Introductory talk: Legendrian knots and constructible sheaves.

SU, Tao . YMSC, 2021.10.12.

Goal: some background for the main talk.

- microsupport of (constructible) sheaves
- constructible sheaves from Legendrian knots.
- microlocal monodromy
- (— Irregular Riemann-Hilbert over curves).

Main ref: Shende-Treumann-Zaslow, Legendrian knots and constructible sheaves.
2016.

1. Microsupport

- M = smooth manifold. of dim n
- k = base field (more generally, comm. ring)
- $\mathcal{F}$ = constructible sheaf on M.

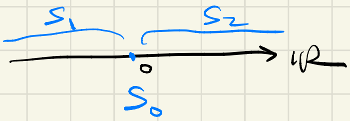
I.e.: $\mathcal{F}$ is a cochain complex of sheaves of k-modules s.t.

- the cohomology sheaf $\mathcal{H}^i(\mathcal{F})$ is bdd ($\mathcal{H}^i(\mathcal{F})=0$, for $|i|\gg 0$)
- $\exists$ (nice) stratification $\mathcal{S}=\{S_\alpha\}$ of M s.t.
  $\mathcal{H}^i(\mathcal{F})|_{S_\alpha}$ is locally constant (with perfect stalks), $\forall\alpha$

  ($\forall S_\alpha \Rightarrow \overline{S_\alpha} = \bigcup_{\text{some}} S_\beta$).

E.g. 1: $\mathcal{F} = k_M$ $\quad (\mathcal{S} = \{M\})$

E.g. 2: $M =$



$$= \underset{\underset{(-\infty,0)}{\parallel}}{S_1} \sqcup \underset{\underset{\{0\}}{\parallel}}{S_0} \sqcup \underset{\underset{(0,+\infty)}{\parallel}}{S_2} \qquad (\text{stratification}) \; \mathcal{S}$$

$$\mathcal{F}_1 = k_{(-\infty,0]} = i_! k, \text{ where } i : (-\infty, 0] \hookrightarrow M = \mathbb{R}$$

$\Longrightarrow$

$$\begin{tikzcd} (\mathcal{F}_1)_{-\varepsilon} \arrow[d, equal] & (\mathcal{F}_1)_0 \arrow[l, "1"'] \arrow[r] \arrow[d, equal] & (\mathcal{F}_1)_\varepsilon \arrow[d, equal] \\ k & k & 0 \end{tikzcd} \qquad (\varepsilon > 0)$$

$$\mathcal{F}_2 = k_{(-\infty,0)} = j_! k, \quad j : (-\infty, 0) \hookrightarrow M = \mathbb{R}.$$

$\Longrightarrow$

$$\begin{tikzcd} (\mathcal{F}_2)_{-\varepsilon} \arrow[d, equal] & (\mathcal{F}_2)_0 \arrow[l, "0"'] \arrow[r, "0"] \arrow[d, equal] & (\mathcal{F}_2)_\varepsilon \arrow[d, equal] \\ k & 0 & 0 \end{tikzcd}$$

Slogan: "Microsupport captures the codirections along which the local sections fail to propagate"

Let $x \in M$, $\xi \in T_x^* M$, say $\xi \neq 0$.

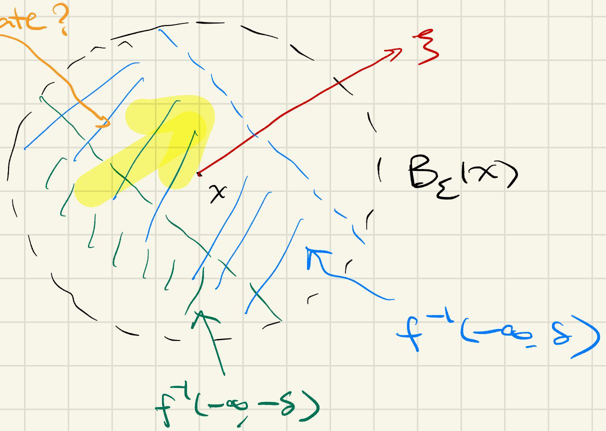
Def: If $\exists$ a small ball $B_\varepsilon(x) \subset M$ and a smooth function $f : B_\varepsilon(x) \to \mathbb{R}$ with $f(x) = 0$, $df(x) = \xi$, such that

$$R\Gamma(f^{-1}(-\infty, \delta); \mathcal{F}) \longrightarrow R\Gamma(f^{-1}(-\infty, -\delta); \mathcal{F})$$

is not a quasi-isom, for some $0 < \delta \ll 1$.

