Nonlinear Dirac equations and surfaces of prescribed mean curvature

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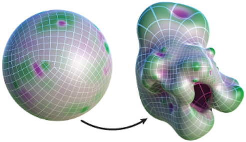


Brief review of the Yamabe problem

Yamabe (1960)

Given closed Riemannian manifold (M, g) of dimension $m \geq 3$, find a metric conformal to g with constant scalar curvature.

- Any metric conformal to g can be written as $\tilde{g} = e^{2f} g$
- If S and $\tilde{S}$ denote the scalar curvature of g and $\tilde{g}$, respectively, they satisfy the transformation law:

$$\tilde{S} = e^{2f}(S - 2(m-1)\Delta f - (m-1)(m-2)|\nabla f|^2),$$

  in which $-\Delta f$ denotes the Laplacian of f and ∇f its covariant derivative, defined with respect to the metric g.
- Make the substitution $e^{2f} = u^{p-2}$, with $p = \frac{2m}{m-2}$, the scalar curvature satisfies the equation

$$\tilde{S} = u^{1-p}\Big(-\frac{4(m-1)}{m-2}\Delta u + S u\Big)$$

Let $m \geq 3$, we focus on the equation of the form

$$-c_m \Delta u + S u = \tilde{S} u^{p-1} \quad \text{on } (M, g),$$

where $c_m = \frac{4(m-1)}{m-2}$, $p = \frac{2m}{m-2}$.

- $\tilde{g} = u^{p-2} g = u^{\frac{4}{m-2}} g$ has constant scalar curvature $\tilde{S} = \lambda$ iff $u > 0$ satisfies the Yamabe equation

$$-c_m \Delta u + S u = \lambda u^{p-1} \quad \text{on } (M, g). \tag{1}$$

  The problem is to prove the existence of a real number λ and a C^2 function u, strictly positive everywhere, satisfying (1).

  - Positiveness of a solution to Eq. (1) follows from the Maximum Principle.

  - The existence is a consequence of Variational Methods.

- consider

$$Q_g(u) = \frac{\int_M \tilde{S}\, d\mathrm{vol}_{\tilde{g}}}{\left(\int_M d\mathrm{vol}_{\tilde{g}}\right)^{\frac{2}{p}}} = \frac{\int_M c_m|\nabla u|^2 + S\, u^2 d\mathrm{vol}_g}{\left(\int_M |u|^p d\mathrm{vol}_g\right)^{\frac{2}{p}}}, \qquad u \in H^1(M)$$

then, for any $\varphi \in C^\infty(M)$,

$$\frac{d}{dt} Q_g(u+t\varphi)\Big|_{t=0} = \frac{2}{\|u\|_p^2} \int_M \Big(-c_m \Delta u + S\, u - \frac{Q_g(u)}{\|u\|_p^{p-2}} |u|^{p-2} u\Big) \varphi\, d\mathrm{vol}_g$$

where

$$\|u\|_p = \Big(\int_M |u|^p d\mathrm{vol}_g\Big)^{\frac{1}{p}}$$

  - $u > 0$ is a critical point of Q_g iff it satisfies the Yamabe equation (1) with $\lambda = Q_g(u)/\|u\|_p^{p-2}$.
  - λ: the Lagrange multiplier if we restrict ourselves with $\|u\|_p \equiv 1$.

Consider $\quad Q_g(u) = \dfrac{\int_M c_m |\nabla u|^2 + S u^2 d\mathrm{vol}_g}{\left(\int_M |u|^p d\mathrm{vol}_g\right)^{\frac{2}{p}}}$ on $S_p := \{\|u\|_p = 1\}$.

Sobolev inequality:

$$\begin{aligned}\|u\|^2_{H^1(M)} &= \int_M |\nabla u|^2 + u^2 d\mathrm{vol}_g \\ &= \frac{1}{c_m} Q_g(u) + \int_M \Big(1 - \frac{S}{c_m}\Big) u^2 d\mathrm{vol}_g \\ &\le \frac{1}{c_m} Q_g(u) + C\|u\|_p^2\end{aligned}$$

$\Longrightarrow$ $Q_g(u)$ is bounded from below and coercive on S_p

Set

$$\lambda(M) = \inf\{Q_g(u) : \ u \in S_p\}.$$

If $\lambda(M)$ is attained, then it also plays the role of the Lagrange multiplier in the Yamabe equation.

