SU(1,2) Higgs bundle, Spectral Data and Limiting Configuration

Xuesen Na

Department of Mathematics, University of Maryland

The Chern Institute of Mathematics (CIM), Nankai University

Introduction: Higgs bundle and Hitchin equation

X closed Riemann surface of genus ≥ 2. Fix ω a Kähler form on X.

Introduction: Higgs bundle and Hitchin equation

X closed Riemann surface of genus ≥ 2. Fix ω a Kähler form on X.

Definition

A **Higgs bundle** is a pair $(\mathcal{E}, \Phi)$ of holomorphic objects: $\mathcal{E}$ is a holomorphic vector bundle on X; Φ a holomorphic 1-form valued in endomorphisms: $\Phi \in H^0(X, \mathrm{End}(\mathcal{E}) \otimes K_X)$. K_X is canonical line bundle of X, i.e. the holomorphic cotangent bundle.

Introduction: Higgs bundle and Hitchin equation

X closed Riemann surface of genus ≥ 2. Fix ω a Kähler form on X.

Definition

A **Higgs bundle** is a pair $(\mathcal{E}, \Phi)$ of holomorphic objects: $\mathcal{E}$ is a holomorphic vector bundle on X; Φ a holomorphic 1-form valued in endomorphisms: $\Phi \in H^0(X, \mathrm{End}(\mathcal{E}) \otimes K_X)$. K_X is canonical line bundle of X, i.e. the holomorphic cotangent bundle.
$(\mathcal{E}, \Phi)$ is **(semi)stable** if for each proper Φ-invariant subbundle $\mathcal{E}' \subset \mathcal{E}$ we have

$$\mu(\mathcal{E}') = \frac{\mathrm{deg}\mathcal{E}'}{\mathrm{rank}\mathcal{E}'} < (\leq) \mu(\mathcal{E}) = \frac{\mathrm{deg}\mathcal{E}}{\mathrm{rank}\mathcal{E}}$$

where $\deg \mathcal{E} = \deg \det \mathcal{E}$.

Introduction: Higgs bundle and Hitchin equation

X closed Riemann surface of genus ≥ 2. Fix ω a Kähler form on X.

Definition

A **Higgs bundle** is a pair $(\mathcal{E}, \Phi)$ of holomorphic objects: $\mathcal{E}$ is a holomorphic vector bundle on X; Φ a holomorphic 1-form valued in endomorphisms: $\Phi \in H^0(X, \mathrm{End}(\mathcal{E}) \otimes K_X)$. K_X is canonical line bundle of X, i.e. the holomorphic cotangent bundle.

$(\mathcal{E}, \Phi)$ is **(semi)stable** if for each proper Φ-invariant subbundle $\mathcal{E}' \subset \mathcal{E}$ we have

$$\mu(\mathcal{E}') = \frac{\deg \mathcal{E}'}{\mathrm{rank} \mathcal{E}'} < (\leq) \mu(\mathcal{E}) = \frac{\deg \mathcal{E}}{\mathrm{rank} \mathcal{E}}$$

where $\deg \mathcal{E} = \deg \det \mathcal{E}$.

$(\mathcal{E}, \Phi)$ is **polystable** if $(\mathcal{E}, \Phi) = \oplus_i (\mathcal{E}_i, \Phi_i)$, where $(\mathcal{E}_i, \Phi_i)$ stable with same slopes.

Introduction: Higgs bundle and Hitchin equation

- Topologically smooth complex vector bundles E over compact Riemann surface is classified by **rank** r **and degree** d, where degree is defined by $\deg \Lambda^r E = \deg \det E$, or by Chern-Weil theory with choice of a connection ∇ with curvature F_∇,

$$\deg E = \int_X \frac{\sqrt{-1}}{2\pi} \mathrm{tr} F_\nabla$$

Introduction: Higgs bundle and Hitchin equation

- Topologically smooth complex vector bundles E over compact Riemann surface is classified by **rank** r **and degree** d, where degree is defined by $\deg \Lambda^r E = \deg \det E$, or by Chern-Weil theory with choice of a connection ∇ with curvature F_∇,

$$\deg E = \int_X \frac{\sqrt{-1}}{2\pi} \mathrm{tr} F_\nabla$$

- Mumford invented geometric invariant theory (GIT) in 60s and studied moduli of vector bundles $\mathcal{N}_{r,d}$ of rank r and degree d over compact Riemann surfaces and introduced notion of (slope) stability

- Narasimhan & Seshadri ('65) identified $\mathcal{N}_{r,d}$ with the character variety $\mathrm{Hom}\left(\hat{\pi}_1, U(n)\right) /\!\!/ U(n)$ of irreducible projective unitary representation of fundamental group

Introduction: Higgs bundle and Hitchin equation

- Narasimhan & Seshadri ('65) identified $\mathcal{N}_{r,d}$ with the character variety $\text{Hom}\,(\hat{\pi}_1, U(n)) /\!\!/ U(n)$ of irreducible projective unitary representation of fundamental group
- Donaldson ('82) further identified this with the moduli space of projectively flat irred. unitary conn. on underlying bundle E. Later generalized by Uhlenbeck-Yau ('86) characterizing stability over compact Kähler manifold (X, ω) with existence of Hermitian-Yang-Mills connection

$$\sqrt{-1}F_\nabla \wedge \omega^{n-1} = \mu \text{Id}_E \omega^n$$

Introduction: Higgs bundle and Hitchin equation

- Hitchin ('87) studied a dimension reduction of self-dual Yang-Mills equation $F_A = *F_A$ from $d = 4$ to $d = 2$, which retains conformal symmetry and is naturally defined on Riemann surfaces, leading to Hitchin equation. The moduli space $\mathcal{M}$ of its solutions has very rich geometry and contains $T^*\mathcal{N}_{2,d}$ as open dense subset.

Introduction: Higgs bundle and Hitchin equation

- Hitchin ('87) studied a dimension reduction of self-dual Yang-Mills equation $F_A = *F_A$ from $d = 4$ to $d = 2$, which retains conformal symmetry and is naturally defined on Riemann surfaces, leading to Hitchin equation. The moduli space $\mathcal{M}$ of its solutions has very rich geometry and contains $T^*\mathcal{N}_{2,d}$ as open dense subset.
- Hitchin showed that the moduli space $\mathcal{M}$ of solutions is equivalent to that of the pair $(\mathcal{E}, \Phi)$ with above stability condition, along the same line as Donaldson-Uhlenbeck-Yau. A wide range of similar results are now referred to collectively as Hitchin-Kobayashi correspondence.

Introduction: Higgs bundle and Hitchin equation

- Hitchin ('87) studied a dimension reduction of self-dual Yang-Mills equation $F_A = *F_A$ from $d = 4$ to $d = 2$, which retains conformal symmetry and is naturally defined on Riemann surfaces, leading to Hitchin equation. The moduli space $\mathcal{M}$ of its solutions has very rich geometry and contains $T^*\mathcal{N}_{2,d}$ as open dense subset.
- Hitchin showed that the moduli space $\mathcal{M}$ of solutions is equivalent to that of the pair $(\mathcal{E}, \Phi)$ with above stability condition, along the same line as Donaldson-Uhlenbeck-Yau. A wide range of similar results are now referred to collectively as Hitchin-Kobayashi correspondence.
- The name 'Higgs bundle': Gauge fields A_3, A_4 in the direction of reduction gives a field similar to that of Higgs boson in the standard model of particle physics, which he calls Higgs field.

Introduction: Higgs bundle and Hitchin equation

There are two equivalent formulations of Hitchin equation for $(\mathcal{E}, \Phi)$:

Introduction: Higgs bundle and Hitchin equation

There are two equivalent formulations of Hitchin equation for $(\mathcal{E}, \Phi)$:

- Fix hermitian metric h_0 on E underlying complex vector bundle. Hitchin equation is an equation of the pair (A, Φ):

$$\begin{cases} \sqrt{-1}\Lambda\left(F_A + [\Phi \wedge \Phi^\dagger]\right) = \mu(E)\mathrm{Id}_E \\ d_A''\Phi = 0 \end{cases}$$

where F_A curvature 2-form and the holomorphic structure of $\mathcal{E}$ is given by $\bar{\partial}_E = d_A''$.

Introduction: Higgs bundle and Hitchin equation

There are two equivalent formulations of Hitchin equation for $(\mathcal{E}, \Phi)$:

- Fix hermitian metric h_0 on E underlying complex vector bundle. Hitchin equation is an equation of the pair (A, Φ):

$$\begin{cases} \sqrt{-1}\Lambda\left(F_A + [\Phi \wedge \Phi^\dagger]\right) = \mu(E)\mathrm{Id}_E \\ d_A''\Phi = 0 \end{cases}$$

  where F_A curvature 2-form and the holomorphic structure of $\mathcal{E}$ is given by $\bar{\partial}_E = d_A''$.
- Fix holomorphic structure $\bar{\partial}_E$, Hitchin equation is an equation of hermitian metric h on E:

$$\sqrt{-1}\Lambda\left(R(h) + [\Phi \wedge \Phi_h^\dagger]\right) = \mu(E)\mathrm{Id}_E$$

  where $R(h) = F_{\nabla_h}$ curvature 2-form of Chern connection $A = A(\bar{\partial}_E, h)$.

Introduction: Higgs bundle and Hitchin equation

There are two equivalent formulations of Hitchin equation for $(\mathcal{E}, \Phi)$:

- Fix hermitian metric h_0 on E underlying complex vector bundle. Hitchin equation is an equation of the pair (A, Φ):

$$\begin{cases} \sqrt{-1}\Lambda\left(F_A + [\Phi \wedge \Phi^\dagger]\right) = \mu(E)\mathrm{Id}_E \\ d''_A \Phi = 0 \end{cases}$$

  where F_A curvature 2-form and the holomorphic structure of $\mathcal{E}$ is given by $\bar{\partial}_E = d''_A$.
- Fix holomorphic structure $\bar{\partial}_E$, Hitchin equation is an equation of hermitian metric h on E:

$$\sqrt{-1}\Lambda\left(R(h) + [\Phi \wedge \Phi_h^\dagger]\right) = \mu(E)\mathrm{Id}_E$$

  where $R(h) = F_{\nabla_h}$ curvature 2-form of Chern connection $A = A(\bar{\partial}_E, h)$.

In this talk we will adopt the second. The tuple $(E, \bar{\partial}_E, \Phi, h)$ is called harmonic bundle.

Introduction: Higgs bundle and Hitchin equation

There is also a version of Higgs bundle for principal bundles (eqv. for vector bundle but with some additional structure).

Introduction: Higgs bundle and Hitchin equation

There is also a version of Higgs bundle for principal bundles (eqv. for vector bundle but with some additional structure).

Definition

$G^{\mathbb{C}}$ complex reductive Lie group. A $G^{\mathbb{C}}$-Higgs bundle is a pair (P, Φ), P a holomorphic principal $G^{\mathbb{C}}$-bundle and Φ a holomorphic 1-form valued in the adjoint bundle, $\Phi \in H^0(X, P \times_{\mathrm{Ad}} \mathfrak{g}^{\mathbb{C}} \otimes K)$.

There is also a version of Higgs bundle for principal bundles (eqv. for vector bundle but with some additional structure).

Definition

$G^{\mathbb{C}}$ complex reductive Lie group. A $G^{\mathbb{C}}$-Higgs bundle is a pair (P, Φ), P a holomorphic principal $G^{\mathbb{C}}$-bundle and Φ a holomorphic 1-form valued in the adjoint bundle, $\Phi \in H^0(X, P \times_{\mathrm{Ad}} \mathfrak{g}^{\mathbb{C}} \otimes K)$.

For $G^{\mathbb{C}} \subset GL(n, \mathbb{C})$ we may associate with any $G^{\mathbb{C}}$-Higgs bundle a (vector) Higgs bundle (E, Φ).

Introduction: Higgs bundle and Hitchin equation

There is also a version of Higgs bundle for principal bundles (eqv. for vector bundle but with some additional structure).

Definition

$G^{\mathbb{C}}$ complex reductive Lie group. A $G^{\mathbb{C}}$-Higgs bundle is a pair (P, Φ), P a holomorphic principal $G^{\mathbb{C}}$-bundle and Φ a holomorphic 1-form valued in the adjoint bundle, $\Phi \in H^0(X, P \times_{\text{Ad}} \mathfrak{g}^{\mathbb{C}} \otimes K)$.

For $G^{\mathbb{C}} \subset GL(n, \mathbb{C})$ we may associate with any $G^{\mathbb{C}}$-Higgs bundle a (vector) Higgs bundle (E, Φ).

Examples $G^{\mathbb{C}} = SL(n, \mathbb{C}) \rightsquigarrow (\mathcal{E}, \Phi)$ with rank$\mathcal{E} = n$ with a trivialization $\det \mathcal{E} \xrightarrow{\sim} \mathcal{O}_X$ and tr$\Phi = 0$; $G^{\mathbb{C}} = Sp(2n, \mathbb{C}) \rightsquigarrow (\mathcal{E}, \Phi)$ with E of rank 2n with a holomorphic simplectic form Ω and Φ satisfied $\Omega(\Phi v, w) = -\Omega(v, \Phi w)$.