Then we say $\xi \in T_x^*M$ is characteristic/singular w.r.t. $\mathcal{F}$

picture:



Informal def:

The microsupport/singular support of $\mathcal{F}$ is

$$\underset{\underset{T^*M}{\cap}}{SS(\mathcal{F})} := \overline{\{\text{characteristic covectors of } \mathcal{F}\} \cup \operatorname{supp}(\mathcal{F})}$$

Fact: $\mathcal{F}$ constructible sheaf on M

$\Rightarrow$ $SS(\mathcal{F}) \subset T^*M$ is a conic closed Lagrangian (possibly singular) subset.

conic: $\mathbb{R}_{>0} \curvearrowright T^*M$: $t\cdot(x,\xi) \longmapsto (x, t\xi)$

$\Rightarrow$ $\mathbb{R}_{>0}\cdot SS(\mathcal{F}) = SS(\mathcal{F})$.

$\Rightarrow$ $\underbrace{SS(\mathcal{F}) \cap S^*M}_{\text{microsupport at infinity}}$ is a closed Legendrian subset (possibly singular) $\Downarrow$ ((n-1)-dim'l) ⊛

Here: $S^*M = (T^*M - \underset{\substack{\uparrow \\ \text{0-section}}}{0_M})/\mathbb{R}_{>0} \simeq \underset{\substack{\uparrow \\ \text{unit cotangent bdle}}}{ST^*M}$

pick a Riemann metric on M.

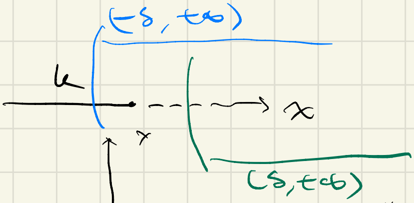
- $\Lambda \hookrightarrow S^*M$ is <u>Legendrian</u> if: $\alpha_L|_{\Lambda^{sm}} = 0$, where $\alpha_L = \sum p_i dq_i$ is the Liouville 1-form restricted to S^*M ($\rightsquigarrow$ contact 1-form).

<u>E.g.-1</u>: $F = k_M \Rightarrow SS(F) = 0_M \hookrightarrow T^*M$ Lagrangian.

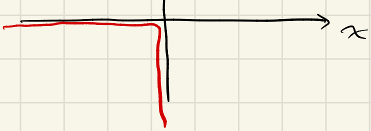
$SS(F) \cap S^*M = \emptyset$.

<u>E.g.-2</u>: $M = \mathbb{R}$.

- $F_1 = \underline{k}_{(-\infty,0]} =$



$\Rightarrow$ $SS(F_1) =$



$T^*\mathbb{R}_x = \mathbb{R}^2_{x,\xi}$

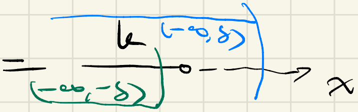
For instance: $f = -x \Rightarrow df(0) = -dx$

$$\begin{tikzcd} \rightsquigarrow & R\Gamma(f^{-1}\underbrace{(-\infty, \delta)}_{\overset{\shortparallel}{(-\delta, +\infty)}}; F_1) \arrow[r] \arrow[d, equal] & R\Gamma(f^{-1}\underbrace{(-\infty, -\delta)}_{\overset{\shortparallel}{(\delta, +\infty)}}; F_1) \arrow[d, equal] \\ & k \arrow[r, "0"'] & 0 \end{tikzcd}$$

not a <u>quasi-isom</u>

$\Rightarrow (0, -dx) \in SS(F)$.

- $\mathcal{F}_2 = \underline{k}_{(-\infty,0)} =$



$\Rightarrow SS(\mathcal{F}_2) =$

$T^*\mathbb{R}_x = \mathbb{R}^2_{x,y}$

For instance: $f = x$

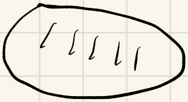
$$\begin{tikzcd} \rightsquigarrow \quad R\Gamma(f^{-1}\underbrace{(-\infty,\delta)}_{\overset{\shortparallel}{(-\infty,\delta)}};\mathcal{F}_2) \arrow[r] \arrow[d, equal] & R\Gamma(f^{-1}\underbrace{(-\infty,-\delta)}_{\overset{\shortparallel}{(-\infty,-\delta)}};\mathcal{F}_2) \arrow[d, equal] \\ 0 \arrow[r, "0"] & k \end{tikzcd}$$

not a quasi-isom

$\Rightarrow (0, dx) \in SS(\mathcal{F}_2)$.

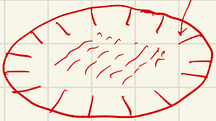
Eg-3: (draw the microsupport)

1). $i: Z =$



$\overset{\text{closed}}{\hookrightarrow} M = \mathbb{R}^2$

$\Rightarrow SS(\underline{k}_Z) =$



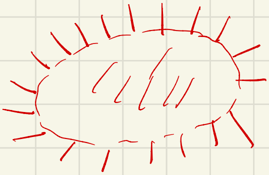
lives in the conormal bdle of ∂Z

$\hookrightarrow T^*\mathbb{R}^2 \simeq T\mathbb{R}^2$

2). $j: U =$

$\underset{\text{open}}{\hookrightarrow} M = \mathbb{R}^2$

$\Rightarrow SS(\underline{k}_U) =$



$\hookrightarrow T^*\mathbb{R}^2 \simeq T\mathbb{R}^2$.