$\lambda(M) \le 0$ is ease to solve!

Theorem (Aubin 1976)

If M is any closed Riemannian manifold of dimension $m \geq 3$, then

$$\lambda(M) \leq \lambda(S^m) = m(m-1)\omega_m^{\frac{2}{m}},$$

where ω_m is the volume of the unit sphere S^m

Conformal invariance of Q: $\lambda(S^m) = \inf_{\varphi \neq 0} \frac{c_m \int_{\mathbb{R}^m} |\nabla\varphi|^2 dx}{\left(\int_{\mathbb{R}^m} |\varphi|^p dx\right)^{\frac{2}{p}}}, \quad p = \frac{2m}{m-2}$

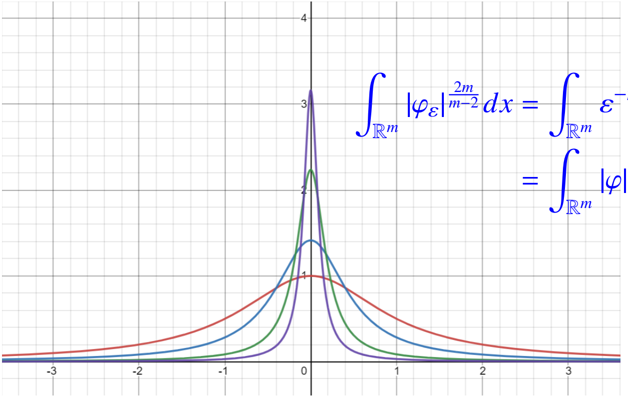
Extremal function:

$$\varphi(x) = \frac{[m(m-2)]^{\frac{m-2}{4}}}{[1+|x|^2]^{\frac{m-2}{2}}}, \quad x \in \mathbb{R}^m.$$

Rescaling, cut-off and transplant:

$$\varphi_\varepsilon(x) = \varepsilon^{-\frac{m-2}{2}}\varphi(x/\varepsilon) \quad \text{and} \quad u_\varepsilon = \eta\varphi_\varepsilon$$

where η is a cut-off function such that $\eta(x) \equiv 1$ for $|x| < \delta$ and $\eta(x) \equiv 0$ for $|x| > 2\delta$. Then $Q_g(u_\varepsilon) = \lambda(S^m) + o_\varepsilon(1)$.



$$\int_{\mathbb{R}^m} |\varphi_\varepsilon|^{\frac{2m}{m-2}} dx = \int_{\mathbb{R}^m} \varepsilon^{-m}\Big|\varphi\Big(\frac{x}{\varepsilon}\Big)\Big|^{\frac{2m}{m-2}} dx = \int_{\mathbb{R}^m} |\varphi|^{\frac{2m}{m-2}} dx$$

Theorem (Aubin 1976)

The Yamabe problem can be solved on any closed manifold M with $\lambda(M) < \lambda(S^m)$.

Subcritical perturbation:

$$-c_m \Delta u + S u = \lambda_{q_n} u^{q_n - 1}, \quad u > 0$$

the idea is to choose $2 < q_n < p = \frac{2m}{m-2}$ and let $q_n \nearrow p$.

Theorem (Aubin 1976)

If M has dimension $m \geq 6$ and is not locally conformally flat, then $\lambda(M) < \lambda(S^m)$.

Theorem (Schoen 1984)

If M has dimension $3, 4$ or 5, or if M is locally conformally flat, then $\lambda(M) < \lambda(S^m)$ unless M is conformal to the standard sphere.