Introduction: Higgs bundle and Hitchin equation

Theorem (Hitchin '87, Simpson '88)

An SL(n,$\mathbb{C}$)-Higgs bundle $(\mathcal{E}, \Phi)$ admits unique solution h to Hitchin equation iff it is polystable

Introduction: Higgs bundle and Hitchin equation

Theorem (Hitchin '87, Simpson '88)

An SL(n,$\mathbb{C}$)-Higgs bundle $(\mathcal{E}, \Phi)$ admits unique solution h to Hitchin equation iff it is polystable

In other words there is a unique harmonic bundle structure $(E, \bar{\partial}_E, \Phi, h)$ on a polystable Higgs bundle $(E, \bar{\partial}_E, \Phi)$. This will also provide a flat connection $\nabla = \nabla_h + \Phi + \Phi_h^{\dagger}$

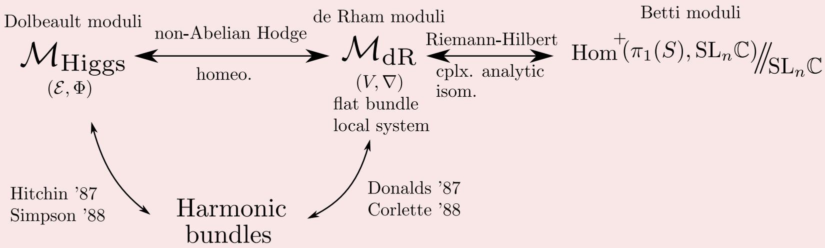
Introduction: Higgs bundle and Hitchin equation

Theorem (Hitchin '87, Simpson '88)

An SL(n,$\mathbb{C}$)-Higgs bundle $(\mathcal{E}, \Phi)$ admits unique solution h to Hitchin equation iff it is polystable

In other words there is a unique harmonic bundle structure $(E, \bar{\partial}_E, \Phi, h)$ on a polystable Higgs bundle $(E, \bar{\partial}_E, \Phi)$. This will also provide a flat connection $\nabla = \nabla_h + \Phi + \Phi_h^\dagger$

Introduction: Higgs bundle and Hitchin equation



Introduction: spectral curve and spectral data

Definition

The Hitchin map for $G^{\mathbb{C}}$-Higgs bundle is given by

$$\mathrm{Hit}: \mathcal{M}_{G^{\mathbb{C}}} \to \mathcal{B}_{G^{\mathbb{C}}} = \bigoplus_{i=1}^{k} H^0(X, K^{\otimes d_j})$$
$$(P,\Phi) \mapsto (p_1(\Phi), \dots, p_k(\Phi))$$

where $\{p_1, \dots, p_r\}$ is a basis of $G^{\mathbb{C}}$-invariant polynomials on $\mathfrak{g}^{\mathbb{C}}$ and $d_j = \deg p_j$. For $G^{\mathbb{C}} = \mathrm{SL}(n, \mathbb{C})$, Hit simply takes coefficients of characteristic polynomial.

Introduction: spectral curve and spectral data

Definition

The Hitchin map for $G^{\mathbb{C}}$-Higgs bundle is given by

$$\begin{aligned}\mathrm{Hit}: \mathcal{M}_{G^{\mathbb{C}}} \to \mathcal{B}_{G^{\mathbb{C}}} = \bigoplus_{i=1}^{k} H^0(X, K^{\otimes d_i})\\ (P, \Phi) \mapsto (p_1(\Phi), \ldots, p_k(\Phi))\end{aligned}$$

where $\{p_1, \ldots, p_r\}$ is a basis of $G^{\mathbb{C}}$-invariant polynomials on $\mathfrak{g}^{\mathbb{C}}$ and $d_j = \deg p_j$. For $G^{\mathbb{C}} = \mathrm{SL}(n, \mathbb{C})$, Hit simply takes coefficients of characteristic polynomial.

Hit is a proper and surjective map and as remarked by Hitchin (1987) 'Somewhat miraculously' $\dim \mathcal{B}_{G^{\mathbb{C}}} = \dim \mathcal{M}_{G^{\mathbb{C}}}/2$, Hit makes $\mathcal{M}_{G^{\mathbb{C}}}$ into an integrable system – **Hitchin system**. Almost all integrable systems in classical mechanics may be realized as special cases.

Introduction: spectral curve and spectral data

Let $\pi : \mathrm{Tot}(K) \to X$ be the canonical line bundle and $p \in \mathcal{B}$ a polynomial. $\lambda \in \pi^* K$ the tautological section. The spectral curve $\pi : \Sigma_p \to X$ given by the zero locus of $p(\lambda)$.

Introduction: spectral curve and spectral data

Let $\pi : \mathrm{Tot}(K) \to X$ be the canonical line bundle and $p \in \mathcal{B}$ a polynomial. $\lambda \in \pi^* K$ the tautological section. The spectral curve $\pi : \Sigma_p \to X$ given by the zero locus of $p(\lambda)$.

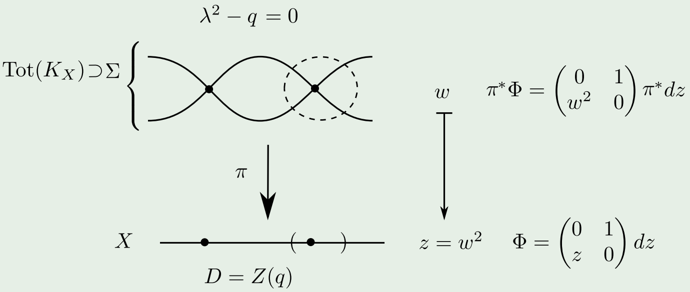
In other words, for a rank r Higgs bundle $(E, \Phi) \mapsto p$ under Hitchin map, Σ_p marks the eigenvalues of Higgs field Φ. $\pi : \Sigma_p \to X$ is an r-sheeted branched covering.

Introduction: spectral curve and spectral data

Let $\pi : \mathrm{Tot}(K) \to X$ be the canonical line bundle and $p \in \mathcal{B}$ a polynomial. $\lambda \in \pi^* K$ the tautological section. The spectral curve $\pi : \Sigma_p \to X$ given by the zero locus of $p(\lambda)$.

In other words, for a rank r Higgs bundle $(E, \Phi) \mapsto p$ under Hitchin map, Σ_p marks the eigenvalues of Higgs field Φ. $\pi : \Sigma_p \to X$ is an r-sheeted branched covering.

Consider $SL(2, \mathbb{C})$ Higgs bundle with Hit:$(\mathcal{E}, \Phi) \mapsto q = \det \Phi$ simple zeros



Introduction: spectral curve and spectral data

Theorem (Beauville-Narasimhan-Ramanan '89)

For p with Σ_p an integral (i.e. reduced and irreducible) curve, there is a natural equivalence between

- Pairs $(\mathcal{E}, \Phi)$ with $\mathrm{char}(\Phi) = p$, and
- rank-one torsion free sheaves on Σ_p (Note: not necessarily an invertible sheaf, i.e. line bundles for singular Σ_p)

Introduction: spectral curve and spectral data

Theorem (Beauville-Narasimhan-Ramanan '89)

For p with Σ_p an integral (i.e. reduced and irreducible) curve, there is a natural equivalence between

- Pairs $(\mathcal{E}, \Phi)$ with $\text{char}(\Phi) = p$, and
- rank-one torsion free sheaves on Σ_p (Note: not necessarily an invertible sheaf, i.e. line bundles for singular Σ_p)

The extra data for constructing a pair $(\mathcal{E}, \Phi)$ after fixing $p \in \mathcal{B}$ is called spectral data, equivalently spectral data gives description of the Hitchin fiber.

Introduction: spectral curve and spectral data

Theorem (Beauville-Narasimhan-Ramanan '89)

For p with Σ_p an integral (i.e. reduced and irreducible) curve, there is a natural equivalence between

- Pairs $(\mathcal{E}, \Phi)$ with char$(\Phi) = p$, and
- rank-one torsion free sheaves on Σ_p (Note: not necessarily an invertible sheaf, i.e. line bundles for singular Σ_p)

The extra data for constructing a pair $(\mathcal{E}, \Phi)$ after fixing $p \in \mathcal{B}$ is called spectral data, equivalently spectral data gives description of the Hitchin fiber.

In particular the data is given by the 'eigen-line-subbundle'

$$0 \longrightarrow L(-\Delta) \longrightarrow \pi^* E \xrightarrow{\pi^*\Phi - \lambda} \pi^*(E \otimes K_X)$$

Conversely we recover $(\mathcal{E}, \Phi)$ from (L, λ) by applying direct image functor / pushforward π_*. For G =SL(n,$\mathbb{C}$) when Σ_p is smooth curve, the Hitchin fiber can be identified with Prym(Σ_p, X), an abelian variety.

Introduction: G-Higgs bundle

- Non-abelian Hodge correspondence opens door to study the character varieties $\mathrm{Hom}^{+}(\pi, G) /\!\!/ G$ by Higgs bundles. The work of Hitchin (1992) for $G = \mathrm{SL}(n, \mathbb{R})$ motivated the notion of G-Higgs bundle for G **real form** of complex Lie group $G^{\mathbb{C}}$.

Introduction: G-Higgs bundle

- Non-abelian Hodge correspondence opens door to study the character varieties $\mathrm{Hom}^+(\pi, G) /\!\!/ G$ by Higgs bundles. The work of Hitchin (1992) for $G = \mathrm{SL}(n, \mathbb{R})$ motivated the notion of G-Higgs bundle for G **real form** of complex Lie group $G^{\mathbb{C}}$.
- $\mathfrak{g}^{\mathbb{C}}$ Lie algebra of reductive Lie group $G^{\mathbb{C}}$. Recall we have

$$\left\{ \begin{array}{l} \text{real form } \mathfrak{g} \\ \text{of } \mathfrak{g}^{\mathbb{C}} \end{array} \right\} \xrightarrow{1-1} \left\{ \begin{array}{l} \text{conjugacy cls. of} \\ \text{antihol'c involution} \end{array} \right\} \xrightarrow{1-1} \left\{ \begin{array}{l} \text{conj. cls. of} \\ \text{hol'c involutions } \theta \end{array} \right\}$$

θ is the Cartan involution of the real form and its eigenspace decomposition is Cartan decomposition. Let $\mathfrak{g}$ (resp. G) a non-compact real form with Cartan decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ (Killing form negative definite on $\mathfrak{h}$). H: maximal compact subgroup corresp to $\mathfrak{h}$.

Introduction: G-Higgs bundle

- Non-abelian Hodge correspondence opens door to study the character varieties $\mathrm{Hom}^+(\pi, G) /\!\!/ G$ by Higgs bundles. The work of Hitchin (1992) for $G =$SL$(n, \mathbb{R})$ motivated the notion of G-Higgs bundle for G **real form** of complex Lie group $G^{\mathbb{C}}$.
- $\mathfrak{g}^{\mathbb{C}}$ Lie algebra of reductive Lie group $G^{\mathbb{C}}$. Recall we have

$$\left\{\begin{array}{l}\text{real form } \mathfrak{g} \\ \text{of } \mathfrak{g}^{\mathbb{C}}\end{array}\right\} \xrightarrow{1-1} \left\{\begin{array}{l}\text{conjugacy cls. of} \\ \text{antihol'c involution}\end{array}\right\} \xrightarrow{1-1} \left\{\begin{array}{l}\text{conj. cls. of} \\ \text{hol'c involutions } \theta\end{array}\right\}$$

  θ is the Cartan involution of the real form and its eigenspace decomposition is Cartan decomposition. Let $\mathfrak{g}$ (resp. G) a non-compact real form with Cartan decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ (Killing form negative definite on $\mathfrak{h}$). H: maximal compact subgroup corresp to $\mathfrak{h}$.
- $[\mathfrak{h}, \mathfrak{m}] \subseteq \mathfrak{m}$, restriction of adjoint rep'n gives **isotropy representation** $\iota : H^{\mathbb{C}} \to \mathrm{GL}(\mathfrak{m}^{\mathbb{C}})$

Introduction: G-Higgs bundle

Definition (Bradlow, García-Prada & Gothen 2006)

A G-Higgs bundle is a pair (P, Φ): P is a holomorphic principal-$H^{\mathbb{C}}$, $\Phi \in H^0(X, P \times_{\mathrm{Ad}} \mathfrak{m}^{\mathbb{C}} \otimes K)$.

Definition (Bradlow, García-Prada & Gothen 2006)

A G-Higgs bundle is a pair (P, Φ): P is a holomorphic principal-$H^{\mathbb{C}}$, $\Phi \in H^0(X, P \times_{\text{Ad}} \mathfrak{m}^{\mathbb{C}} \otimes K)$.

- The Hitchin map is given by $\mathcal{M} \to \mathcal{B} = \bigoplus_{i=1}^{a} H^0(X, K^{m_i})$ by evaluating Higgs field at a basis of the ring of polynomial $H^{\mathbb{C}}$-invariants. m_i are exponents of G and a is real rank.

Introduction: G-Higgs bundle

Definition (Bradlow, García-Prada & Gothen 2006)

A G-Higgs bundle is a pair (P, Φ): P is a holomorphic principal-$H^{\mathbb{C}}$, $\Phi \in H^0(X, P \times_{\mathrm{Ad}} \mathfrak{m}^{\mathbb{C}} \otimes K)$.

- The Hitchin map is given by $\mathcal{M} \to \mathcal{B} = \bigoplus_{i=1}^{a} H^0(X, K^{m_i})$ by evaluating Higgs field at a basis of the ring of polynomial $H^{\mathbb{C}}$-invariants. m_i are exponents of G and a is real rank.
- As a first step in extension of work of Mazzeo-Swoboda-Weiss-Witt (2014) to G-Higgs bundle, we will consider G with real rank one, e.g. $SU(1, n)$, $SO(1, n)$. We will consider the simplest of these, the Lie group $G =$SU(1,2).

Introduction: G-Higgs bundle

Definition (Bradlow, García-Prada & Gothen 2006)

A G-Higgs bundle is a pair (P, Φ): P is a holomorphic principal-$H^{\mathbb{C}}$, $\Phi \in H^0(X, P \times_{\mathrm{Ad}} \mathfrak{m}^{\mathbb{C}} \otimes K)$.

- The Hitchin map is given by $\mathcal{M} \to \mathcal{B} = \bigoplus_{i=1}^{a} H^0(X, K^{m_i})$ by evaluating Higgs field at a basis of the ring of polynomial $H^{\mathbb{C}}$-invariants. m_i are exponents of G and a is real rank.
- As a first step in extension of work of Mazzeo-Swoboda-Weiss-Witt (2014) to G-Higgs bundle, we will consider G with real rank one, e.g. $SU(1,n)$, $SO(1,n)$. We will consider the simplest of these, the Lie group $G =$SU(1,2).
- $H = S\left(U(1) \times U(2)\right)$, $H^{\mathbb{C}} = S\left(GL(1) \times GL(2)\right)$ and $\mathfrak{m}^{\mathbb{C}}$ consists of matrices $\begin{pmatrix} 0 & x_1 & x_2 \\ x_3 & 0 & 0 \\ x_4 & 0 & 0 \end{pmatrix}$, $x_j \in \mathbb{C}$.