$\circledast$ suggests the following definition:

given a (smooth) Legendrian submanifold

$\Lambda \hookrightarrow S^*M$.

$\rightsquigarrow$

Def: $Sh_\Lambda(M;k) :=$ the dg category of constructible sheaves $\mathcal{F}$ on M s.t. $SS(\mathcal{F}) \cap S^*M \subset \Lambda$, "localized" at quasi-isomorphisms.

Thm (Guillermou–Kashiwara–Schapira 2012).

$Sh_\Lambda(M;k)$ is a categorical Legendrian isotopy invariant of Λ.

Interesting case: $M = \Sigma$ is a surface.

$\Lambda \hookrightarrow S^*\Sigma$ is a Legendrian link

$\rightsquigarrow$ get computable Legendrian invariants

2. Constructible sheaves from Legendrian knots

Focus on the special case: $M = \mathbb{R}^2_{x,z}$

open contact embedding

$$\begin{tikzcd}
\Lambda \arrow[rr] \arrow[dr] & & (\mathbb{R}^3_{x,y,z}, \alpha = dz - y\,dx) \arrow[r, phantom, "\simeq"] \arrow[d, "\pi_{x,z}"] & S^{*,-}M \arrow[r, hook] & S^*M \\
& \pi_{xz}(\Lambda) \arrow[r, hook] & \mathbb{R}^2_{x,z} & &
\end{tikzcd}$$

$\uparrow$ oriented Legendrian link (i.e. $\alpha|_\Lambda = 0$)

$S^{*,-}M = \{(x,z,y,t) \in S^*M : t<0\}$

$\pi_{x,z}$ = front projection

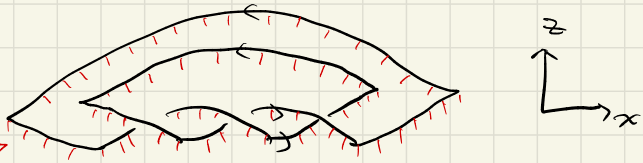
$\pi_{xz}(\Lambda)$ = <u>front diagram</u> of Λ

<u>Note</u>: $dz - y\,dx = 0$ on $\Lambda \Rightarrow y = \frac{dz}{dx}$

$\Rightarrow \Lambda$ can be recovered from $\pi_{xz}(\Lambda)$.

E.g. of a <u>front diagram</u>:

$\pi_{xz}(\Lambda) =$



downward covectors give the lift Λ.

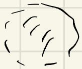
(the right-handed Legendrian trefoil knot $T_{2,3}$)

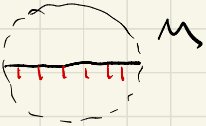
<u>Q</u>: Given a Legendrian link $\Lambda \hookrightarrow \mathbb{R}^3_{x,y,z} \simeq S^{*,-}\mathbb{R}^2_{x,z}$,

what does $\mathcal{F} \in Sh_\Lambda(\mathbb{R}^2_{x,z}; k)$ look like?

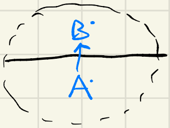
<u>A</u>: reduce to local problems $\rightsquigarrow \exists$ a combinatorial description

via quiver representations.

Local models (STZ) : $M = D^2_{x,z} =$ , then :

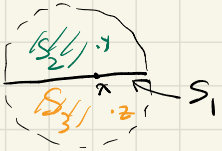
① : $\Lambda =$ 

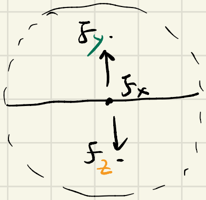
$\Rightarrow$

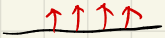
$sh_\Lambda(M;k) \simeq$ " 

" $= Rep(A_2)$

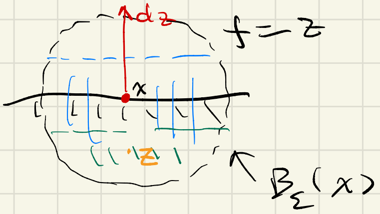
Idea : $\mathcal{F} \in sh_\Lambda(M;k) \rightsquigarrow SS(\mathcal{F}) \cap S^*M \subset \Lambda$

$\Rightarrow$ $\mathcal{F}$ is constructible w.r.t $S_\Lambda =$ 

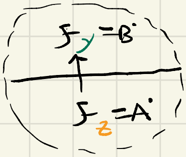
So $\mathcal{F} \Rightarrow$ 

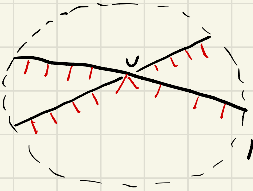
Moreover, $\Lambda \subset S^{*,-}M \Rightarrow$ upward covectors  are smooth wrt $\mathcal{F}$

$$\Rightarrow \begin{tikzcd} \mathcal{F}_x \arrow[d, "\wr"] \\ \mathcal{F}_z \end{tikzcd} \text{ is quasi-isom.}$$

