $M^n \cong N^{n-1} \times \mathbb{R}$ if there exists a geodesic line on M^n.
- Yau 1976: Each complete manifold has infinite volume.

Gromov-Hausdorff distance

- (Gromov-Hausdorff distance) Let (X, d_1) and (Y, d_2) be two compact metric spaces, the Gromov-Hausdorff metric d_{GH} defines as follow

$$d_{GH}\Big((X, d_1), (Y, d_2)\Big) := \inf_{(Z,d)} \inf\{\epsilon : \;\; X \subset B_\epsilon(Y), \; Y \subset B_\epsilon(X)\}$$

where $(X, d_1), (Y, d_2) \hookrightarrow (Z, d)$ (*isometric embedding*).

- If $\lim_{i\to\infty} d_{GH}\Big((X_i, d_i), (X, d)\Big) = 0$, then we say $(X_i, d_i) \xrightarrow{d_{GH}} (X, d)$.

- Can define Pointed Gromov-Hausdorff convergence for noncompact metric spaces.

Examples

- Collapsing: $(\mathbb{T}^2, g_r) = \mathbb{S}^1 \times \mathbb{S}^1_r$, if $r \to 0$, then $(\mathbb{T}^2, g_r) \xrightarrow{d_{GH}} \mathbb{S}^1$.
- Tangent cone: Let (M^n, g) be smooth. Blowing up at $x \in M$ we get GH-limit $\mathbb{R}^n$.
- Singular Limit: Blowing down of Eguchi-Hanson metrics, we get GH-limit $\mathbb{R}^4/\mathbb{Z}_2$.

Gromov's precompactness

Theorem 1 (Gromov, 1981)

The space $\mathcal{M}(n,\Lambda,D) = \{(M^n,g) : \mathrm{Ric} \geq -(n-1)\Lambda g, \mathrm{diam}(M,g) \leq D\}$ *is precompact under Gromov-Hausdorff topology. i.e., any sequence* $(M_i,g_i) \in \mathcal{M}(n,\Lambda,D)$ *has a subsequence and a metric space* (X,d) *such that*

$$(M^n_{i'}, g_{i'}) \xrightarrow{d_{GH}} (X,d).$$

- What can be said about the convergence and the structure of X?
- Roughly, "GH-convergence" is "L^∞-convergence".

VS L^∞-convergence

Let $f_i \in C^\infty(\mathbb{R}^n)$ and $f_i \to f$ in L^∞ norm on $B_1(0^n)$. Regularity of f? Improve the convergence?

- Improve the convergence: If $|\nabla^2 f_i| \leq C$ uniformly, then $f_i \to f$ in $C^{1,\alpha}$-sense for any $0 < \alpha < 1$ and $f \in C^{1,\alpha}$. (By Arzela-Ascoli theorem)
- Improve the convergence: If $\Delta f_i = 0$ then $f_i \to f$ in smooth sense and f is smooth.
- Improve the convergence: If $|\Delta f_i| \leq C$ uniformly, then $f_i \to f$ in $C^{1,\alpha}$-sense for any $0 < \alpha < 1$ and $f \in C^{1,\alpha}$.
- Super-harmonic: If $\Delta f_i \geq 0$, Can not improve the regularity.

Improved GH-convergence

Let $(M_i^n, g_i) \xrightarrow{d_{GH}} (X, d)$ satisfy $\mathrm{Ric}_{g_i} \geq -(n-1)g_i$ and $\mathrm{diam}(M_i, g_i) \leq D$.

Note that $\mathrm{Rm} \approx \nabla^2 g_{ij}$ and $\mathrm{Ric}_{ij} \approx \Delta g_{ij}$

- Cheeger-Gromov: If $|\mathrm{Rm}_{g_i}| \leq \Lambda$ and $\mathrm{Vol}(M_i, g_i) \geq V > 0$, then d_{GH}-convergence is $C^{1,\alpha}$-convergence for any $0 < \alpha < 1$ and X is smooth.