SU(1,2) Higgs bundle, stability and spectral data

An SU(1,2) Higgs bundle is a rank three Higgs bundle

$$\left(L \oplus F, \Phi = \begin{pmatrix} 0 & \gamma \\ \beta & 0 \end{pmatrix}\right)$$

$\mathrm{rank} L = 1$, $\mathrm{rank} F = 2$, $L = \det F^*$, $\beta : L \to F \otimes K$, $\gamma : F \to LK$.
Alternatively the data is contained in the triple of holomorphic objects (F, β, γ):

$$LK^{-1} \xrightarrow{\beta} F \xrightarrow{\gamma} LK$$

SU(1,2) Higgs bundle, stability and spectral data

An SU(1,2) Higgs bundle is a rank three Higgs bundle

$$\left(L \oplus F, \Phi = \begin{pmatrix} 0 & \gamma \\ \beta & 0 \end{pmatrix}\right)$$

rank$L = 1$, rank$F = 2$, $L = \det F^*$, $\beta : L \to F \otimes K$, $\gamma : F \to LK$.
Alternatively the data is contained in the triple of holomorphic objects (F, β, γ):

$$LK^{-1} \xrightarrow{\beta} F \xrightarrow{\gamma} LK$$

The Hitchin equation for hermitian metric h on F,

$$R(h) + \beta \wedge \beta_h^{\dagger} + \gamma_h^{\dagger} \wedge \gamma = 0$$

By 'decoupled equation' we mean:

$$R(h) = 0, \;\; \beta \wedge \beta_h^{\dagger} + \gamma_h^{\dagger} \wedge \gamma = 0$$

SU(1,2) Higgs bundle, stability and spectral data

G-Higgs bundles (principal-$H^{\mathbb{C}}$ bundles) over X are topologically classified by a characteristic class in $\pi_1(H^{\mathbb{C}}) = \pi_1(H) = \pi_1(G)$. In case of $G =$U(p,q), SU(p,q) this is given by the Toledo invariant, which is constant on connected components of the moduli space. For SU(1,2) Higgs bundle the Toledo invariant is $2 \deg L = -2 \deg F$.

SU(1,2) Higgs bundle, stability and spectral data

G-Higgs bundles (principal-$H^{\mathbb{C}}$ bundles) over X are topologically classified by a characteristic class in $\pi_1(H^{\mathbb{C}}) = \pi_1(H) = \pi_1(G)$. In case of $G =$U(p,q), SU(p,q) this is given by the Toledo invariant, which is constant on connected components of the moduli space. For SU(1,2) Higgs bundle the Toledo invariant is $2\deg L = -2\deg F$.

By direct generalization of the result of $G =$SL(2,$\mathbb{R}$) the Toledo invariant satisfies a Milnor-Wood type inequality (Domic & Toledo '87):

$$|2d| \leq \text{rank}(G/H)(g-1) = 2(g-1)$$

In fact by work of Bradlow, Garcia-Prada & Gothen '03, the stability and polystability of SU(p,q) Higgs bundle agrees with that of underlying SL($p+q,\mathbb{C}$)-Higgs bundle, therefore give further information of β, γ we may enhance the Toledo inequality for SU(1,2) Higgs bundle!

SU(1,2) Higgs bundle, stability and spectral data

The data of (F, β, γ) in $\mathcal{B}$ is determined by $q = \gamma \circ \beta$ which we assume has **simple zeros** $D = \left\{p_1, \ldots, p_{4g-4}\right\}$. For each $p \in D$,

SU(1,2) Higgs bundle, stability and spectral data

The data of (F, β, γ) in $\mathcal{B}$ is determined by $q = \gamma \circ \beta$ which we assume has **simple zeros** $D = \left\{p_1, \ldots, p_{4g-4}\right\}$. For each $p \in D$,

- either $\beta(p) = 0$, denote $p \in D_\beta$, $d_\beta := |D_\beta|$
- or $\gamma(p) = 0$, denote $p \in D_\gamma$, $d_\gamma := |D_\gamma|$
- or neither, denote $p \in D_r$, $d_r := |D_r|$

SU(1,2) Higgs bundle, stability and spectral data

The data of (F, β, γ) in $\mathcal{B}$ is determined by $q = \gamma \circ \beta$ which we assume has **simple zeros** $D = \{p_1, \ldots, p_{4g-4}\}$. For each $p \in D$,

- either $\beta(p) = 0$, denote $p \in D_\beta$, $d_\beta := |D_\beta|$
- or $\gamma(p) = 0$, denote $p \in D_\gamma$, $d_\gamma := |D_\gamma|$
- or neither, denote $p \in D_r$, $d_r := |D_r|$

SU(1,2) Higgs bundle is (semi)stable iff Φ-invariant proper subbundles of the form $L' \oplus F' \subset E = L \oplus F$ has less slope. Equivalently:

SU(1,2) Higgs bundle, stability and spectral data

The data of (F, β, γ) in $\mathcal{B}$ is determined by $q = \gamma \circ \beta$ which we assume has **simple zeros** $D = \{p_1, \dots, p_{4g-4}\}$. For each $p \in D$,

- either $\beta(p) = 0$, denote $p \in D_\beta$, $d_\beta := |D_\beta|$
- or $\gamma(p) = 0$, denote $p \in D_\gamma$, $d_\gamma := |D_\gamma|$
- or neither, denote $p \in D_r$, $d_r := |D_r|$

SU(1,2) Higgs bundle is (semi)stable iff Φ-invariant proper subbundles of the form $L' \oplus F' \subset E = L \oplus F$ has less slope. Equivalently:

Proposition

(F, β, γ) is (semi)stable iff

$$-(g-1) + \frac{d_\gamma}{2} < (\leq) \deg L < (\leq)(g-1) - \frac{d_\beta}{2}$$

and polystable iff either stable or $F \cong LK^{-1}(D_\beta) \oplus LK(-D_\gamma)$.

SU(1,2) Higgs bundle, stability and spectral data

The data of (F, β, γ) in $\mathcal{B}$ is determined by $q = \gamma \circ \beta$ which we assume has **simple zeros** $D = \{p_1, \ldots, p_{4g-4}\}$. For each $p \in D$,

- either $\beta(p) = 0$, denote $p \in D_\beta$, $d_\beta := |D_\beta|$
- or $\gamma(p) = 0$, denote $p \in D_\gamma$, $d_\gamma := |D_\gamma|$
- or neither, denote $p \in D_r$, $d_r := |D_r|$

SU(1,2) Higgs bundle is (semi)stable iff Φ-invariant proper subbundles of the form $L' \oplus F' \subset E = L \oplus F$ has less slope. Equivalently:

Proposition

(F, β, γ) is (semi)stable iff

$$-(g-1) + \frac{d_\gamma}{2} < (\leq) \deg L < (\leq)(g-1) - \frac{d_\beta}{2}$$

and polystable iff either stable or $F \cong LK^{-1}(D_\beta) \oplus LK(-D_\gamma)$.

We will also call partition $\underline{D}$ (semi)stable.

SU(1,2) Higgs bundle, stability and spectral data

Given a stable Higgs bundle $(\mathcal{E}, \Phi)$, the pair $(\mathcal{E}, t\Phi)$ for $t \in \mathbb{C}^\times$ is also stable $\rightsquigarrow$ a $\mathbb{C}^\times$-action on $\mathcal{M}_{\text{Higgs}}$. For $|t| = 1$, this gives a S^1-action whose fixed points are the critical points of the Morse function $(A, \Phi) \mapsto ||\Phi||^2_{L^2}$ and was used by Hitchin '87 to study topology of $\mathcal{M}_{\text{Higgs}}$.

SU(1,2) Higgs bundle, stability and spectral data

Given a stable Higgs bundle $(\mathcal{E}, \Phi)$, the pair $(\mathcal{E}, t\Phi)$ for $t \in \mathbb{C}^\times$ is also stable $\rightsquigarrow$ a $\mathbb{C}^\times$-action on $\mathcal{M}_{\text{Higgs}}$. For $|t| = 1$, this gives a S^1-action whose fixed points are the critical points of the Morse function $(A, \Phi) \mapsto ||\Phi||^2_{L^2}$ and was used by Hitchin '87 to study topology of $\mathcal{M}_{\text{Higgs}}$.

On the other hand for $t \in \mathbb{R}_+$ we will be focusing on the limiting behavior of unique solution h_t as $t \to \infty$.

SU(1,2) Higgs bundle, stability and spectral data

Given a stable Higgs bundle $(\mathcal{E}, \Phi)$, the pair $(\mathcal{E}, t\Phi)$ for $t \in \mathbb{C}^\times$ is also stable $\rightsquigarrow$ a $\mathbb{C}^\times$-action on $\mathcal{M}_{\text{Higgs}}$. For $|t| = 1$, this gives a S^1-action whose fixed points are the critical points of the Morse function $(A, \Phi) \mapsto ||\Phi||^2_{L^2}$ and was used by Hitchin '87 to study topology of $\mathcal{M}_{\text{Higgs}}$.

On the other hand for $t \in \mathbb{R}_+$ we will be focusing on the limiting behavior of unique solution h_t as $t \to \infty$.

Goal

Given family of stable SU(1,2) Higgs bundles $(F, t\beta, t\gamma)$, study behavior of unique solution h_t, a hermian metric on the fixed rank two bundle F as $t \to \infty$.

Limiting configuration

Theorem (Mazzeo-Swoboda-Weiss-Witt '14)

$(\mathcal{E}, t\Phi)$ family of SL(2,$\mathbb{C}$) Higgs bundle, $q = \det \Phi$ has *simple zeros* D. We have uniform convergence

$$h_t \to h_\infty \text{ as } t \to \infty$$

where h_∞ solves decoupled equation

$$R(h) = 0, \ [\Phi \wedge \Phi_h^\dagger] = 0$$

Conversely any solution to decoupled equation arise as such a limit.

Limiting configuration

Theorem (Mazzeo-Swoboda-Weiss-Witt '14)

$(\mathcal{E}, t\Phi)$ family of $SL(2,\mathbb{C})$ Higgs bundle, $q = \det \Phi$ has *simple zeros* D. We have uniform convergence

$$h_t \to h_\infty \text{ as } t \to \infty$$

where h_∞ solves decoupled equation

$$R(h) = 0, \; [\Phi \wedge \Phi_h^\dagger] = 0$$

Conversely any solution to decoupled equation arise as such a limit.

In particular, the moduli of limiting configuration is a torus of $\dim_{\mathbb{R}} = 6g - 6$.

Limiting configuration

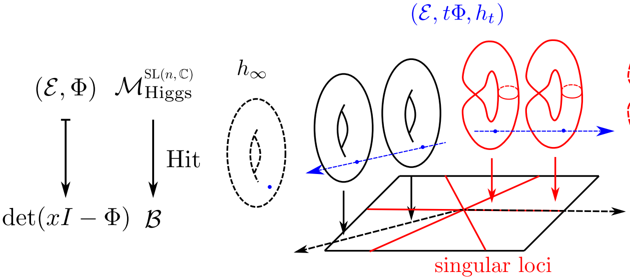
- Collier & Li (2014) describes the limiting behavior for cyclic $SL(n,\mathbb{R})$-Higgs bundle;

Limiting configuration

- Collier & Li (2014) describes the limiting behavior for cyclic SL(n,$\mathbb{R}$)-Higgs bundle;
- Mochizuki (2016): general case for SL(2,$\mathbb{C}$), reduce to $\Sigma = \mathrm{Im}(\omega) \cup \mathrm{Im}(-\omega)$ with $q = -\omega^2$; Fredrickson (2018): extends to SL(n,$\mathbb{C}$) Higgs bundle where ramification index of Σ is at most 2

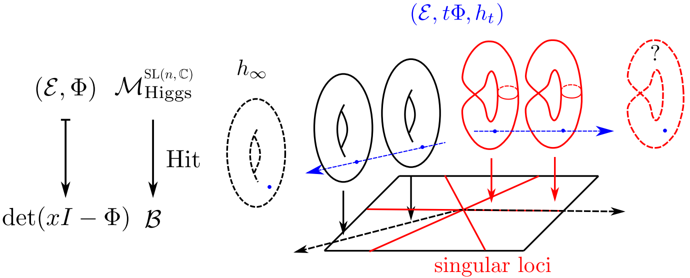
Limiting configuration

- Collier & Li (2014) describes the limiting behavior for cyclic SL(n,$\mathbb{R}$)-Higgs bundle;
- Mochizuki (2016): general case for SL(2,$\mathbb{C}$), reduce to $\Sigma = \mathrm{Im}(\omega) \cup \mathrm{Im}(-\omega)$ with $q = -\omega^2$; Fredrickson (2018): extends to SL(n,$\mathbb{C}$) Higgs bundle where ramification index of Σ is at most 2



Limiting configuration

- Collier & Li (2014) describes the limiting behavior for cyclic SL(n,$\mathbb{R}$)-Higgs bundle;
- Mochizuki (2016): general case for SL(2,$\mathbb{C}$), reduce to $\Sigma = \mathrm{Im}(\omega) \cup \mathrm{Im}(-\omega)$ with $q = -\omega^2$; Fredrickson (2018): extends to SL(n,$\mathbb{C}$) Higgs bundle where ramification index of Σ is at most 2



- To study the space of limiting config for SU(1,2) Higgs bundle, it is necessary to get a better description of the Hitchin fiber / spectral data.

SU(1,2) Higgs bundle, stability and spectral data

Given stable SU(1,2) Higgs bundle (F, β, γ)

Idea: transform data in $LK^{-1} \xrightarrow{\beta} F \xrightarrow{\gamma} LK$, to get F as a Hecke modification of fixed bundle $V = L^{-2}K \oplus LK$.

SU(1,2) Higgs bundle, stability and spectral data

Given stable SU(1,2) Higgs bundle (F,β,γ)

Idea: transform data in $LK^{-1} \xrightarrow{\beta} F \xrightarrow{\gamma} LK$, to get F as a Hecke modification of fixed bundle $V = L^{-2}K \oplus LK$.

For this, we may first dualize to get $F^* \xrightarrow{\beta^t} L^{-1}K$ then use the canonical map $\phi_F : F^* \otimes \Lambda^2 F \to F$, which is an isom for rank two bundle F, to get composition

$$\iota_1 : F \xrightarrow{\phi_F^{-1}} F^* \otimes \Lambda^2 F = F^* \otimes L^{-1} \xrightarrow{\beta^t \otimes 1} L^{-1}K \otimes L^{-1} = L^{-2}K$$

SU(1,2) Higgs bundle, stability and spectral data

Given stable SU(1,2) Higgs bundle (F, β, γ)

Idea: transform data in $LK^{-1} \xrightarrow{\beta} F \xrightarrow{\gamma} LK$, to get F as a Hecke modification of fixed bundle $V = L^{-2}K \oplus LK$.