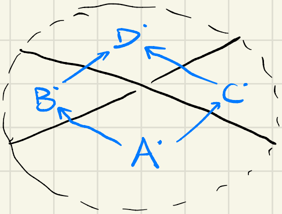


$$\rightsquigarrow \text{"can assume"} \quad \begin{tikzcd} \mathcal{F}_x \arrow[d, "id"] \\ \mathcal{F}_z \end{tikzcd} .$$

So $\mathcal{F} \Leftrightarrow$ 

② $\Lambda =$ 

M $\Rightarrow$

$\mathrm{sh}_\Lambda(M;k) \simeq$ " 

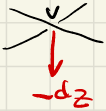
" s.t.

$$\begin{tikzcd} & D^\cdot & \\ B^\cdot \arrow[ur] & \circlearrowleft & C^\cdot \arrow[ul] \\ & A^\cdot \arrow[ul] \arrow[ur] & \end{tikzcd}$$

$\mathrm{Tot}\left(A \to B\oplus C \xrightarrow{\text{difference}} D\right)$

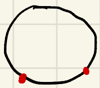
is acyclic.

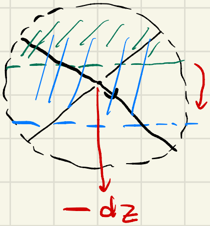
Idea for the acyclic condition:

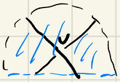
$\Lambda_v \hookrightarrow S^*_v M \Rightarrow$ the downward covector 

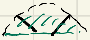
is smooth w.r.t. $\mathcal{F} \in \mathrm{sh}_\Lambda(M;k)$

$\|$ $\quad\wr$

$\cdot\ \cdot \hookrightarrow$  S^1

$\Rightarrow$ 

$\mathcal{F}$ propagates: $R\Gamma($ 

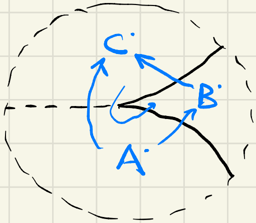
$;\mathcal{F}) \xrightarrow{\sim} R\Gamma($  $;\mathcal{F})$

$\wr$ $\qquad\qquad \wr$

$A \qquad\qquad$

$$\begin{tikzcd} B\times_D C \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & C \arrow[d] \\ B \arrow[r] & D \end{tikzcd}$$

③ : $\Lambda =$  M $\Rightarrow$

$\mathrm{sh}_\Lambda(M;k) \simeq$ " 

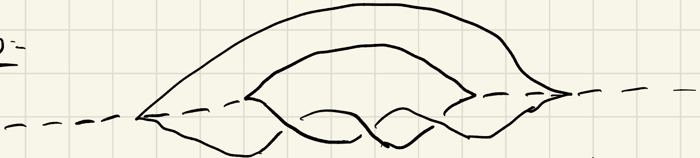
" s.t.

$$\begin{tikzcd} C^\cdot \\ A^\cdot \arrow[u, "\wr"', "\text{quasi-isom.}"] \end{tikzcd}$$

Global case: $\Lambda \hookrightarrow \mathbb{R}^3_{x,y,z} \simeq S^{*,-}M \hookrightarrow S^*M$, $M = \mathbb{R}^2_{x,z}$.

$\pi_{xz}(\Lambda)$ induces a stratification $\mathcal{S}$ of $M = \mathbb{R}^2_{x,z}$:

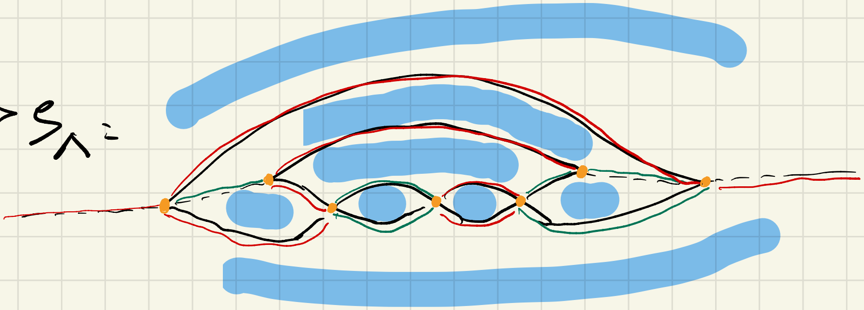
e.g.:



$\Lambda \underset{\xi}{\rightsquigarrow} \Lambda^+$

* add a few horizontal dashed lines connecting the cusps, if necessary.

$\rightsquigarrow \mathcal{S}_\Lambda$:



· 0-dim'l strata of $\mathcal{S}$ = ●'s

= singularities (cusps, crossings)

· 1-dim'l strata of $\mathcal{S}$ = ⌒ or ⌒ (arcs)

· 2-dim'l strata of $\mathcal{S}$ = ▬'s. (regions)

* ensures that $\mathcal{S}$ is a regular cell complex:

1) each stratum of $\mathcal{S}$ is contractible

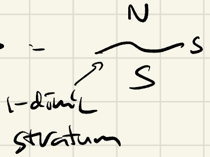
2) the star of each stratum is contractible.