- Blow-up for the Yamabe equation $-c_m \Delta u + S u = \lambda u^{p-1}$
  - the unknown function $u > 0$ appears as the conformal factor, i.e., $\tilde{g} = u^{\frac{4}{m-2}} g$

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- $*$ $u_n \in C^2(M)$, $u_n > 0$

$$\begin{cases} -c_m \Delta u_n + S u_n = \lambda u_n^{p-1} + o_n(1) \quad \text{in } H^{-1}(M) \\ Vol\Big(M, u_n^{\frac{4}{m-2}} g\Big) = \int_M u_n^{\frac{2m}{m-2}} d\mathrm{vol}_g \to constant > 0 \\ \max_M u_n \to +\infty \end{cases}$$

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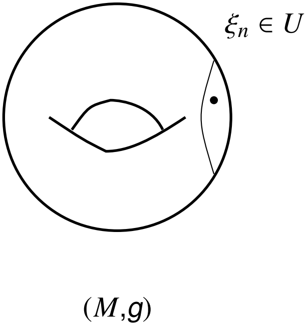
If we choose $\xi_n \in M$ with $u_n(\xi_n) = \max_M u_n$, the blow-up phenomenon describes concentration behavior of the conformal metrics $u_n^{\frac{4}{m-2}} g$ around ξ_n. (spike layer)

- Blow-up



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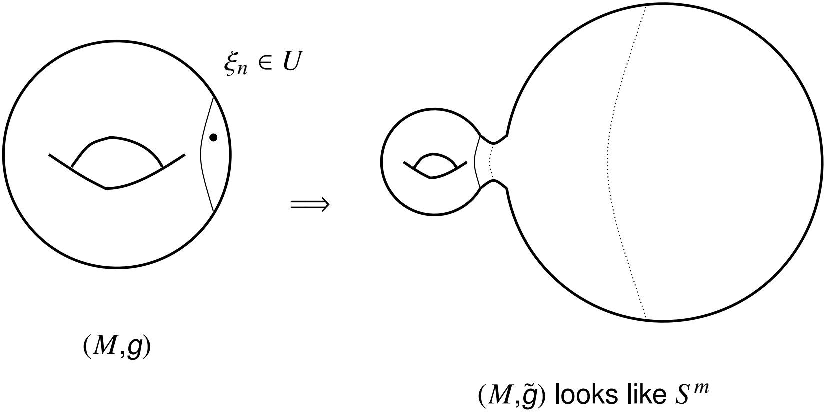

If the background manifold M is a sphere, then blow-up still gives a sphere (conformally).

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$\Longrightarrow$ value of $Q_g(u_n)$ converges to $\lambda(S^m)$

$\lambda(M) < \lambda(S^m)$, i.e. $\inf Q_g < \lambda(S^m) \Longrightarrow$ Blow-up will not occur

$\Longrightarrow$ compactness

Spinorial Yamabe problem and related topics

In the setting of Spin Geometry, a problem analogous to the Yamabe problem has received increasing attention in recent years. Bernd Ammann *et al* provide a framework in which variational methods may be employed.

References

[1] B. Ammann, A variational problem in conformal spin geometry, Habilitationsschift, Universität Hamburg, (2003).

[2] B. Ammann, J.-F. Grossjean, E. Humbert, B. Morel, A spinorial analogue of Aubin's inequality, Math. Z. 260 (2008), 127-151.

Spinorial Yamabe problem and related topics

1. Dirac operator introduced by M.F. Atiyah in 1962.

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  - $\mathbb{C}\ell(V)$ the complex Clifford algebra of V with its multiplication being denoted by "$\cdot$". Precisely,

$$\mathbb{C}\ell(V, g) = T(V)/I(V, g),$$

where $T(V) = \oplus_{s\geq 0} \otimes_{j=0}^{s} V$ is the tensor algebra of V, and $I(V, g)$ the ideal generated by all elements of the form $x \otimes x + g(x, x)$, for $x \in V$.

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  - Clifford representation

$$\mathbb{C}\ell(V, g) \cong \begin{cases} \mathcal{M}(2^k; \mathbb{C}) & m = 2k \\ \mathcal{M}(2^k; \mathbb{C}) \oplus \mathcal{M}(2^k; \mathbb{C}) & m = 2k + 1 \end{cases}$$

$\mathbb{C}\ell(V)$ has an irreducible module, the spinor module $\mathbb{S}_m$, with $\dim_{\mathbb{C}} \mathbb{S}_m = 2^{[\frac{m}{2}]}$.