For this, we may first dualize to get $F^* \xrightarrow{\beta^t} L^{-1}K$ then use the canonical map $\phi_F : F^* \otimes \Lambda^2 F \to F$, which is an isom for rank two bundle F, to get composition

$$\iota_1 : F \xrightarrow{\phi_F^{-1}} F^* \otimes \Lambda^2 F = F^* \otimes L^{-1} \xrightarrow{\beta^t \otimes 1} L^{-1}K \otimes L^{-1} = L^{-2}K$$

and let $\iota_2 = \gamma : F \to LK$. This gives an SES of coherent sheaves on X:

$$0 \to F \xrightarrow{\iota = \iota_1 \oplus \iota_2} V \xrightarrow{\pi} \mathcal{O}_D \to 0$$

with $\mathcal{O}_D$ skyscraper sheaf of length 1 at $p \in D$.

SU(1,2) Higgs bundle, stability and spectral data

It turns out that the above idea can be used to construct a family of SU(1,2) Higgs bundles parametrized by a fiber bundle over the Jacobian variety with **local universal property**.

SU(1,2) Higgs bundle, stability and spectral data

It turns out that the above idea can be used to construct a family of SU(1,2) Higgs bundles parametrized by a fiber bundle over the Jacobian variety with **local universal property**.

Fix $|d| < g - 1$ and let $\mathscr{L}$ be a Poincaré line bundle over $X \times \mathrm{Pic}^d X$ and $\mathcal{V} = \mathscr{L}^{-2}\mathrm{pr}_X^* K \oplus \mathscr{L}\mathrm{pr}_X^* K$ and let $\iota_x : \mathrm{Pic}^d X \to X \times \mathrm{Pic}^d X$ be $\ell \mapsto (x, \ell)$, the fiber bundle is given by

$$\mathcal{P} = \mathcal{P}_1 \underset{\mathrm{Pic}^d X}{\times} \dots \underset{\mathrm{Pic}^d X}{\times} \mathcal{P}_{4g-4}, \text{ where } \mathcal{P}_j = \mathbb{P}\left(\iota_{x_j}^* \mathcal{V}^*\right).$$

with fiber $\cong \left(\mathbb{P}^1\right)^{4g-4}$. Since $\mathcal{V}$ is decomposable, there is a well-defined $\mathbb{C}^\times$-action given fiber-wise by $[x_0 : x_1] \mapsto [cx_0 : x_1]$, and

SU(1,2) Higgs bundle, stability and spectral data

It turns out that the above idea can be used to construct a family of SU(1,2) Higgs bundles parametrized by a fiber bundle over the Jacobian variety with **local universal property**.

Fix $|d| < g-1$ and let $\mathscr{L}$ be a Poincaré line bundle over $X \times \mathrm{Pic}^d X$ and $\mathcal{V} = \mathscr{L}^{-2}\mathrm{pr}_X^* K \oplus \mathscr{L}\mathrm{pr}_X^* K$ and let $\iota_x : \mathrm{Pic}^d X \to X \times \mathrm{Pic}^d X$ be $\ell \mapsto (x, \ell)$, the fiber bundle is given by

$$\mathcal{P} = \mathcal{P}_1 \underset{\mathrm{Pic}^d X}{\times} \cdots \underset{\mathrm{Pic}^d X}{\times} \mathcal{P}_{4g-4}, \text{ where } \mathcal{P}_j = \mathbb{P}\left(\iota_{x_j}^* \mathcal{V}^*\right).$$

with fiber $\cong \left(\mathbb{P}^1\right)^{4g-4}$. Since $\mathcal{V}$ is decomposable, there is a well-defined $\mathbb{C}^\times$-action given fiber-wise by $[x_0 : x_1] \mapsto [cx_0 : x_1]$, and

Theorem (N '21)

For $|d| < g-1$, $q \in H^0(X, K_X^2)$ with simple zeros $D = x_1 + \ldots + x_{4g-4}$, $\mathcal{P}$ and $\mathbb{C}^\times$-action as above, there is an appropriate linearization using an ample line bundle $\mathscr{L}'$ over $\mathcal{P}$ such that

$$\mathrm{Hit}_d^{-1}(q) \cong \mathcal{P} /\!/ \mathbb{C}^\times$$

SU(1,2) Higgs bundle, stability and spectral data

Given a stable SU(1,2) Higgs bundle (F, β, γ) corresponding to some point $\underline{\ell} \in \mathcal{P}$ in the fiber $\mathbb{P}\left(\mathcal{V}^*_{x_1}\right) \times \ldots \times \mathbb{P}\left(\mathcal{V}^*_{x_{4g-4}}\right)$ over $L \in \mathrm{Pic}^d X$. Write $\underline{\ell} = (\ell_1, \ldots, \ell_{4g-4})$ with $\ell_j \in \left(\mathcal{V}^*_{x_j}\right) \cong \mathbb{P}^1$, we have a correspondence

SU(1,2) Higgs bundle, stability and spectral data

Given a stable SU(1,2) Higgs bundle (F, β, γ) corresponding to some point $\underline{\ell} \in \mathcal{P}$ in the fiber $\mathbb{P}\left(\mathcal{V}^*_{x_1}\right) \times \ldots \times \mathbb{P}\left(\mathcal{V}^*_{x_{4g-4}}\right)$ over $L \in \mathrm{Pic}^d X$. Write $\underline{\ell} = (\ell_1, \ldots, \ell_{4g-4})$ with $\ell_j \in \left(\mathcal{V}^*_{x_j}\right) \cong \mathbb{P}^1$, we have a correspondence

- $\ell_j = [0:1]$ iff $\iota : F \to V = L^{-2}K \oplus LK$ at p_j lands in second summand, iff $\beta(p_j) = 0$
- $\ell_j = [1:0]$ iff $\iota : F \to V = L^{-2}K \oplus LK$ at p_j lands in first summand, iff $\gamma(p_j) = 0$
- $\ell_j \neq [0:1], [1:0]$ iff $\iota : F \to V = L^{-2}K \oplus LK$ at p_j does not land in either axes, iff $\beta(p_j), \gamma(p_j) \neq 0$.

SU(1,2) Higgs bundle, stability and spectral data

Given a stable SU(1,2) Higgs bundle (F, β, γ) corresponding to some point $\underline{\ell} \in \mathcal{P}$ in the fiber $\mathbb{P}\left(\mathcal{V}^*_{x_1}\right) \times \ldots \times \mathbb{P}\left(\mathcal{V}^*_{x_{4g-4}}\right)$ over $L \in \mathrm{Pic}^d X$. Write $\underline{\ell} = (\ell_1, \ldots, \ell_{4g-4})$ with $\ell_j \in \left(\mathcal{V}^*_{x_j}\right) \cong \mathbb{P}^1$, we have a correspondence

- $\ell_j = [0:1]$ iff $\iota : F \to V = L^{-2}K \oplus LK$ at p_j lands in second summand, iff $\beta(p_j) = 0$
- $\ell_j = [1:0]$ iff $\iota : F \to V = L^{-2}K \oplus LK$ at p_j lands in first summand, iff $\gamma(p_j) = 0$
- $\ell_j \neq [0:1], [1:0]$ iff $\iota : F \to V = L^{-2}K \oplus LK$ at p_j does not land in either axes, iff $\beta(p_j), \gamma(p_j) \neq 0$.

Let $n_1(\underline{\ell})$ (resp. $n_2(\underline{\ell})$) be the number of $[0:1]$ (resp. $[1:0]$) in its components, the stable locus in $\mathrm{Hit}^{-1}(q)$ is then given by the $\mathbb{C}^\times$ quotient of

$$\mathcal{Y} = \left\{ \underline{\ell} = (\ell_1, \ldots, \ell_{4g-4}) \in \mathcal{P} \;\middle|\; \begin{array}{l} \ell_j \in \mathcal{P}_j, \\ n_1(\underline{\ell}) < 2(g-1+d), \\ n_2(\underline{\ell}) < 2(g-1-d), \end{array} \right\}$$

SU(1,2) Higgs bundle, stability and spectral data

The partition of D into three cases (zero of β; zero of γ; neither) then gives a stratification

$$\mathcal{Y} = \coprod_{\underline{D} \text{ stable}} \mathcal{Y}_{\underline{D}},$$

SU(1,2) Higgs bundle, stability and spectral data

The partition of D into three cases (zero of β; zero of γ; neither) then gives a stratification

$$\mathcal{Y} = \coprod_{\underline{D}\text{ stable}} \mathcal{Y}_{\underline{D}}\,,$$

Each stratum is of the form $(\mathbb{C}^{\times})^{d_r}$:

$$\mathcal{Y}_{\underline{D}}\Big|_L := \mathcal{F}_{L,D_r} \cong \prod_{x\in D_r} \left(L^3\big|_x\right)^{\times} = \left\{\underline{b} = (b_x) \,\middle|\, b_x \in L^3\big|_x - \{0\}\,,\ x \in D_r\right\}.$$

In the following we fix coord nbhd $(D_j;\zeta_j)$ with $p_j \in D_r$ and $q = \zeta_j\,(\zeta_j)^2$.

SU(1,2) Higgs bundle, stability and spectral data

The partition of D into three cases (zero of β; zero of γ; neither) then gives a stratification

$$\mathcal{Y} = \coprod_{\underline{D} \text{ stable}} \mathcal{Y}_{\underline{D}}\,,$$

Each stratum is of the form $(\mathbb{C}^\times)^{d_r}$:

$$\mathcal{Y}_{\underline{D}}\big|_L := \mathcal{F}_{L,D_r} \cong \prod_{x\in D_r} \left(L^3\big|_x\right)^\times = \left\{\underline{b} = (b_x)\big|\ b_x \in L^3\big|_x - \{0\}\,,\ x \in D_r\right\}.$$

In the following we fix coord nbhd $(D_j; \zeta_j)$ with $p_j \in D_r$ and $q = \zeta_j\,(\zeta_j)^2$.

Corollary

The stable Higgs bundle (F, β, γ) correspond to $\underline{b} \in \mathcal{F}_{L,D_r}$ iff there are compatible hol'c frames $s_{0,j}$ of $L = \det F^*$ and $\left\{s_{1,j}, s_{2,j}\right\}$ of F over D_j with

- $s_{0,j}^{\otimes 3} = b_{p_j}$ for $p_j \in D_r$
- under such frames $\beta = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ \zeta_j\end{pmatrix} d\zeta_j$ and $\gamma = \frac{1}{\sqrt{2}}\begin{pmatrix}\zeta_j & 1\end{pmatrix} d\zeta_j$.

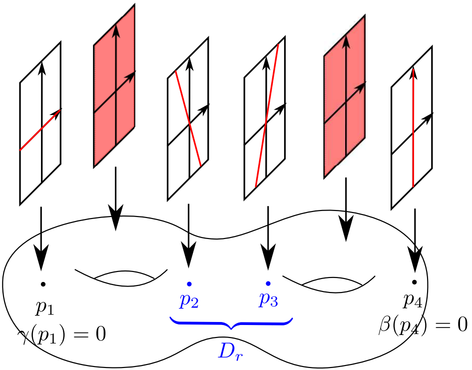
Hecke transformation representation

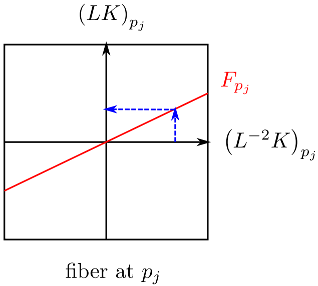
The tuple $\underline{b}$ specifies the subsheaf by matching the two summands in $V = L^{-2}K \oplus LK$ over D_r, will be called **fiber-matching parameters**. These also give (up to equivalence) holomorphic frames in which Higgs field has standard form

Hecke transformation representation

The tuple $\underline{b}$ specifies the subsheaf by matching the two summands in $V = L^{-2}K \oplus LK$ over D_r, will be called **fiber-matching parameters**. These also give (up to equivalence) holomorphic frames in which Higgs field has standard form

$$F \subset V = L^{-2}K \oplus LK$$





Parabolic weights

Recall notion of **filtered bundle**. Let $\mathcal{O}_X(*D)$ be sheaf of meromorphic functions with possibly poles at D and $\mathcal{E}$ a locally free $\mathcal{O}_X(*D)$ module of finite rank. A filtered bundle structure $\mathcal{P}_*\mathcal{E}$ on $\mathcal{E}$ is a family of coherent $\mathcal{O}_X$-submod of $\mathcal{E}$ labelled by tuple of real numbers $\underline{a}$,

Parabolic weights

Recall notion of **filtered bundle**. Let $\mathcal{O}_X(*D)$ be sheaf of meromorphic functions with possibly poles at D and $\mathcal{E}$ a locally free $\mathcal{O}_X(*D)$ module of finite rank. A filtered bundle structure $\mathcal{P}_*\mathcal{E}$ on $\mathcal{E}$ is a family of coherent $\mathcal{O}_X$-submod of $\mathcal{E}$ labelled by tuple of real numbers $\underline{a}$,

- $\mathcal{P}_{\underline{a}}\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{O}_X(*D) = \mathcal{E}$.

Parabolic weights

Recall notion of **filtered bundle**. Let $\mathcal{O}_X(*D)$ be sheaf of meromorphic functions with possibly poles at D and $\mathcal{E}$ a locally free $\mathcal{O}_X(*D)$ module of finite rank. A filtered bundle structure $\mathcal{P}_*\mathcal{E}$ on $\mathcal{E}$ is a family of coherent $\mathcal{O}_X$-submod of $\mathcal{E}$ labelled by tuple of real numbers $\underline{a}$,

- $\mathcal{P}_{\underline{a}}\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{O}_X(*D) = \mathcal{E}$.
- Stalk $\mathcal{P}_{a_P}\mathcal{E}_P$ of $\mathcal{P}_{\underline{a}}\mathcal{E}$ at $p \in D$ depends only on $a_P \in \mathbb{R}$.