↑

($s_\alpha \in \mathcal{S}$, $\mathrm{star}(s_\alpha) = \bigcup_{s_\alpha \subseteq \overline{s}_\beta} s_\beta$)

⇝ quiver with relations $\overline{Q}_S$:

- Vertices = regions in S
- Arrows :

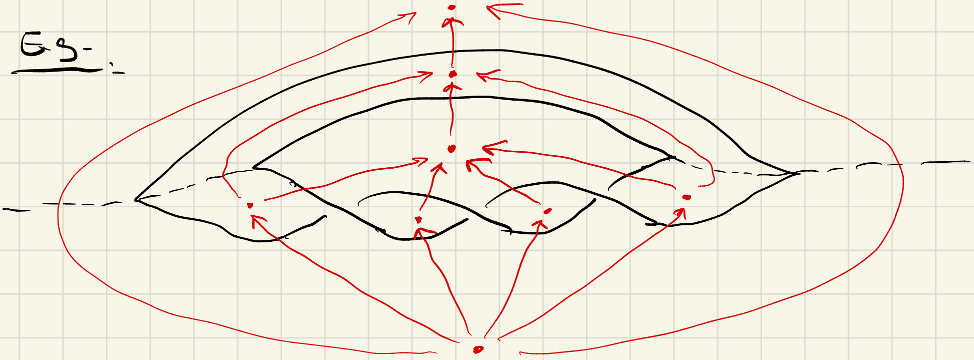


$$\begin{tikzcd} N \\ S \arrow[u, "e_s"] \end{tikzcd}$$

⟹ (upward arrow)

- relations : all "squares" commute.

E.g. :



$\overline{Q}_S$

Prop (STZ) : There exists a natural dg equivalence :

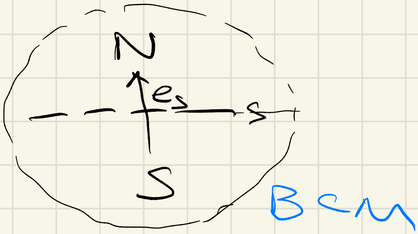
$$\Gamma_S : Sh_\Lambda(M;k) \xrightarrow{\sim} Rep_\Lambda(\overline{Q}_S) \hookrightarrow Rep(\overline{Q}_S).$$

where :

- $Rep(\overline{Q}_S)$ is the dg category of representations of $\overline{Q}_S$ (with values in perfect complexes of k-modules), "localized" at quasi-isomorphisms.

- $Rep_\Lambda(\overline{Q}_S)$ is the full dg subcategory of $Rep(\overline{Q}_S)$ consisting of $F \in Rep(\overline{Q}_S)$ s.t.

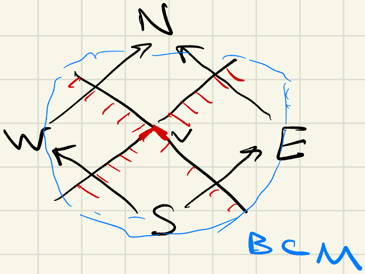
① : Near a dashed arc



$$\Longrightarrow \quad F(e_s) = F(S) \simeq F(N).$$

$$\left(SS(\mathcal{F}|_B) \cap S^*B = \emptyset \right)$$

② : Near a crossing v :



$$\Longrightarrow \quad \begin{tikzcd} & F(N) & \\ F(W) \arrow[ur] & \circlearrowright & F(E) \arrow[ul] \\ & F(S) \arrow[ul] \arrow[ur] & \end{tikzcd}$$

$$\& \ \mathrm{Tot}\left(F(S) \to F(W) \oplus F(E) \xrightarrow{\text{difference}} F(N)\right) \text{ is } \underline{\text{acyclic}}$$

proof: Use local models. □

3. Microlocal monodromy (= microlocal stalk up to a deg shift)

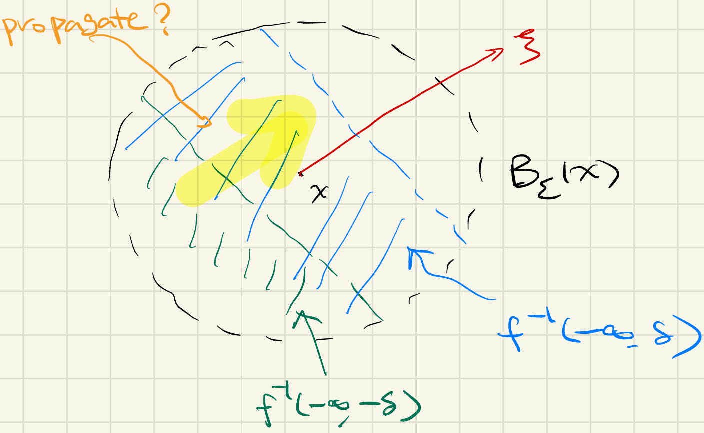
Slogan: "microlocal stalk refines the micro-support by measuring how much the local sections fail to propagate along a given singular covector".