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$\mathbb{C}\ell(V)$ has an irreducible module, the spinor module $\mathbb{S}_m$, with $\dim_{\mathbb{C}} \mathbb{S}_m = 2^{[\frac{m}{2}]}$. The Clifford representation is the following action

$$\mathbb{C}\ell(V) \otimes \mathbb{S}_m \to \mathbb{S}_m, \quad \xi \otimes \psi \mapsto \xi \cdot \psi.$$

Spinorial Yamabe problem and related topics

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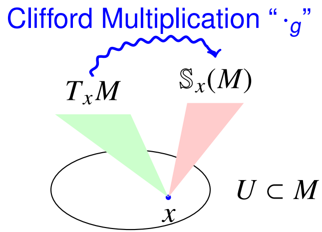
- The Dirac operator

  Suppose (M, g, σ) is a spin manifold of dimension $m \geq 2$, with a fixed spin structure σ.



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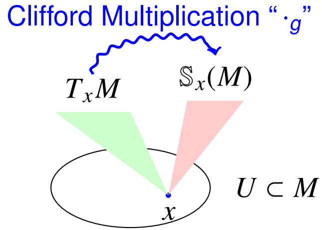

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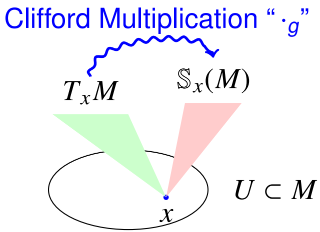

  Let $\{e_1, \ldots, e_m\} \subset T_U M$ be an (local) orthonormal frame, the Dirac operator is defined as

$$D_g\psi = \sum_{k=1}^{m} e_k \cdot_g \nabla_{e_k}\psi \quad \text{for smooth section } \psi \in \mathbb{S}(M)$$

  where ∇ is the lifted Levi-Civita connection (on the spinor bundle).

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Difference:

- Dirac operator is a first order differential operator
- Dirac operator acts on spinors (which are complex vectors)
- the spectrum of a Dirac operator accumulates both $+\infty$ and $-\infty$

$$-\infty \leftarrow \cdots \leq \lambda_{-2} \leq \lambda_{-1} \leq 0 < \lambda_1 \leq \lambda_2 \leq \cdots \rightarrow +\infty$$

- Bär-Hijazi-Lott invariant

$$\lambda_{min}^+(M,[g],\sigma) := \inf_{\tilde{g}\in[g]} \lambda_1^+(\tilde{g})\mathrm{Vol}(M,\tilde{g})^{\frac{1}{m}}$$

  - $\lambda_1^+(\tilde{g})$ the smallest positive eigenvalue of the Dirac operator $D_{\tilde{g}}$
  - $\tilde{g} \in [g] := \{f^2 g : \ f \in C^\infty(M), \ f > 0\}$

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  * If D_g is invertible, then $\lambda_{min}^+(M,[g],\sigma) > 0$. (J. Lott 1986)
  * $\lambda_{min}^+(M,[g],\sigma)^2 \geq \frac{m}{4(m-1)}\lambda(M)$. (O. Hijazi 1986)
  * $\lambda_{min}^+(S^2,[g_{S^2}],\sigma) = 2\sqrt{\pi}$. (C. Bär 1992)

- Bär-Hijazi-Lott invariant

$$\lambda^+_{min}(M,[g],\sigma) := \inf_{\tilde{g}\in[g]} \lambda^+_1(\tilde{g})\mathrm{Vol}(M,\tilde{g})^{\frac{1}{m}}$$

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* $\lambda^+_{min}(M,[g],\sigma)^2 \geq \frac{m}{4(m-1)}\lambda(M)$. (O. Hijazi 1986)
* $\lambda^+_{min}(S^2,[g_{S^2}],\sigma) = 2\sqrt{\pi}$. (C. Bär 1992)
* $\lambda^+_{min}(M,[g],\sigma)>0$ for any closed spin manifold. (B. Ammann 2003)
* $\lambda^+_{min}(M,[g],\sigma) \leq \lambda^+_{min}(S^m,[g_{S^m}],\sigma) = \frac{m}{2}\omega_m^{\frac{1}{m}}$, where ω_m stands for the volume of S^m. (Ammann et al 2006-2008)