Parabolic weights

Recall notion of **filtered bundle**. Let $\mathcal{O}_X(*D)$ be sheaf of meromorphic functions with possibly poles at D and $\mathcal{E}$ a locally free $\mathcal{O}_X(*D)$ module of finite rank. A filtered bundle structure $\mathcal{P}_*\mathcal{E}$ on $\mathcal{E}$ is a family of coherent $\mathcal{O}_X$-submod of $\mathcal{E}$ labelled by tuple of real numbers $\underline{a}$,

- $\mathcal{P}_{\underline{a}}\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{O}_X(*D) = \mathcal{E}$.
- Stalk $\mathcal{P}_{a_P}\mathcal{E}_P$ of $\mathcal{P}_{\underline{a}}\mathcal{E}$ at $p \in D$ depends only on $a_P \in \mathbb{R}$.
- $\mathcal{P}_a\mathcal{E}_P \subset \mathcal{P}_b\mathcal{E}_P$ iff $a \leq b$ and there is $\epsilon > 0$ such that $\mathcal{P}_a\mathcal{E}_P = \mathcal{P}_{a+\epsilon}\mathcal{E}_P$.

Parabolic weights

Recall notion of **filtered bundle**. Let $\mathcal{O}_X(*D)$ be sheaf of meromorphic functions with possibly poles at D and $\mathcal{E}$ a locally free $\mathcal{O}_X(*D)$ module of finite rank. A filtered bundle structure $\mathcal{P}_*\mathcal{E}$ on $\mathcal{E}$ is a family of coherent $\mathcal{O}_X$-submod of $\mathcal{E}$ labelled by tuple of real numbers $\underline{a}$,

- $\mathcal{P}_{\underline{a}}\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{O}_X(*D) = \mathcal{E}$.
- Stalk $\mathcal{P}_{a_P}\mathcal{E}_P$ of $\mathcal{P}_{\underline{a}}\mathcal{E}$ at $p \in D$ depends only on $a_P \in \mathbb{R}$.
- $\mathcal{P}_a\mathcal{E}_P \subset \mathcal{P}_b\mathcal{E}_P$ iff $a \leq b$ and there is $\epsilon > 0$ such that $\mathcal{P}_a\mathcal{E}_P = \mathcal{P}_{a+\epsilon}\mathcal{E}_P$.
- For $n \in \mathbb{Z}$, $\mathcal{P}_a\mathcal{E}_P \otimes_{\mathcal{O}_{X,P}} \mathcal{O}_X(nP)_P = \mathcal{P}_{a+n}\mathcal{E}_P$.

Parabolic weights

A hermitian metric h is **adapted** to $\mathcal{P}_*\mathcal{E}$ if

$$s \in \mathcal{P}_{\underline{a}}\mathcal{E} \Leftrightarrow |s|_h = O\left(|z_P|^{-a_P-\epsilon}\right) \ \forall \epsilon > 0$$

where (U, z_P) centered at $P \in D$, s a holc. section of $\mathcal{E}$. Below essentially follow from Hodge theory (later we will see an explicit construction) :

Parabolic weights

A hermitian metric h is **adapted** to $\mathcal{P}_*\mathcal{E}$ if

$$s \in \mathcal{P}_{\underline{a}}\mathcal{E} \Leftrightarrow |s|_h = O(|z_P|^{-a_P-\epsilon}) \ \forall \epsilon > 0$$

where (U, z_P) centered at $P \in D$, s a holc. section of $\mathcal{E}$. Below essentially follow from Hodge theory (later we will see an explicit construction) :

Fact

$\mathcal{L}$ a line bundle. For $\deg \mathcal{L} + \sum_j \lambda_j = 0$, $\exists$ herm. metric h on $X - D$ with $R(h) = 0$ and for any $s_j(p_j) \neq 0$

$$\log |s_j|_h = \lambda_j \log |\zeta_j| + O(1)$$

h is unique up to positive const, called the **harmonic metric** adapted to **parabolic line bundle** $(\mathcal{L}, \underline{\lambda})$.

Decoupled equation

Decoupled solution: a solution to decoupled equation with logarithmic Chern conn.: $\partial_h : F \to F \otimes \Omega_X^{(1,0)}(\log D)$.

Decoupled equation

Decoupled solution: a solution to decoupled equation with logarithmic Chern conn.: $\partial_h : F \to F \otimes \Omega_X^{(1,0)}(\log D)$.

Lemma

Any decoupled solution have the form

$$h_\infty = \iota^* \left(h_L^{-2} h_K \oplus h_L h_K \right)$$

where $\iota : F \to L^{-2}K \oplus LK$ the Hecke transf. as above, $|q|_{h_K} \equiv 1$ and h_L is a harmonic metric adapted to $(L, \underline{\lambda})$.

Decoupled equation

Decoupled solution: a solution to decoupled equation with logarithmic Chern conn.: $\partial_h : F \to F \otimes \Omega_X^{(1,0)}(\log D)$.

Lemma

Any decoupled solution have the form

$$h_\infty = \iota^* \left(h_L^{-2} h_K \oplus h_L h_K \right)$$

where $\iota : F \to L^{-2}K \oplus LK$ the Hecke transf. as above, $|q|_{h_K} \equiv 1$ and h_L is a harmonic metric adapted to $(L, \underline{\lambda})$.

Let $\underline{D}$ be stable. It will be useful to define a tuple $\underline{\lambda}$ to be **admissible** if

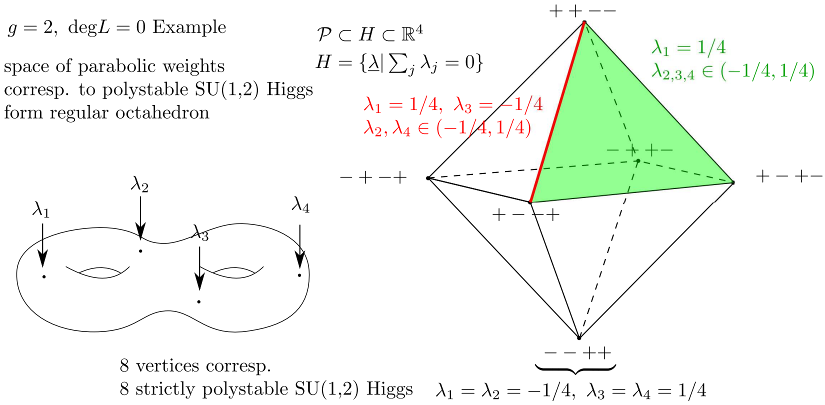
- $\lambda_j = 1/4$ for $p_j \in D_\beta$
- $\lambda_j = -1/4$ for $p_j \in D_\gamma$
- $-1/4 < \lambda_j < 1/4$ for $p_j \in D_r$.

Admissible parabolic weights

Let $\mathcal{P}_{\underline{D}}$ be the space of admissible weights corresp to $\underline{D}$. The union over all semistable $\underline{D}$ gives a polytope in a hyperplane $H \subset \mathbb{R}^{4g-4}$.

Admissible parabolic weights

Let $\mathcal{P}_{\underline{D}}$ be the space of admissible weights corresp to $\underline{D}$. The union over all semistable $\underline{D}$ gives a polytope in a hyperplane $H \subset \mathbb{R}^{4g-4}$.



Limiting configuration of SU(1,2) Hitchin equation

Let (F, β, γ) be stable SU(1,2) Higgs bundle and $\iota : F \to V = L^{-2}K \oplus LK$ the corresp Hecke transf. Let $S_a = \mathrm{diag}(a^{-2}, a)$ endom. on V hence F outside D. Fix a base point $p_0 \in X - D$ and $v_0 \in L|_{p_0}$.

Limiting configuration of SU(1,2) Hitchin equation

Let (F, β, γ) be stable SU(1,2) Higgs bundle and $\iota : F \to V = L^{-2}K \oplus LK$ the corresp Hecke transf. Let $S_a = \mathrm{diag}(a^{-2}, a)$ endom. on V hence F outside D. Fix a base point $p_0 \in X - D$ and $v_0 \in L|_{p_0}$.

Theorem 2 (N '21)

On any compact subset of $X - D$ we have uniform convergence

$$S^*_{|v_0|_{\det h_t^{-1}}} h_t \to h_\infty \text{ as } t \to \infty$$

where $h_\infty = \iota^* \left(h_L^{-2} h_K \oplus h_L h_K \right)$, h_K satisfy $|q|_{h_K} \equiv 1$ and h_L is a harmonic metric adapted to $(L, \underline{\lambda_\infty})$ with $\underline{\lambda_\infty}$ admissible and

$$\lambda_{\infty,j} = -\frac{1}{d_r}\left(\deg L + \frac{1}{4}\left(d_\beta - d_\gamma\right)\right), \ \forall p_j \in D_r$$

Limiting configuration of SU(1,2) Hitchin equation

Let (F, β, γ) be stable SU(1,2) Higgs bundle and $\iota : F \to V = L^{-2}K \oplus LK$ the corresp Hecke transf. Let $S_a = \mathrm{diag}(a^{-2}, a)$ endom. on V hence F outside D. Fix a base point $p_0 \in X - D$ and $v_0 \in L|_{p_0}$.

Theorem 2 (N '21)

On any compact subset of $X - D$ we have uniform convergence

$$S^*_{|v_0|_{\det h_t^{-1}}} h_t \to h_\infty \text{ as } t \to \infty$$

where $h_\infty = \iota^*\left(h_L^{-2} h_K \oplus h_L h_K\right)$, h_K satisfy $|q|_{h_K} \equiv 1$ and h_L is a harmonic metric adapted to $(L, \underline{\lambda_\infty})$ with $\underline{\lambda_\infty}$ admissible and

$$\lambda_{\infty,j} = -\frac{1}{d_r}\left(\deg L + \frac{1}{4}\left(d_\beta - d_\gamma\right)\right), \ \forall p_j \in D_r$$

The parabolic weight characterizing limiting configurations are in the barycenter of the simplex corresponding to $\underline{D}$.

Strategy of proof

Step 1 Construct local model solution with Higgs field in standard forms via good filtered Higgs bundle on $\mathbb{P}^1$ with parabolic weights at ∞ determined by λ_j. Stability of local model $\Leftrightarrow$ admissibility of $\underline{\lambda}$

Strategy of proof

Step 1 Construct local model solution with Higgs field in standard forms via good filtered Higgs bundle on $\mathbb{P}^1$ with parabolic weights at ∞ determined by λ_j. Stability of local model $\Leftrightarrow$ admissibility of $\underline{\lambda}$

Step 2 Glue local models in $|\zeta_j| \leq R/3$ to an decoupled solution of same weight in $|\zeta_j| > R$ by smooth interpolation on annuli. There is a global compatibility condition depending on t.

Strategy of proof

Step 1 Construct local model solution with Higgs field in standard forms via good filtered Higgs bundle on $\mathbb{P}^1$ with parabolic weights at ∞ determined by λ_j. Stability of local model $\Leftrightarrow$ admissibility of $\underline{\lambda}$

Step 2 Glue local models in $|\zeta_j| \leq R/3$ to an decoupled solution of same weight in $|\zeta_j| > R$ by smooth interpolation on annuli. There is a global compatibility condition depending on t.
For $t \gg 1$, there is t-compatible family $\underline{\lambda}(t) \to \underline{\lambda_\infty}$, giving approx. soln h_t^{app}

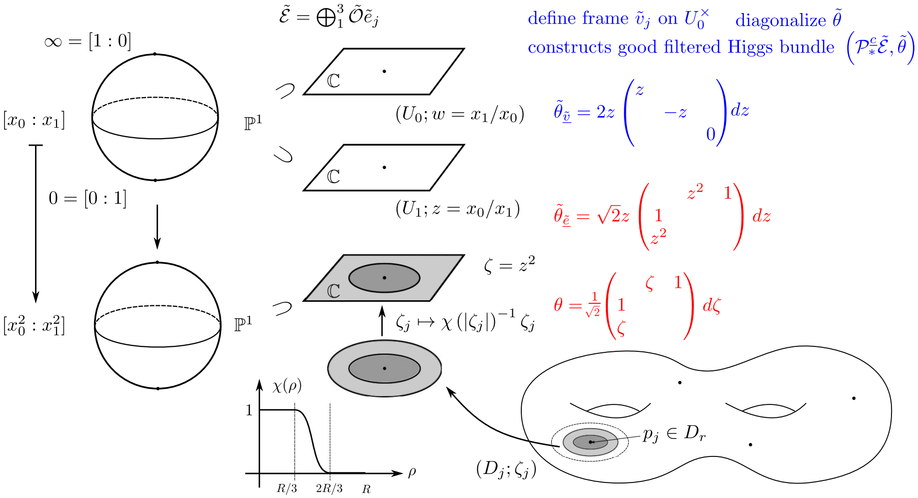
Strategy of proof

Step 1 Construct local model solution with Higgs field in standard forms via good filtered Higgs bundle on $\mathbb{P}^1$ with parabolic weights at ∞ determined by λ_j. Stability of local model $\Leftrightarrow$ admissibility of $\underline{\lambda}$

Step 2 Glue local models in $|\zeta_j| \leq R/3$ to an decoupled solution of same weight in $|\zeta_j| > R$ by smooth interpolation on annuli. There is a global compatibility condition depending on t.
For $t \gg 1$, there is t-compatible family $\underline{\lambda}(t) \to \underline{\lambda_\infty}$, giving approx. soln h_t^{app}

Step 3 Use small perturbation to find solution h_t nearby

Local model solution



Good filtered Higgs bundle and harmonic metric

A **filtered Higgs bundle** is a pair $(\mathcal{P}_*\mathcal{E}, \theta : \mathcal{E} \to \mathcal{E} \otimes \Omega^1_X)$. It is called **unramifiedly good** if there is a finite collection of germs of meromorphic functions at P, $\mathcal{I}(P) \subset \mathcal{O}_X(*D)_P$ and a decomposition $\mathcal{P}_a\mathcal{E}_P = \bigoplus_{f \in \mathcal{I}(P)} \mathcal{P}_a\mathcal{E}_{P,f}$ such that

Good filtered Higgs bundle and harmonic metric

A **filtered Higgs bundle** is a pair $(\mathcal{P}_*\mathcal{E}, \theta : \mathcal{E} \to \mathcal{E} \otimes \Omega^1_X)$. It is called **unramifiedly good** if there is a finite collection of germs of meromorphic functions at P, $\mathcal{I}(P) \subset \mathcal{O}_X(*D)_P$ and a decomposition $\mathcal{P}_a\mathcal{E}_P = \bigoplus_{f \in \mathcal{I}(P)} \mathcal{P}_a\mathcal{E}_{P,f}$ such that

$$(\theta - (df)\mathrm{Id})\,\mathcal{P}_a\mathcal{E}_{P,f} \subset \mathcal{P}_a\mathcal{E}_{P,f} \otimes \Omega^1_X(\log D)_P$$

Good filtered Higgs bundle and harmonic metric

A **filtered Higgs bundle** is a pair $(\mathcal{P}_*\mathcal{E}, \theta : \mathcal{E} \to \mathcal{E} \otimes \Omega^1_X)$. It is called **unramifiedly good** if there is a finite collection of germs of meromorphic functions at P, $\mathcal{I}(P) \subset \mathcal{O}_X(*D)_P$ and a decomposition $\mathcal{P}_a\mathcal{E}_P = \bigoplus_{f \in \mathcal{I}(P)} \mathcal{P}_a\mathcal{E}_{P,f}$ such that

$$(\theta - (df)\mathrm{Id})\,\mathcal{P}_a\mathcal{E}_{P,f} \subset \mathcal{P}_a\mathcal{E}_{P,f} \otimes \Omega^1_X(\log D)_P$$

Degree of a filtered bundle is defined by

$$\deg \mathcal{P}_*\mathcal{E} = \deg \mathcal{P}_{\underline{c}}\mathcal{E} - \sum_{P \in D} \sum_{c_P - 1 < a \leq c_P} a \dim_{\mathbb{C}} \left(\mathcal{P}_a\mathcal{E}_P / \mathcal{P}_{<a}\mathcal{E}_P\right) .$$

Stability of good filtered Higgs bundle is defined similarly to that of Higgs bundle with this notion of degree.