- M = smooth mfld of dim n.
- $\Lambda \hookrightarrow S^*M$ smooth Legendrian.
- $\mathcal{F} \in sh_\Lambda(M;k)$.

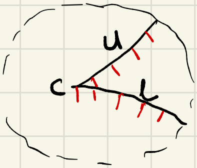
Defn/prop: $\forall (x,\xi) \in \Lambda$. Take any smooth function $f : B_\varepsilon(x) \to \mathbb{R}$ s.t. $f(x)=0$, $df(x)=\xi$, then the microlocal stalk:

$$\mathcal{F}_{(x,\xi)} := \mathrm{cone}\left(R\Gamma(f^{-1}(-\infty,\delta);\mathcal{F}) \longrightarrow R\Gamma(f^{-1}(-\infty,-\delta);\mathcal{F})\right),\ 0<\delta\ll 1$$

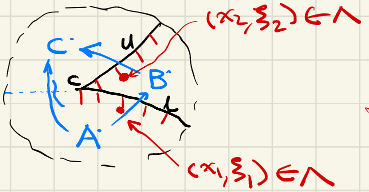
is independent of the choices of ε, f, δ.



observe:

$\Lambda =$ 

$M = D^2_{x,z}$

$\rightsquigarrow \mathcal{F} \in sh_\Lambda(M;k) \iff$ F : 

$$\Rightarrow \quad \mathcal{F}|_{(x_1,\xi_1)} = \mathrm{cone}(A \to B)$$

$\uparrow\wr$ (⇐ the octahedral axiom)

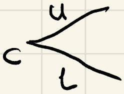
$$\mathcal{F}|_{(x_2,\xi_2)}[-1] = \mathrm{cone}(B \to C)[-1].$$

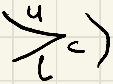
$\rightsquigarrow$ **Defn**: $\Lambda \hookrightarrow \mathbb{R}^3_{x,y,z} \simeq S^{*,-}M$ Legendrian link, $M = \mathbb{R}^2_{x,z}$.

$\rightsquigarrow$ front diagram $\pi_{xz}(\Lambda) \hookrightarrow M = \mathbb{R}^2_{x,z}$

A $\mathbb{Z}$-valued Maslov potential is:

$$\mu : \{\underbrace{\text{connected components}}_{\text{strands}} \text{ of } \pi_{xz}(\Lambda) - \{\text{cusps}\}\} \longrightarrow \mathbb{Z}.$$

s.t. 

(or 

) $\Rightarrow \mu(u) = \mu(l) + 1$.

Assume such a μ exists ($\iff r(\Lambda) = 0$)

$$\Rightarrow \quad \mathcal{F}|_{(x_1,\xi_1)}[-\mu(l)] \cong \mathcal{F}|_{(x_2,\xi_2)}[-\mu(u)]$$

$\rightsquigarrow$ **Defn/Prop**: $\exists$ a natural dg functor (microlocal monodromy):

$$\mu mon : sh_\Lambda(M;k) \longrightarrow Loc(\Lambda) = Rep(\pi_1(\Lambda))$$

$$\mathcal{F} \longmapsto \Big(\underset{\text{generic}}{(x,\xi)} \longmapsto \mathcal{F}|_{(x,\xi)}[-\mu(x)] \Big)$$

Defn: The dg category of microlocal rank 1 constructible sheaves in $sh_\Lambda(M;k)$ is:

$$\begin{tikzcd} \boxed{\mathcal{C}_1(\Lambda;k)} \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & sh_\Lambda(M;k)_0 \arrow[r, hook] \arrow[d, "\mu mon"'] & sh_\Lambda(M;k) \\ \{\underline{k}\} \arrow[r, hook] & Loc(\Lambda) & \end{tikzcd}$$

$0 \uparrow$ compact support

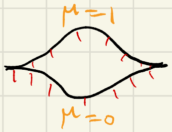
Fact: $\mathcal{C}_1(\Lambda;k)$ is a categorical Legendrian isotopy inv of (Λ,μ).

Defn: the STZ moduli stack $\mathfrak{M}_1(\Lambda)$ is the (underived) moduli stack of objects in $\mathcal{C}_1(\Lambda;k)$

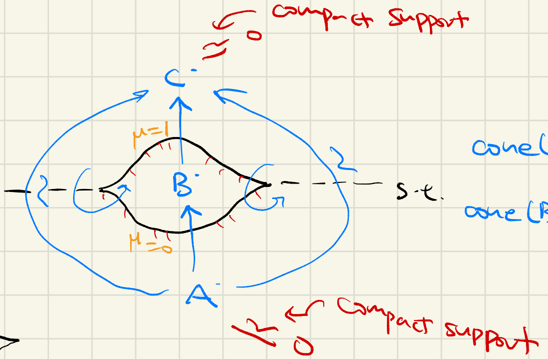
⇝ associated moduli space (suitably defined) $M_1(\Lambda)$.