- Anderson-Cheeger-Colding: If $|\mathrm{Ric}_{g_i}| \leq \Lambda$ and X is a smooth n-manifold, then d_{GH}-convergence is $C^{1,\alpha}$-convergence for any $0 < \alpha < 1$.

- For $\mathrm{Ric} \geq -(n-1)g$: Cannot improve the convergence. What can we say about the limit?

Ricci Limit Space: Cheeger-Colding

Noncollapsing: Let $(M_i^n, g_i, p_i) \xrightarrow{d_{GH}} (X, d, p)$ satisfy $\mathrm{Ric}_i \geq -(n-1)g_i$ and $\mathrm{Vol}(B_1(p_i)) \geq \mathrm{v} > 0$.

- Volume Convergence: $\mathrm{Vol}(M_i^n) \to \mathrm{Vol}(X)$.
- Tangent cones are metric cones, i.e., $Y = C(Z) = Z \times [0, \infty)/Z \times \{0\}$ for some metric space Z.
- Splitting: If $\mathrm{Ric}_i \geq -\lambda_i g_i \to 0$ and X contains one geodesic line, then X splits off a factor $\mathbb{R} \times X_1$.

Regular-Singular Decomposition: Cheeger-Colding

Let $(M_i^n, g_i, p_i) \xrightarrow{d_{GH}} (X, d, p)$ satisfy $\mathrm{Ric}_i \geq -(n-1)g_i$ and $\mathrm{Vol}(B_1(p_i)) \geq$ v > 0.

- Regular-Singular Decomposition: $X = \mathcal{R} \cup \mathcal{S}$.
- Regular set $\mathcal{R}$: Tangent cone is $\mathbb{R}^n$.
- Local Regularity of $\mathcal{R}$: For any $x \in \mathcal{R}$, exists r_x such that $B_{r_x}(x)$ is bi-Hölder to $\mathbb{R}^n$.(Conjecturally, locally bi-Lipschitz)
- Singular set $\mathcal{S}$: $\mathcal{S} := X \setminus \mathcal{R}$.
- Structure of Singular set: $\dim \mathcal{S} \leq n-2$. ($\mathcal{S}$ may be not closed and may be dense in X)

Regularity on Bounded Ricci curvature-Limit Space I

Let $(M_i^n, g_i, p_i) \xrightarrow{d_{GH}} (X, d, p)$ satisfy $\mathrm{Vol}(B_1(p_i)) \geq \mathrm{v} > 0$ and $|\mathrm{Ric}| \leq n-1$. Then:

- Anderson-Cheeger-Colding 1997: $\mathcal{S}$ is closed
- Colding-Naber 2012: $\mathcal{R}$ is convex
- Anderson, Bando-Kasue-Nakajima, Tian, 1989: If $\int_{B_1(p_i)} |\mathrm{Rm}|^{n/2} \leq \Lambda$ then $\mathcal{S} \cap B_1(p)$ is a finite set
- Cheeger-Colding-Tian 2002, Cheeger 2003, Chen-Donaldson 2014: If $\int_{B_1(p_i)} |\mathrm{Rm}|^q \leq \Lambda$ for $q < n/2$ then $H^{n-2q}(\mathcal{S} \cap B_1(p)) \leq C(n, \mathrm{v}, q, \Lambda)$

Regularity on Bounded Ricci curvature-Limit Space II

Let $(M_i^n, g_i, p_i) \xrightarrow{d_{GH}} (X, d, p)$ satisfy $\mathrm{Vol}(B_1(p_i)) \geq \mathrm{v} > 0$ and $|\mathrm{Ric}| \leq n-1$. Then:

- $|\mathrm{Ric}| \leq \Lambda$ implies $|\mathrm{Rm}| \leq C(\Lambda)$ for $n = 3$, then $\mathcal{S} = \emptyset$.
- Codimension four conjecture: $\dim \mathcal{S} \leq n - 4$ solved by Tian, Cheeger for Kähler case, general case by Cheeger-Naber 2015.
- Finite measure conjecture: $H^{n-4}(B_1(p) \cap \mathcal{S}) \leq C(n, \mathrm{v})$ solved by Jiang-Naber 2021.