Good filtered Higgs bundle and harmonic metric

A **filtered Higgs bundle** is a pair $(\mathcal{P}_*\mathcal{E}, \theta : \mathcal{E} \to \mathcal{E} \otimes \Omega_X^1)$. It is called **unramifiedly good** if there is a finite collection of germs of meromorphic functions at P, $\mathcal{I}(P) \subset \mathcal{O}_X(*D)_P$ and a decomposition $\mathcal{P}_a\mathcal{E}_P = \bigoplus_{f \in \mathcal{I}(P)} \mathcal{P}_a\mathcal{E}_{P,f}$ such that

$$(\theta - (df)\mathrm{Id})\,\mathcal{P}_a\mathcal{E}_{P,f} \subset \mathcal{P}_a\mathcal{E}_{P,f} \otimes \Omega_X^1(\log D)_P$$

Degree of a filtered bundle is defined by

$$\deg \mathcal{P}_*\mathcal{E} = \deg \mathcal{P}_{\underline{c}}\mathcal{E} - \sum_{P \in D} \sum_{c_P - 1 < a \le c_P} a \dim_{\mathbb{C}} \left(\mathcal{P}_a\mathcal{E}_P / \mathcal{P}_{<a}\mathcal{E}_P\right).$$

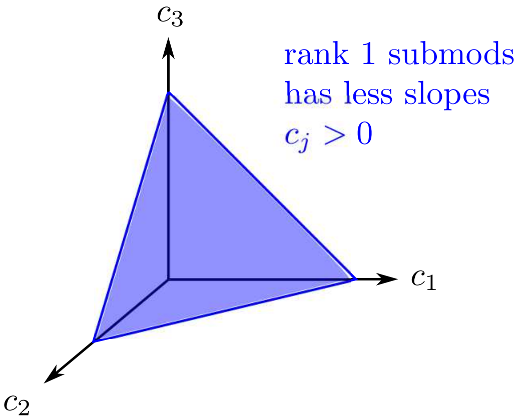
Stability of good filtered Higgs bundle is defined similarly to that of Higgs bundle with this notion of degree.

Theorem (Biqard-Boalch '04)

A stable unramifiedly good filtered Higgs bundle $(\mathcal{P}_*\mathcal{E}, \theta)$ with $\deg \mathcal{P}_*\mathcal{E} = 0$ admits a harmonic metric adapted to it. This metric is unique up to mult. by positive constant.

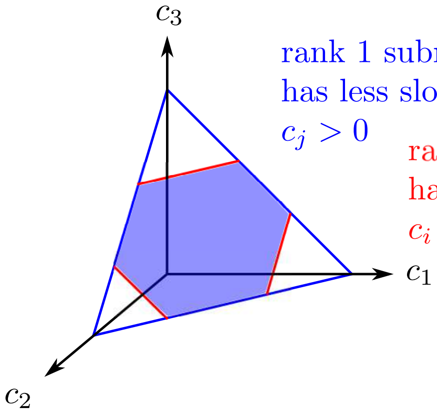
Admissibility and local model stability

$$\deg \mathcal{P}_* \mathcal{E} = 0 \Leftrightarrow c_1 + c_2 + c_3 = 3$$



Admissibility and local model stability

$$\deg \mathcal{P}_*\mathcal{E} = 0 \Leftrightarrow c_1 + c_2 + c_3 = 3$$

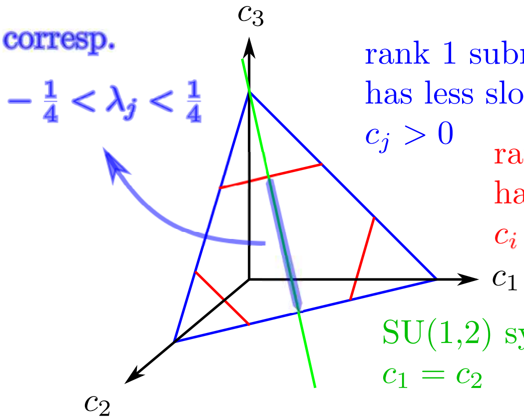


rank 1 submods
has less slopes
$c_j > 0$

rank 2 submods
has less slopes
$c_i + c_j > 1 \; i \neq j$

Admissibility and local model stability

$$\deg \mathcal{P}_*\mathcal{E} = 0 \Leftrightarrow c_1 + c_2 + c_3 = 3$$



rank 1 submods
has less slopes
$c_j > 0$

rank 2 submods
has less slopes
$c_i + c_j > 1 \; i \neq j$

SU(1,2) symmetry
$c_1 = c_2$

Local model symmetries

The weights $\underline{c} = (c_1, c_2, c_3)$ for which unramifiedly good filtered Higgs bundle $(\mathcal{P}^{\underline{c}}_* \mathcal{E}, \tilde{\theta})$ on $\mathbb{P}^1$ with at ∞ is stable with $\deg \mathcal{P}^{\underline{c}}_* \mathcal{E} = 0$ lies in a regular hexagon.

Local model symmetries

The weights $\underline{c} = (c_1, c_2, c_3)$ for which unramifiedly good filtered Higgs bundle $(\mathcal{P}_*^{\underline{c}}\mathcal{E}, \tilde{\theta})$ on $\mathbb{P}^1$ with at ∞ is stable with $\deg \mathcal{P}_*^{\underline{c}}\mathcal{E} = 0$ lies in a regular hexagon.
For such weight by Biquard-Boalch theorem there exists harmonic metric $\tilde{h}_t^{\underline{c}}$ uniq up to positive const.

Local model symmetries

The weights $\underline{c} = (c_1, c_2, c_3)$ for which unramifiedly good filtered Higgs bundle $(\mathcal{P}_*^{\underline{c}}\mathcal{E}, \tilde{\theta})$ on $\mathbb{P}^1$ with at ∞ is stable with $\deg \mathcal{P}_*^{\underline{c}}\mathcal{E} = 0$ lies in a regular hexagon.
For such weight by Biquard-Boalch theorem there exists harmonic metric $\tilde{h}_t^{\underline{c}}$ uniq up to positive const.
For $c_1 = c_2$ the harmonic metric respects $H^{\mathbb{C}}$-symmetry of the vector bundle.
Then $1/2 < c_1 < 3/2$. Normalize by $\det \tilde{h}_t^{\underline{c}} \equiv 1$. To get $\log\det \tilde{h}_t^{\underline{c}} = -2\lambda \log|\zeta| + O(1)$ at ∞ we have

$$\underline{c} = (1 + 2\lambda, 1 + 2\lambda, 1 - 4\lambda) \text{ with } \lambda \in (-1/4, 1/4)$$

where

$$-\frac{1}{4} < \lambda < \frac{1}{4}$$

Local model symmetries

Standard form of Higgs field

$$\tilde{\theta} = \sqrt{2}z \begin{pmatrix} & z^2 & 1 \\ 1 & & \\ z^2 & & \end{pmatrix} dz$$

have some symmetries, by uniqueness of $\tilde{h}_t^c$ we get corresp symmetries of local model. First note it descends to $h_{t,\lambda}$ down the branched covering, denote its local form by matix $H_{t,\lambda}$.

Local model symmetries

Standard form of Higgs field

$$\tilde{\theta} = \sqrt{2}z \begin{pmatrix} & z^2 & 1 \\ 1 & & \\ z^2 & & \end{pmatrix} dz$$

have some symmetries, by uniqueness of $\tilde{h}_t^c$ we get corresp symmetries of local model. First note it descends to $h_{t,\lambda}$ down the branched covering, denote its local form by matix $H_{t,\lambda}$. We have rotational symmetry:

$$H_{t,\lambda}(\zeta e^{i\phi}) = g_\phi^\dagger H_{t,\lambda}(\zeta) g_\phi, \;\; g_\phi = \begin{pmatrix} e^{-i\phi/2} & \\ & e^{i\phi/2} \end{pmatrix}$$

Local model symmetries

Standard form of Higgs field

$$\tilde{\theta} = \sqrt{2}z \begin{pmatrix} & z^2 & 1 \\ 1 & & \\ z^2 & & \end{pmatrix} dz$$

have some symmetries, by uniqueness of $\tilde{h}_t^c$ we get corresp symmetries of local model. First note it descends to $h_{t,\lambda}$ down the branched covering, denote its local form by matix $H_{t,\lambda}$. We have rotational symmetry:

$$H_{t,\lambda}(\zeta e^{i\phi}) = g_\phi^\dagger H_{t,\lambda}(\zeta) g_\phi, \; g_\phi = \begin{pmatrix} e^{-i\phi/2} & \\ & e^{i\phi/2} \end{pmatrix}$$

as well as scaling law

$$H_{1,\lambda}(t^{2/3}\zeta) = \Gamma_t^\dagger H_{t,\lambda}(\zeta)\Gamma_t, \; \Gamma_t = \begin{pmatrix} t^{1/3} & \\ & t^{-1/3} \end{pmatrix}$$

Local model symmetries

Standard form of Higgs field

$$\tilde{\theta} = \sqrt{2}z\begin{pmatrix} & z^2 & 1 \\ 1 & & \\ z^2 & & \end{pmatrix}dz$$

have some symmetries, by uniqueness of $\tilde{h}_t^c$ we get corresp symmetries of local model. First note it descends to $h_{t,\lambda}$ down the branched covering, denote its local form by matix $H_{t,\lambda}$. We have rotational symmetry:

$$H_{t,\lambda}(\zeta e^{i\phi}) = g_\phi^\dagger H_{t,\lambda}(\zeta)g_\phi, \ g_\phi = \begin{pmatrix} e^{-i\phi/2} & \\ & e^{i\phi/2} \end{pmatrix}$$

as well as scaling law

$$H_{1,\lambda}(t^{2/3}\zeta) = \Gamma_t^\dagger H_{t,\lambda}(\zeta)\Gamma_t, \ \Gamma_t = \begin{pmatrix} t^{1/3} & \\ & t^{-1/3} \end{pmatrix}$$

in particular we have scalar radial functions $h_{1,\lambda}$, $h_{2,\lambda} \in \mathbb{R}$, and $h_{3,\lambda} \in \mathbb{C}$:

$$H_{t,\lambda}(\rho e^{i\theta}) = \begin{pmatrix} \rho h_{1,\lambda}(t^{2/3}\rho) & h_{3,\lambda}(t^{2/3}\rho)e^{-i\theta} \\ \overline{h_{3,\lambda}(t^{2/3}\rho)e^{i\theta}} & \frac{1}{\rho}h_{2,\lambda}(t^{2/3}\rho) \end{pmatrix}$$

Local model explicit form

Note in particular if $\lambda = 0$ the Hitchin equation reduces to an ODE of a single radial function $f = h_1$:

$$(\rho\partial_\rho)^2 \log f = 2\rho^3 \left(f - \frac{1}{f}\right)$$

Local model explicit form

Note in particular if $\lambda = 0$ the Hitchin equation reduces to an ODE of a single radial function $f = h_1$:

$$(\rho\partial_\rho)^2 \log f = 2\rho^3 \left(f - \frac{1}{f}\right)$$

With $f(\rho) = \exp\left(2\psi\left(4\rho^{3/2}/3\right)\right)$, ψ satisfies a Painlevé III equation $\psi'' + \frac{1}{x}\psi' - \frac{1}{2}\sinh(2\psi) = 0$. The unique solution in this case is characterized by $\psi(x) \sim x^{1/2}e^{-x}$ as $x \to \infty$. For points in D_β, D_γ we use local frame in which Higgs field and local model has the following form with $\log\det H_t = -2\lambda \log|\zeta| + O(1)$

Local model explicit form

Note in particular if $\lambda = 0$ the Hitchin equation reduces to an ODE of a single radial function $f = h_1$:

$$(\rho\partial_\rho)^2 \log f = 2\rho^3 \left(f - \frac{1}{f}\right)$$

With $f(\rho) = \exp\left(2\psi\left(4\rho^{3/2}/3\right)\right)$, ψ satisfies a Painlevé III equation $\psi'' + \frac{1}{x}\psi' - \frac{1}{2}\sinh(2\psi) = 0$. The unique solution in this case is characterized by $\psi(x) \sim x^{1/2}e^{-x}$ as $x \to \infty$. For points in D_β, D_γ we use local frame in which Higgs field and local model has the following form with $\log\det H_t = -2\lambda \log|\zeta| + O(1)$

$$\Phi = \begin{pmatrix} & 1 & 0 \\ \zeta & & \\ 0 & & \end{pmatrix} d\zeta, \quad H_t = \begin{pmatrix} \frac{1}{c\sqrt{|\zeta|}} e^{-\psi\left(\frac{8}{3}t|\zeta|^{3/2}\right)} & \\ & c^2 \end{pmatrix}, \quad \lambda = 1/4 \text{ for } p \in D_\beta$$

$$\Phi = \begin{pmatrix} & \zeta & 0 \\ 1 & & \\ 0 & & \end{pmatrix} d\zeta, \quad H_t = \begin{pmatrix} \frac{1}{c}\sqrt{|\zeta|} e^{\psi\left(\frac{8}{3}t|\zeta|^{3/2}\right)} & \\ & c^2 \end{pmatrix}, \quad \lambda = -1/4 \text{ for } p \in D_\gamma$$