(⇝ the Betti moduli space in the second talk)

E.g.: $\Lambda =$



$\mathcal{F} \in \mathcal{C}_1(\Lambda;k) \iff$



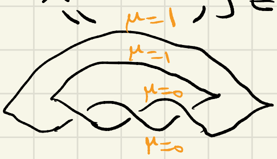
s.t. cone$(A\to B) \simeq k$

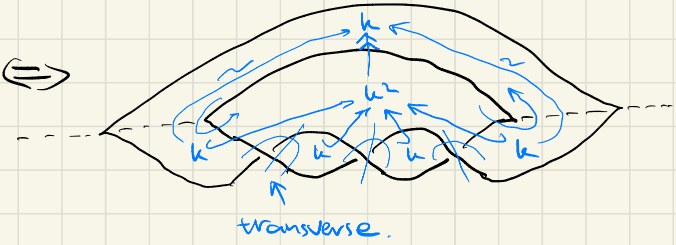
cone$(B\to C)[-1] \simeq k$

$$\iff \begin{tikzcd} C^\bullet = 0 \\ B^\bullet = k \arrow[u] \\ A^\bullet = 0 \arrow[u] \end{tikzcd}$$

$B^\bullet = k =$

Ex: $\Lambda =$, $\mathcal{F} \in \mathcal{C}_1(\Lambda;k) \iff$



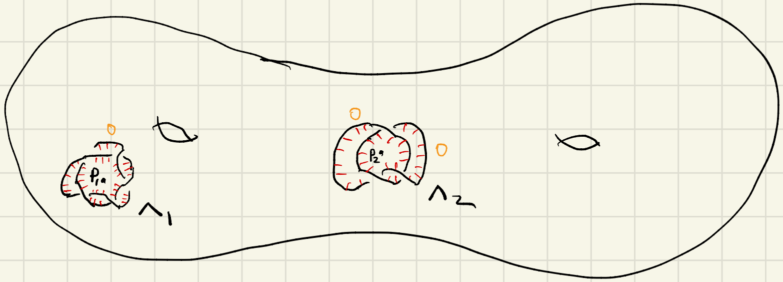


4. Irregular Riemann-Hilbert correspondence over curves

$M =$ Riemann surface Σ with punctures $p_1, \dots, p_k$.

$\Lambda = \cup \Lambda_i$ Legendrian links with Λ_i near p_i in S^*M.

e.g.



$\Lambda = \Lambda_1 \cup \Lambda_2$, $M = \Sigma_2$ with 2 punctures p_1, p_2.

⇝ the same construction above gives

$$\begin{tikzcd} \mu\mathrm{mon}: \mathcal{C}_B(\Sigma, \{p_i\}, \{\Lambda_i\}) := Sh_{\cup\Lambda_i}(\Sigma\setminus\{p_i\};\mathbb{C})_0 \arrow[r] & \mathcal{L}oc(\Lambda) = \prod \mathcal{L}oc(\Lambda_i) \\ C_B(\Sigma, \{p_i\}, \{\Lambda_i\}) \arrow[u, hook] \arrow[r] \arrow[ur, phantom, "\urcorner^h", very near start] & Loc(\Lambda) = \prod Loc(\Lambda_i) \arrow[u, hook] \end{tikzcd}$$

local systems in deg 0

A microlocal formulation of irregular RH over curves:

irregular connections

local solution sheaf with stokes filtration

stokes Legendrian defined by τ_i

$$\begin{tikzcd} C_{dR}(\Sigma, \{p_i\}, \{\tau_i\}) \arrow[r, "St", "\sim"'] \arrow[d, "\prod_i F_i"] \arrow[dr, phantom, "\circlearrowright"] & C_B(\Sigma, \{p_i\}, \{\Lambda_i\}) \arrow[d, "\mu\mathrm{mon}"] \\ \prod_i \left(\begin{array}{l} \text{formal connections} \\ \text{over } \mathbb{C}((u)) \text{ with} \\ \text{prescribed formal type } \tau_i \end{array} \right) \arrow[r, "\prod St_i", "\sim"'] & \prod_i Loc(\Lambda_i) \end{tikzcd}$$

formal completion at p_i.

deRham side | Betti side

⇝ Slogan: "Betti moduli space / wild character variety

= moduli space of constructible sheaves with controlled micro-support."

Illustration by example

- $\Sigma = \mathbb{C}P^1, \quad k=1, \quad p_1 = \infty$.