- Bär-Hijazi-Lott invariant

Ammann's idea: finding a critical metric of BHL-invariant is equivalent to prove the existence of a spinor field $\psi \in C^\infty(M, \mathbb{S}(M))$ minimizing the functional defined by

$$J_g(\phi) = \frac{\left(\int_M |D_g \phi|^{\frac{2m}{m+1}} d\mathrm{vol}_g \right)^{\frac{m+1}{m}}}{\left| \int_M (D_g \phi, \phi) d\mathrm{vol}_g \right|},$$

whose Euler-Lagrange equation is

$$D_g \psi = \lambda^+_{min}(M, [g], \sigma) |\psi|_g^{m^*-2} \psi, \quad \text{where } m^* := \frac{2m}{m-1}.$$

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One may also consider the normalized equation

$$D_g\psi = |\psi|_g^{\frac{2}{m-1}}\psi, \qquad m^* - 2 = \frac{2}{m-1}$$

- Bär-Hijazi-Lott invariant

A compactness result (Ammann 2009)

Assume that

$$\lambda^+_{min}(M,[g],\sigma) < \lambda^+_{min}(S^m,[g_{S^m}],\sigma).$$

The ground state energy solution of the perturbed equation (with $\varepsilon > 0$, i.e. subcritical exponent)

$$D_g\psi = |\psi|_g^{\frac{2}{m-1}-\varepsilon}\psi$$

will converge in the C^1-topology as $\varepsilon \to 0$ to a solution of

$$D_g\psi = |\psi|_g^{\frac{2}{m-1}}\psi.$$

Moreover, $\lambda^+_{min}(M,[g],\sigma)$ is attained.

Open problem: the strict inequality for $\lambda^+_{min}(M,[g],\sigma)$.

Spinorial Yamabe problem and related topics

- Spinorial Weierstraß representation

  Goal or Geometric starting point: a global representation for surfaces in $\mathbb{R}^3$ of arbitrary mean curvature H.

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  - Simple description:

Spinorial Weierstraß representation

$$\left\{\begin{array}{l}\text{Solutions of} \\ D_g\psi = H\psi, \\ |\psi|_g = 1 \text{ on } M\end{array}\right\} \Longleftrightarrow \left\{\begin{array}{l}\text{Periodic isometric immersions of} \\ \widetilde{M} \text{ in } \mathbb{R}^3 \text{ with mean curvature } H\end{array}\right\}$$

History involved:

Kenmotsu (1979), Abresch (1987), Trautman (1992,1995), Kusner & Schmitt (1993, 1995, 1996), Konopelchenko (1996), Taimanov (1996,1997), Bär (1998), Friedrich (1998)

Spinorial Yamabe problem and related topics

- Spinorial Weierstraß representation

The approach by Trautman/Bär/Friedrich
— restricting spinors to hypersurfaces

For Riemannian spin manifolds $M^m \hookrightarrow N^{m+1}$

$$\nabla_X^M \varphi = \nabla_X^N \varphi - \frac{1}{2} W(X) \cdot \varphi$$

$\varphi : M \to \mathbb{S}(M) \subset \mathbb{S}(N)$ spinor field defined on M
$X \in TM \subset TN$
W is the Weingarten map of M in N
$W(X) \cdot \varphi \in \mathbb{S}(M)$ is Clifford multiplication of $W(X) \in TM$ and $\varphi \in \mathbb{S}(M)$

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W is the Weingarten map of M in N
$W(X) \cdot \varphi \in \mathbb{S}(M)$ is Clifford multiplication of $W(X) \in TM$ and $\varphi \in \mathbb{S}(M)$

Consequence: If we restrict a parallel spinor φ of N to M, then

$$D_g \varphi^* = H \varphi^*, \quad |\varphi^*|_g \equiv constant \ (= 1 \text{ w.l.o.g})$$

Spinorial Yamabe problem and related topics

- Spinorial Weierstraß representation

  Converse:

$m = 2$ Friedrich's approach: Assume φ solves

$$D_g\varphi = H\varphi, \quad |\varphi|_g \equiv 1.$$

Then there is a symmetric W with

$$\nabla_X\varphi = -\frac{1}{2}W(X)\cdot\varphi$$

Further, W satisfies the Gauß and Codazzi equations. This yields an immersion $\widetilde{M} \hookrightarrow \mathbb{R}^3$.