Regularity on Bounded Ricci Curvature-Manifold

Let (M^n, g, p) be pointed Riemannian manifold with $|\mathrm{Ric}| \leq n-1$ and $\mathrm{Vol}(B_1(p)) \geq \mathrm{v} > 0$.

- Cheeger-Naber 2015: $⨍_{B_1(p)} |\mathrm{Rm}|^{2-\epsilon} \leq C_\epsilon(n, \mathrm{v})$ for any $n \geq 5$, $0 < \epsilon < 1$ and $⨍_{B_1(p)} |\mathrm{Rm}|^2 \leq C(\mathrm{v})$ for $n = 4$ based on Chern-Gauss-Bonnet formula.

- L^2-Conjecture: $⨍_{B_1(p)} |\mathrm{Rm}|^2 \leq C(n, \mathrm{v})$ solved by Jiang-Naber 2021.

Remarks:

- L^2-estimate is sharp.

- Chern-Weil theory: $\int_M |\mathrm{Rm}|^2 \leq C$ for Kähler manifolds with topological restrictions.

- In the cases of Harmonic map and minimal hypersurface, the best is a weak L^2, no strong L^2.

Singular Set on Lower Ricci Curvature

Let $(M_i^n, g_i, p_i) \xrightarrow{d_{GH}} (X, d, p)$ satisfy $\text{Ric}_i \geq -(n-1)g_i$ and $\text{Vol}(B_1(p_i)) \geq v > 0$.

- Cheeger-Colding 1997: $\dim \mathcal{S} \leq n-2$.
- $H^{n-2}(\mathcal{S})$ could be infinite.
- Cheeger-Naber's Quantitative Estimate 2013: Decomposition $\mathcal{S} = \cup_{\epsilon>0}\mathcal{S}_\epsilon$ such that $\text{Vol}(B_r(\mathcal{S}_\epsilon) \cap B_1(p)) \leq C(n, \text{v}, \epsilon, \eta) r^{2-\eta}$ for any $0 < \eta, \epsilon, r < 1$.

Structure on Lower Ricci Curvature

Let $(M_i^n, g_i, p_i) \xrightarrow{d_{GH}} (X, d, p)$ satisfy $\mathrm{Vol}(B_1(p_i)) \geq \mathrm{v} > 0$ and $\mathrm{Ric}_{g_i} \geq -(n-1)g_i$.

Theorem 2 (Cheeger-Jiang-Naber 2021)

For any $0 < \epsilon < 1$, there exists $\mathcal{S}_\epsilon \subset \mathcal{S}$ such that

- *Decomposition: $X = \mathcal{S}_\epsilon \cup \mathcal{R}_\epsilon$.*
- $\mathrm{Vol}(B_r(\mathcal{S}_\epsilon) \cap B_1(p)) \leq C(\epsilon, n, \mathrm{v}) r^2$ *for any* $0 < \epsilon, r < 1$.
- $H^{n-2}(\mathcal{S}_\epsilon \cap B_1(p)) \leq C(\epsilon, n, \mathrm{v})$ *and $\mathcal{S}_\epsilon$ is $(n-2)$-rectifiable.*
- *$\mathcal{R}_\epsilon$ is $(1-\epsilon)$-bi-Hölder to a smooth n-manifold.*
- *$\mathcal{S}$ is $(n-2)$-rectifiable.*

Remark 1

Recently, for limit space of polarized Kähler manifolds, Liu-Szekelyhidi proved the singular sets are given by a countable union of analytic subvarieties.

Finite diffeomorphism of Ricci flow

Let $(M^n, g_t)_{t\in[1,2]}$ be Ricci flow on closed manifold with bounded scalar curvature $|R| \le \Lambda$ and lower ν-entropy $\nu[g_1, 2] \ge -\Lambda$. Then

Theorem 3 (Jiang 2021)

If $n = 4$, the manifold M has at most $C(\Lambda)$ many diffeomorphism types.