- **formal type of irregular singularity**

$$\tau_1 = \left\{ g_1(x) = -\frac{2}{5} x^{-\frac{5}{2}}, \; g_2(x) = \frac{2}{5} x^{-\frac{5}{2}} \right\} \subset x^{-\frac{1}{N}} \mathbb{C}[x^{-\frac{1}{N}}], \quad N=2.$$

- E.g. $\left(V = \mathcal{O}_{\mathbb{C}P^1}^{\oplus 2}, \nabla\right) \in \mathcal{C}_{dR}(\mathbb{C}P^1, \{\infty\}, \{\tau_1\})$ with

$$\nabla = d - \begin{pmatrix} 0 & 1 \\ z^3 & 0 \end{pmatrix} dz$$

where z = the standard coordinate on $\mathbb{C}P^1$ centered at 0

& $x = z^{-1}$ is the standard coord. centered at $p_1 = \infty$.

Reason: local horizontal sections near $p_1 = \infty$ are solutions to

$$\nabla \begin{pmatrix} f \\ g \end{pmatrix} = 0 \iff \begin{cases} \frac{df}{dz} = g \\ \frac{dg}{dz} = z^3 f \end{cases}$$

$$\iff \left(\frac{d^2}{dz^2} - z^3\right) f = 0$$

$$\iff Lf = 0, \quad L = \delta^2 + \delta - \underbrace{x^{-5}}_{\substack{\uparrow \\ \text{irregular}}}, \quad \delta = x\frac{d}{dx} = -z\frac{d}{dz}$$

$\rightsquigarrow$ linearly indep formal solutions

$$f_{1,2} = \exp\Big(\underbrace{\mp \frac{2}{5} x^{-\frac{5}{2}}}_{\substack{\uparrow \\ \{g_1, g_2\} \\ \cap \\ \tau_1}}\Big) \underbrace{x^{\frac{3}{4}} \sum_{m \geq 0} a_m^{\pm} x^{\frac{m}{2}}}_{\substack{\uparrow \\ \text{at most polynomial growth rate} \\ \text{as } x \to 0.}}, \quad \text{for some } a_m^{\pm} \in \mathbb{C}.$$

$\Big\downarrow$

the growth rate is controlled by

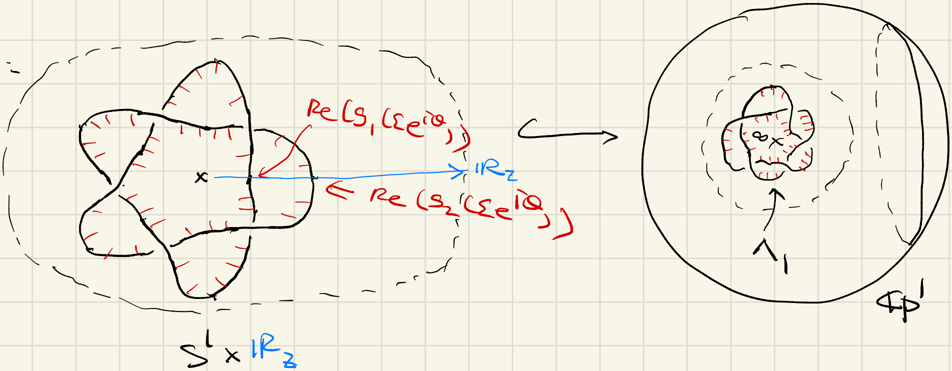
$\mathrm{Re}(g_1)$, $\mathrm{Re}(g_2)$.

- Stokes Legendrian knot Λ_1 :

Fix $0<\varepsilon<<1$, the graphs of the multivalued functions

$$S^1 \ni \theta \longmapsto \mathrm{Re}(g_i(\varepsilon e^{i\theta})) \in \mathbb{R}_z$$

give:



$St(V,\nabla) = ?$

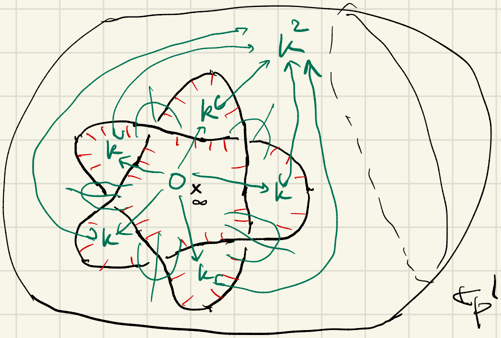
- Away from the disk: $St(V,\nabla) = Sol(V,\nabla)$

$\uparrow$ solution sheaf of local horizontal sections.

- Inside the disk:

$$St(V,\nabla)_{\theta,z} := \bigoplus_{\mathrm{Re}(g_i(\varepsilon e^{i\theta}))<z} \mathbb{C}\cdot f_i$$

$\rightsquigarrow$ $St(V,\nabla) \Longleftrightarrow$



$\rightsquigarrow$ $St(V,\nabla) \in \mathcal{C}_1(\mathbb{CP}^1\setminus\{\infty\},\Lambda_1) \hookrightarrow Sh_{\Lambda_1}(\mathbb{CP}^1\setminus\{\infty\};\mathbb{C})_0$.

$\nwarrow$ microlocal rank 1.

observe: $St(V,\nabla) \Longleftrightarrow$ some $\mathcal{F}\in\mathcal{C}_1(\Lambda;\mathbb{C})$ in EX !