Asymptotics of local model

Let $\underline{b}$ corresp to (F, β, γ) stable. Let frames $\left\{s_{1,j}, s_{2,j}\right\}$ as in Thm 1 near $p_j \in D_r$. The corresp section $s_{0,j} = (s_{1,j} \wedge s_{2,j})^{-1}$ gives rise to a frame $\sigma_{1,j} = (s_{0,j}^{-2} d\zeta_j, 0)$, $\sigma_{2,j} = (0, s_{0,j} d\zeta_j)$ of $V = L^{-2}K \oplus LK$. Let $(h_{t,\lambda})_{\underline{\sigma}} = t^{2/3} M_\lambda(t^{2/3}\rho)$, its entries are radial functions

Asymptotics of local model

Let $\underline{b}$ corresp to (F, β, γ) stable. Let frames $\left\{s_{1,j}, s_{2,j}\right\}$ as in Thm 1 near $p_j \in D_r$. The corresp section $s_{0,j} = (s_{1,j} \wedge s_{2,j})^{-1}$ gives rise to a frame $\sigma_{1,j} = (s_{0,j}^{-2} d\zeta_j, 0)$, $\sigma_{2,j} = (0, s_{0,j} d\zeta_j)$ of $V = L^{-2}K \oplus LK$. Let $(h_{t,\lambda})_{\underline{\sigma}} = t^{2/3} M_\lambda(t^{2/3}\rho)$, its entries are radial functions

Theorem (Mochizuki '16)

Let $\left(E, \bar{\partial}_E, \theta, h\right) = \bigoplus_{j=1}^{N} \left(L_j, \bar{\partial}_{L_j}, f_j dz\right)$ decomp. of harmonic bundle on disk $D = \{|z| < R\}$ into line bundles with $|f_j| < M$. Let $h_j = h|_{L_j}$. Then for any $0 < r < R$ we have C, $c > 0$ depending on rk$E$, M, R, r such that on $\{|z| < r\}$:

$$\left|R(h)\right|_h, \left|R(h_j)\right|_{h_j} \leq Ce^{-cd},$$

$$h(s_i, s_j) \leq Ce^{-cd} |s_i|_h \left|s_j\right|_h \quad \text{for } i \neq j$$

where $d = \min_{i \neq j} \left|f_i - f_j\right|$.

Asymptotics of local model

Let $\underline{b}$ corresp to (F, β, γ) stable. Let frames $\left\{s_{1,j}, s_{2,j}\right\}$ as in Thm 1 near $p_j \in D_r$. The corresp section $s_{0,j} = (s_{1,j} \wedge s_{2,j})^{-1}$ gives rise to a frame $\sigma_{1,j} = (s_{0,j}^{-2} d\zeta_j, 0)$, $\sigma_{2,j} = (0, s_{0,j} d\zeta_j)$ of $V = L^{-2}K \oplus LK$. Let $(h_{t,\lambda})_{\underline{\sigma}} = t^{2/3} M_\lambda(t^{2/3}\rho)$, its entries are radial functions

Theorem (Mochizuki '16)

Let $\left(E, \bar{\partial}_E, \theta, h\right) = \bigoplus_{j=1}^N \left(L_j, \bar{\partial}_{L_j}, f_j dz\right)$ decomp. of harmonic bundle on disk $D = \{|z| < R\}$ into line bundles with $|f_j| < M$. Let $h_j = h|_{L_j}$. Then for any $0 < r < R$ we have C, $c > 0$ depending on rk$E$, M, R, r such that on $\{|z| < r\}$:

$$\left|R(h)\right|_h, \left|R(h_j)\right|_{h_j} \leq Ce^{-cd},$$

$$h(s_i, s_j) \leq Ce^{-cd} |s_i|_h \left|s_j\right|_h \quad \text{for } i \neq j$$

where $d = \min_{i \neq j} \left|f_i - f_j\right|$.

i.e. over locus where Higgs field is semisimple, harmonic metric becomes asymptotically flat and orthogonal, at rate exponential in distance between eigenvalues.

Asymptotics of local model

Using this we see there is $\rho_0 > 1$ such that for all $\lambda \in I \subset\subset (-1/4, 1/4)$ there is C, $c > 0$ such that

$$\left|\partial_\rho^\ell (M_\lambda - M_{\infty,\lambda})\right| \leq Ce^{-c\rho}$$

for $\ell = 0, 1, 2$ and

$$M_{\infty,\lambda} = \begin{pmatrix} \rho^{-1}\mu_\lambda(\rho)^2 & \\ & \rho^{-1}\mu_\lambda(\rho)^{-1} \end{pmatrix}, \ \mu_\lambda(\rho) = \frac{c(\lambda)}{4}\rho^{-2\lambda}$$

where $c : (-1/4, 1/4) \to \mathbb{R}_+$ can be shown to be continuous.

Asymptotics of local model

Using this we see there is $\rho_0 > 1$ such that for all $\lambda \in I \subset\subset (-1/4, 1/4)$ there is C, $c > 0$ such that

$$\left|\partial_\rho^\ell \left(M_\lambda - M_{\infty,\lambda}\right)\right| \leq Ce^{-c\rho}$$

for $\ell = 0, 1, 2$ and

$$M_{\infty,\lambda} = \begin{pmatrix} \rho^{-1}\mu_\lambda(\rho)^2 & \\ & \rho^{-1}\mu_\lambda(\rho)^{-1} \end{pmatrix}, \ \mu_\lambda(\rho) = \frac{c(\lambda)}{4}\rho^{-2\lambda}$$

where $c : (-1/4, 1/4) \to \mathbb{R}_+$ can be shown to be continuous.

Definition

Admissible weight $\underline{\lambda}$ and fiber-matching param. $\underline{b}$ are *t*-compatible if there is $h_L = h_{L,\underline{\lambda},t}$ adapted to $(L, \underline{\lambda})$ such that $\left(h_L^{-2}h_K \oplus h_L h_K\right)_{\underline{\sigma}} = t^{2/3}M_{\infty,\lambda_j}(t^{2/3}\rho)$

Equivalently for $s_{0,j}$ any section with $s_{0,j}^{\otimes 3} = b_{p_j}$ we have

$$\log \left|s_{0,j}\right|^2_{h_{L,\underline{\lambda},t}} = \log \frac{c(\lambda_j)}{4} + \frac{4}{3}\left(\log t\right)\lambda_j + 2\lambda_j \log|\zeta_j| + O(1)$$

Asymptotics of local model

Using this we see there is $\rho_0 > 1$ such that for all $\lambda \in I \subset\subset (-1/4, 1/4)$ there is C, $c > 0$ such that

$$\left|\partial_\rho^\ell \left(M_\lambda - M_{\infty,\lambda}\right)\right| \leq Ce^{-c\rho}$$

for $\ell = 0, 1, 2$ and

$$M_{\infty,\lambda} = \begin{pmatrix} \rho^{-1}\mu_\lambda(\rho)^2 & \\ & \rho^{-1}\mu_\lambda(\rho)^{-1} \end{pmatrix}, \quad \mu_\lambda(\rho) = \frac{c(\lambda)}{4}\rho^{-2\lambda}$$

where $c : (-1/4, 1/4) \to \mathbb{R}_+$ can be shown to be continuous.

Definition

Admissible weight $\underline{\lambda}$ and fiber-matching param. $\underline{b}$ are *t-compatible* if there is $h_L = h_{L,\underline{\lambda},t}$ adapted to $(L, \underline{\lambda})$ such that $\left(h_L^{-2}h_K \oplus h_L h_K\right)_{\underline{\sigma}} = t^{2/3}M_{\infty,\lambda_j}(t^{2/3}\rho)$

Equivalently for $s_{0,j}$ any section with $s_{0,j}^{\otimes 3} = b_{p_j}$ we have

$$\log\left|s_{0,j}\right|^2_{h_{L,\underline{\lambda},t}} = \log\frac{c(\lambda_j)}{4} + \frac{4}{3}\left(\log t\right)\lambda_j + 2\lambda_j \log|\zeta_j| + O(1)$$

Now ▸ recall ...

Construct harmonic metric on parabolic line bundle

For simplicity focus on case $d_\beta = d_\gamma = 0$, $\deg L = 0$ ($\mathscr{P}_{\underline{D}}$ is top dim. simplex, barycenter at origin). We can construct a harmonic metric $h^0_{L,\underline{\lambda}}$ adapted to $(L, \underline{\lambda})$ from Hermitian-Einstein metric $h_{L,\mathrm{HE}}$ satisfying $\sqrt{-1}\Lambda R(h_{L,\mathrm{HE}}) = \deg L$:

$$h^0_{L,\underline{\lambda}} = h_{L,\mathrm{HE}} e^{\varphi_{\underline{\lambda}}}, \ \varphi_{\underline{\lambda}} = \sum_{j=1}^{4g-4} \left(\sum_{\ell=1}^{j} \lambda_\ell \right) G_j$$

Construct harmonic metric on parabolic line bundle

For simplicity focus on case $d_\beta = d_\gamma = 0$, $\deg L = 0$ ($\mathscr{P}_{\underline{D}}$ is top dim. simplex, barycenter at origin). We can construct a harmonic metric $h^0_{L,\underline{\lambda}}$ adapted to $(L, \underline{\lambda})$ from Hermitian-Einstein metric $h_{L,\mathrm{HE}}$ satisfying $\sqrt{-1}\Lambda R(h_{L,\mathrm{HE}}) = \deg L$:

$$h^0_{L,\underline{\lambda}} = h_{L,\mathrm{HE}} e^{\varphi_{\underline{\lambda}}}, \ \varphi_{\underline{\lambda}} = \sum_{j=1}^{4g-4} \left(\sum_{\ell=1}^{j} \lambda_\ell \right) G_j$$

where G_j are bipolar Green's function for $j = 1, \dots, N-1$ satisfying

$$\Delta_\partial G_j = \begin{cases} 0 & \text{on } X - \{p_j, p_{j+1}\}, \ j = 1, \dots, 4g-5 \\ 1 & \text{on } X - \{p_{4g-4}\}, \ j = 4g-4 \end{cases}, \ \begin{cases} G_j - 2\log|\zeta_j| \text{ bdd at } p_j \\ G_j + 2\log|\zeta_{j+1}| \text{ bdd at } p_j \\ G_{4g-4} - 2\log|\zeta_{4g-4}| \text{ bdd at } p_{4g-4} \end{cases}$$

Let $g_{j\ell} = G_j - \log$ terms on D_ℓ.

Migration of the parabolic weight

By uniqueness up to positive const mult. we have $h_{L,\underline{\lambda},t} = h^0_{L,\underline{\lambda}} e^{c_{\underline{\lambda},t}}$ and

$$\log \left| s_{0,j} \right|^2_{h_{L,\underline{\lambda},t}} = c_{\underline{\lambda},t} + \log \left| s_{0,j}(0) \right|^2_{h_{L,\mathrm{HE}}} + \sum_{\ell=1}^{4g-4} \left(\sum_{k=1}^{\ell} \lambda_k \right) g_{\ell j} + 2\lambda_j \log |\zeta_j|$$

Migration of the parabolic weight

By uniqueness up to positive const mult. we have $h_{L,\underline{\lambda},t} = h^0_{L,\underline{\lambda}} e^{c_{\underline{\lambda},t}}$ and

$$\log \left|s_{0,j}\right|^2_{h_{L,\underline{\lambda},t}} = c_{\underline{\lambda},t} + \log \left|s_{0,j}(0)\right|^2_{h_{L,\mathrm{HE}}} + \sum_{\ell=1}^{4g-4}\left(\sum_{k=1}^{\ell}\lambda_k\right) g_{\ell j} + 2\lambda_j \log|\zeta_j|$$

Comparing the two expressions of $\log \left|s_{0,j}\right|^2_{h_{L,\underline{\lambda},t}}$ with log terms removed, t-compatibility is equivalent to:

$$\begin{aligned} c_{\underline{\lambda},t} = {} & \log \frac{c(\lambda_j)}{4} + \frac{4}{3}(\log t)\,\lambda_j \\ & - \log \left|s_{0,j}(0)\right|^2_{h_{L,\mathrm{HE}}} - \sum_{\ell=1}^{4g-4}\left(\sum_{k=1}^{\ell}\lambda_k\right) g_{\ell j} \end{aligned}$$

Migration of the parabolic weight

By uniqueness up to positive const mult. we have $h_{L,\underline{\lambda},t} = h^0_{L,\underline{\lambda}} e^{c_{\underline{\lambda},t}}$ and

$$\log \left|s_{0,j}\right|^2_{h_{L,\underline{\lambda},t}} = c_{\underline{\lambda},t} + \log \left|s_{0,j}(0)\right|^2_{h_{L,\mathrm{HE}}} + \sum_{\ell=1}^{4g-4} \left(\sum_{k=1}^{\ell} \lambda_k \right) g_{\ell j} + 2\lambda_j \log |\zeta_j|$$

Comparing the two expressions of $\log \left|s_{0,j}\right|^2_{h_{L,\underline{\lambda},t}}$ with log terms removed, t-compatibility is equivalent to:

$$\begin{aligned} c_{\underline{\lambda},t} &= \log \frac{c(\lambda_j)}{4} + \frac{4}{3} (\log t)\, \lambda_j \\ &- \log \left|s_{0,j}(0)\right|^2_{h_{L,\mathrm{HE}}} - \sum_{\ell=1}^{4g-4} \left(\sum_{k=1}^{\ell} \lambda_k \right) g_{\ell j} \end{aligned}$$

for all j with $p_j \in D_r$!