Let (M^n, g) satisfy $\mathrm{Ric} \ge -(n-1)\Lambda$, $\mathrm{Vol}(M) \ge V > 0$, $\mathrm{diam}(M, g) \le D$. Consider Ricci flow (M^n, g_t) starting at $g_0 = g$.

- Jiang 2016: If M is Fano, then $|R_{g_t}| \le C(n, \Lambda, V, D, t^{-1})$ for $0 < t \le 1$.
- Simon-Topping 2017: If $n = 3$, $|R_{g_t}| \le C(\Lambda, V, D, t^{-1})$ for $0 < t \le 1$.
- Bamler-Cabezas Rivas-Wilking 2019: If curvature operator $\mathrm{Rm} \ge -1$, then $|R_{g_t}| \le C(n, \Lambda, V, D, t^{-1})$ for $0 < t \le 1$.
- Jiang 2021: There is no uniform $|R_{g_t}| \le C(n, \Lambda, V, D, t^{-1})$ for $n \ge 4$.(If the flow exists for a uniform time.)

Nodal Set with Lower Ricci Curvature

Yau's conjecture: Let (M^n, g) be closed manifold and u be a non-constant eigenfunction: $-\Delta u = \lambda u$, then $0 < C_0(M, g)\sqrt{\lambda} \leq \mathcal{H}^{n-1}(\{u = 0\}) \leq C_1(M, g)\sqrt{\lambda}$.(lower bound solved by Logunov 2018)

- Question: Uniform estimate for $C_0(M, g)$ and $C_1(M, g)$?
- Chu-Ge-Jiang 2020: $C_0(M, g) = C(\Lambda, V, D)$ where $|\mathrm{Ric}| \leq \Lambda$, $\mathrm{Vol}(M, g) \geq V > 0$ and $\mathrm{diam}(M, g) \leq D$.

Theorem 4 (Chu-Ge-Jiang 2020)

Let (M^n, g, p) *be a manifold with* $\mathrm{Ric} \geq -(n-1)$ *and* $\mathrm{Vol}(B_1(p)) \geq \mathrm{v} > 0$. *Let* $\Delta u = 0$ *and* $D_u(p, 2) := \sup_{B_r(x)\subset B_2(p)} \frac{\sup_{B_{2r}(x)}|u|}{\sup_{B_r(x)}|u|} \leq \Lambda$. *Then for any* $0 < r \leq 1$,

1. $\mathrm{Vol}(B_r(Z_u) \cap B_1(p)) \leq C(n, \mathrm{v}, \Lambda)r$, *where* $Z_u = \{u = 0\}$
2. *If* $u(p) = 0$, *then* $\mathcal{H}^{n-1}(Z_u \cap B_1(p)) \geq C(n, \mathrm{v}, \Lambda)$.

- Remark: No apriori $C^{1,\alpha}$-estimate for harmonic function on manifold with Ricci curvature bounds.

Positive Mass for Singular Metrics

Let (M^n, g, Σ) be asymptotic flat manifold with $g \in C^\infty$ away from a closed bounded subset Σ and $R_g \geq 0$ on $M \setminus \Sigma$.

- Question: What are the conditions of g and Σ such that the positive mass theorem still hold.
- Progress by Miao, Shi-Tam, McFeron-Szekelyhidi, Li-Mantoulidis, Lee,...

Theorem 5 (Jiang-Sheng-Zhang, 2021)

If g is locally Lipschitz and $\mathcal{H}^{n-1}(\Sigma) = 0$, the positive mass theorem still hold.

Remarks:

- The result is sharp, which confirms a conjecture of Dan Lee.
- The result improves Shi-Tam, Lee's results and generalize Lee-Lefloch from Spin to non-Spin.
- The rigidity part involves RCD theory.

Thank you for your attention!