Migration of the parabolic weight

Therefore $\underline{\lambda}$ and $\underline{b}$ being t-compatible is equivalent to

$$\pi : \left(\frac{4}{3}\left(\log t\right)\lambda_j + r(\underline{\lambda})\right)_j \mapsto 0$$

where $\pi : \mathbb{R}^{4g-4} \to H = \left\{\underline{\lambda} | \sum_j \lambda_j = 0\right\}$ is the map $(v_j)_j \mapsto \left(v_j - \sum_\ell v_\ell\right)_j$ and

$$r(\underline{\lambda}) = \log \frac{c(\lambda_j)}{4} - \log \left|s_{0,j}(0)\right|_{h_{L,\mathrm{HE}}} - \sum_{\ell=1}^{4g-4}\left(\sum_{k=1}^{\ell}\lambda_k\right) g_{\ell j}(0)$$

Migration of the parabolic weight

Therefore $\underline{\lambda}$ and $\underline{b}$ being t-compatible is equivalent to

$$\pi : \left(\frac{4}{3}(\log t)\,\lambda_j + r(\underline{\lambda})\right)_j \mapsto 0$$

where $\pi : \mathbb{R}^{4g-4} \to H = \left\{\underline{\lambda} | \sum_j \lambda_j = 0\right\}$ is the map $(v_j)_j \mapsto \left(v_j - \sum_\ell v_\ell\right)_j$ and

$$r(\underline{\lambda}) = \log \frac{c(\lambda_j)}{4} - \log \left|s_{0,j}(0)\right|_{h_{L,\mathrm{HE}}} - \sum_{\ell=1}^{4g-4} \left(\sum_{k=1}^{\ell} \lambda_k\right) g_{\ell j}(0)$$

For fixed $\underline{b}$, $\underline{\lambda}$ is t-compatible iff it is fixed by

$$\underline{\lambda} \mapsto -\frac{3}{4\log t}\pi \circ r(\underline{\lambda})$$

$\pi \circ r$ is continuous, so by Brouwer's fixed point theorem for $t \gg 1$ we can get a family of t-compatible weights $\underline{\lambda}(t)$

Perturbation of approximate solution

Consider $\mathcal{H}_{t,h} : \mathrm{Herm}(F, h) \to \mathrm{Herm}(F, h)$ given by

Perturbation of approximate solution

Consider $\mathcal{H}_{t,h} : \mathrm{Herm}(F, h) \to \mathrm{Herm}(F, h)$ given by

$$\mathcal{H}_{t,h} : u \mapsto \sqrt{-1}\Lambda e^{u/2}\left(R(h^{\exp(u)}) + t^2\beta \wedge \beta^{\dagger}_{h^{\exp(u)}} + t^2\gamma^{\dagger}_{h^{\exp(u)}} \wedge \gamma\right) e^{-u/2}$$

and linearize it at $h = h_t^{\mathrm{app}}$, giving $\mathcal{H}_{t,h}(u) = \mathcal{H}_{t,h}(0) + L_t u + R_t(u)$ with L_t a linear operator

Perturbation of approximate solution

Consider $\mathcal{H}_{t,h} : \mathrm{Herm}(F,h) \to \mathrm{Herm}(F,h)$ given by

$$\mathcal{H}_{t,h} : u \mapsto \sqrt{-1}\Lambda e^{u/2} \left(R(h^{\exp(u)}) + t^2 \beta \wedge \beta^{\dagger}_{h^{\exp(u)}} + t^2 \gamma^{\dagger}_{h^{\exp(u)}} \wedge \gamma \right) e^{-u/2}$$

and linearize it at $h = h_t^{\mathrm{app}}$, giving $\mathcal{H}_{t,h}(u) = \mathcal{H}_{t,h}(0) + L_t u + R_t(u)$ with L_t a linear operator

$$L_t(u) = \Delta_h u + t^2 \left\{ \sqrt{-1}\Lambda \left(\beta \wedge \beta^{\dagger}_h - \gamma^{\dagger}_h \wedge \gamma \right), \hat{u} \right\}$$

with $\hat{u} = u + (\mathrm{tr} u)\,\mathrm{Id}$ and $\{A, B\} = AB + BA$.

Perturbation of approximate solution

Consider $\mathcal{H}_{t,h}$: Herm$(F,h) \to$ Herm(F,h) given by

$$\mathcal{H}_{t,h} : u \mapsto \sqrt{-1}\Lambda e^{u/2}\left(R(h^{\exp(u)}) + t^2\beta\wedge\beta^{\dagger}_{h^{\exp(u)}} + t^2\gamma^{\dagger}_{h^{\exp(u)}}\wedge\gamma\right)e^{-u/2}$$

and linearize it at $h = h_t^{\text{app}}$, giving $\mathcal{H}_{t,h}(u) = \mathcal{H}_{t,h}(0) + L_t u + R_t(u)$ with L_t a linear operator

$$L_t(u) = \Delta_h u + t^2\left\{\sqrt{-1}\Lambda\left(\beta\wedge\beta^{\dagger}_h - \gamma^{\dagger}_h\wedge\gamma\right), \hat{u}\right\}$$

with $\hat{u} = u + (\mathrm{tr}u)\,\mathrm{Id}$ and $\{A,B\} = AB + BA$. We have

$$Q_t(u) = ((L_t u, u)) = \left\|d\,(tru)\right\|^2_{L^2} + 2\left\|\bar{\partial}u\right\|^2_{L^2} + 2t^2\left\|\hat{u}\circ\beta\right\|^2_{L^2} + 2t^2\left\|\gamma\circ\hat{u}\right\|^2_{L^2}$$

where $((u,v)) = (u,\hat{v}) = (\hat{u},v)$.

Perturbation of approximate solution

By estimating first eigenvalue of a related operator bounding Q_t from below, we show $L^2 \to L^2$ lower bound of L_t: there is $C > 0$, for $t \gg 1$ and $u \in L^2_2$,

$$\|L_t u\|^2_{L^2, h_t^{\mathrm{app}}} \geq \frac{C}{\log t} \|u\|^2_{L^2, h_t^{\mathrm{app}}}$$

Note that we will need to fix the problem of t-dependeng norm in the estimate.

Perturbation of approximate solution

By estimating first eigenvalue of a related operator bounding Q_t from below, we show $L^2 \to L^2$ lower bound of L_t: there is $C > 0$, for $t \gg 1$ and $u \in L^2_2$,

$$\|L_t u\|^2_{L^2, h_t^{\text{app}}} \geq \frac{C}{\log t} \|u\|^2_{L^2, h_t^{\text{app}}}$$

Note that we will need to fix the problem of t-dependeng norm in the estimate. Using asymptotics of local model proved by Mochizuki's theorem plus careful analysis of the form of the approximate solution, we can show various upper bounds for any $u \in L^2_2\left(\text{Herm}(F, h_t^{\text{app}})\right)$, $t_0 \gg 1$ and $t \geq t_0$:

$$\frac{1}{Ct^{13}} |u|_{h_{t_0}^{\text{app}}} \leq |u|^2_{h_t^{\text{app}}} \leq Ct^{13} |u|_{h_{t_0}^{\text{app}}} \text{ for some } C > 1$$

$$\left\|\Delta_{h_t^{\text{app}}} u - \Delta_{h_{t_0}^{\text{app}}} u\right\|^2_{L^2, h_{t_0}^{\text{app}}} \leq Ct^4 \left(\left\|d_{h_{t_0}^{\text{app}}} u\right\|^2_{L^2, h_{t_0}^{\text{app}}} + \|u\|^2_{L^2, h_{t_0}^{\text{app}}}\right)$$

$$\left\|\left\{\left(\beta \wedge \beta^\dagger_{h_t^{\text{app}}} - \gamma^\dagger_{h_t^{\text{app}}} \wedge \gamma\right), \hat{u}\right\}\right\|_{L^2, h_{t_0}^{\text{app}}} \leq Ct^{11/3} \|u\|_{L^2, h_{t_0}^{\text{app}}}$$

Perturbation of approximate solution

We may use elliptic estimate of $\Delta_{h_{t_0}^{\mathrm{app}}}$ combining with prev. estimates to get

$$\|L_t u\|_{L_2^2} \geq C t^{-20} \|u\|_{L^2}$$

Perturbation of approximate solution

We may use elliptic estimate of $\Delta_{h_{t_0}^{\mathrm{app}}}$ combining with prev. estimates to get

$$\|L_t u\|_{L_2^2} \geq C t^{-20} \|u\|_{L^2}$$

with some more effort we show for $u_0, u_1 \in B(0, r)$ with $r \ll 1$

$$\left\|R_t(u_0) - R_t(u_1)\right\|_{L^2} \lesssim t^{17/3} \|u_0 - u_1\|_{L_2^2}$$

Perturbation of approximate solution

We may use elliptic estimate of $\Delta_{h_{t_0}^{\text{app}}}$ combining with prev. estimates to get

$$\|L_t u\|_{L_2^2} \geq Ct^{-20} \|u\|_{L^2}$$

with some more effort we show for $u_0, u_1 \in B(0,r)$ with $r \ll 1$

$$\left\|R_t(u_0) - R_t(u_1)\right\|_{L^2} \lesssim t^{17/3} \|u_0 - u_1\|_{L_2^2}$$

The huge power of t is no cause of worry since we have

$$\left|\mathcal{H}_{t,h_t^{\text{app}}}(0)\right| = \left|R(h_t^{\text{app}}) + t^2\beta \wedge \beta^\dagger_{h_t^{\text{app}}} + t^2\gamma^\dagger_{h_t^{\text{app}}} \wedge \gamma\right| \leq Ce^{-ct^{2/3}}$$

Perturbation of approximate solution

We may use elliptic estimate of $\Delta_{h_{t_0}^{\mathrm{app}}}$ combining with prev. estimates to get

$$\|L_t u\|_{L_2^2} \geq C t^{-20} \|u\|_{L^2}$$

with some more effort we show for $u_0, u_1 \in B(0, r)$ with $r \ll 1$

$$\left\| R_t(u_0) - R_t(u_1) \right\|_{L^2} \lesssim t^{17/3} \|u_0 - u_1\|_{L_2^2}$$

The huge power of t is no cause of worry since we have

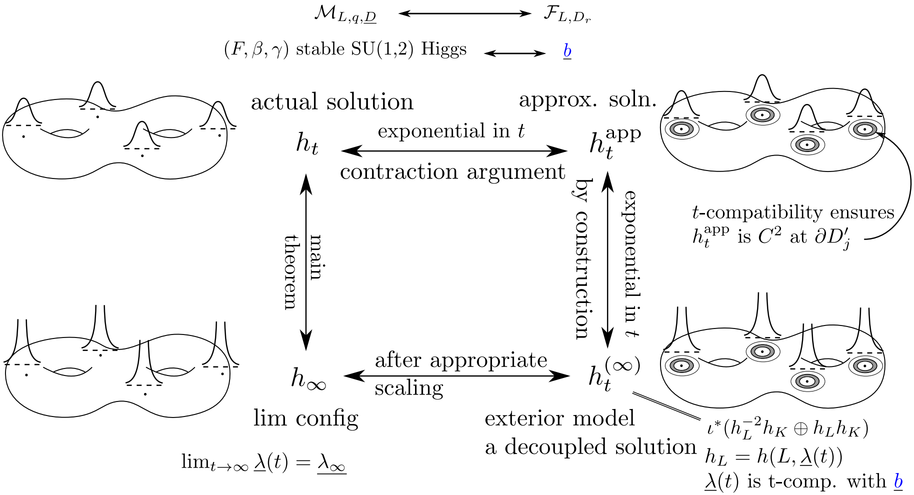
$$\left| \mathcal{H}_{t, h_t^{\mathrm{app}}}(0) \right| = \left| R(h_t^{\mathrm{app}}) + t^2 \beta \wedge \beta^{\dagger}_{h_t^{\mathrm{app}}} + t^2 \gamma^{\dagger}_{h_t^{\mathrm{app}}} \wedge \gamma \right| \leq C e^{-ct^{2/3}}$$

Consider on $L_2^2(\mathrm{End}(F))$,

$$\mathcal{F}_t : u \mapsto -L_t^{-1} \left(2\sqrt{-1} \Lambda e^{u/2} \mathcal{H}_{t, h_t^{\mathrm{app}}}(0) e^{-u/2} + R_t(u) \right)$$

Iterated sequence $u, \mathcal{F}_t u, \mathcal{F}_t^2 u, \ldots$ converges to u_∞ giving an actual solution $h_t = \left(h_t^{\mathrm{app}}\right)^{\exp u_\infty}$ by contraction mapping argument. h_t and h_t^{app} is exponentially close in t.

Perturbation of approximate solution



Thank you !

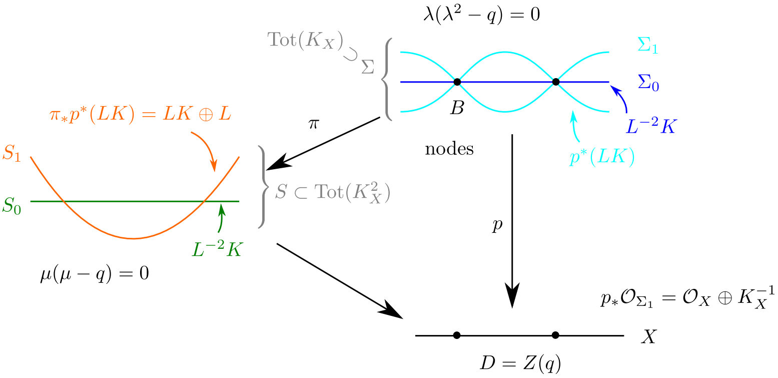
More on SU(1,2) spectral data

Fiber matching

Given a reduced curve $C = C_0 \cup C_1$ with two irreducible non-singular components and simple nodes at $N \subset C$. Let $\iota_j : C_j \to C$, rank one locally free sheaf $\mathcal{F}$ on C are characterized by line bundles $\mathcal{F}_j := \iota_j^* \mathcal{F}$ on C_j, $j = 0, 1$ and $|N|$ parameters at each node $p \in N$. In nbhd around each $p \in N$, suppose $s \in \mathcal{F}$ and $s_j \in \mathcal{F}_j$ the pullback sections, this parameter is given by the ratio $[s_0(p) : s_1(p)]$.
Let $\pi : C_0 \coprod C_1 \to C$ be the normalization. The connection to Hecke transformation viewpoint is given by canonical sheaf map $\mathcal{F} \to \pi_* \pi^* \mathcal{F} = \mathcal{F}_0 \oplus \mathcal{F}_1$. If we have a further map to some non-singular curve X, the latter will be a decomposable bundle.

More on SU(1,2) spectral data

SU(1,2) Fiber matching



Note that the resulting Hecke transformation is of $L \oplus LK \oplus L^{-2}K$ where the fiber matching of rank 2 bundle $L \oplus LK$ with $L^{-2}K$ must occur only in second summand. This is because it is corresponding to $\mathcal{O}_X$ summand of $p_* \mathcal{O}_{\Sigma_1}$, which are not vanishing at branch points.