- Conformal change

  Let g be a Riemannian metric on M^m, $\overline{g} = f^2 g$, then

$$D_{\overline{g}}\varphi = f^{-\frac{m+1}{2}} D_g\Big(f^{\frac{m-1}{2}}\varphi\Big)$$

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$$D_{\overline{g}}\varphi = f^{-\frac{m+1}{2}} D_g\Big(f^{\frac{m-1}{2}}\varphi\Big)$$

Consequence: If $\varphi : M \to \mathbb{S}(M)$ solves the nonlinear equation

$$D_g\varphi = H(\xi)|\varphi|_g^{\frac{2}{m-1}}\varphi, \quad |\varphi|_g > 0$$

then by setting $g_1 = |\varphi|_g^{\frac{4}{m-1}} g$ and $\psi = |\varphi|_g^{-1}\varphi$, one obtains a solution to

$$D_{g_1}\psi = H(\xi)\psi, \quad |\psi|_{g_1} = 1.$$

Spinorial Weierstraß representation $\Rightarrow$ conformal immersion of $\widetilde{M} \hookrightarrow \mathbb{R}^3$.

Spinorial Yamabe problem and related topics

- Geometric problem transformed into a PDE problem:

For $H : M \to \mathbb{R}$ a given (smooth) function

$$
\begin{array}{l} D_g\psi = H(\xi)|\psi|_g^{\frac{2}{m-1}}\psi \\ \text{on } (M, g, \sigma) \end{array}
\begin{cases}
\text{Spinorial Yamabe } (H \equiv constant) \begin{cases} \text{estimates on BHL} \\ \text{existence} \end{cases} \\
\text{Prescribed mean curvature} \begin{cases} \text{suitable criterion for } H \\ \text{existence} \\ \text{zero set of a solution} \end{cases}
\end{cases}
$$

Theorem (Raulot 2009)

If the Dirac operator D_g is invertible, H is positive and satisfies

$$\lambda_{min,H} \cdot \big(\max_M H\big)^{\frac{m-1}{m}} < \frac{m}{2}\omega_m^{\frac{1}{m}}, \quad \lambda_{min,H} := \inf_{\phi\neq 0} \frac{\Big(\int_M H^{-\frac{m-1}{m+1}} |D_g\phi|^{\frac{2m}{m+1}} d\mathrm{vol}_g\Big)^{\frac{m+1}{m}}}{\Big|\int_M (D_g\phi,\phi) d\mathrm{vol}_g\Big|}$$

then there exists a solution to the equation

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1. The strict inequality is an open problem.
2. The strict inequality cannot be valid on S^m in any circumstance.

Study the PDE problem

$$D_g\psi = H(\xi)|\psi|^{\frac{2}{m-1}}\psi \quad \text{on } (S^m, g_{S^m}) \tag{3}$$

is of particular interests: in the 2-dimensional case

Question 1

Given an arbitrary smooth function $H : S^2 \to \mathbb{R}$, what is the condition on H so that there is an isometric immersion $S^2 \hookrightarrow \mathbb{R}^3$ realizing H as its mean curvature?

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Question 2

Assume that a function $H : S^2 \to \mathbb{R}$ is given such that it is the mean curvature of an immersion. Under which condition is this immersion an embedding? And when does it have self-intersections?

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Theorem (C. Bär 1999)

Let (M, g) be compact and connected spin m-manifold and let φ be a solution of

$$D_g\varphi = P\varphi$$

where P is a smooth endomorphism. Then the zero set of φ has at most Hausdorff dimension $m-2$. And if $m = 2$, then the zero set is discrete.

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Can we count the number of the zeros?

Lemma

On a closed spin surface (M^2, g) of genus γ, suppose that $H \in C^\infty$ is positive. Let ψ be a solution of the equation

$$D_g\psi = H(\xi)|\psi|_g^2\psi \quad \text{on } M^2.$$

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$$\#\psi^{-1}(0) \le -1 + \frac{H_{max}\int_{M^2} H(\xi)|\psi|^4 d\mathrm{vol}_g}{4\pi} < -1 + 2 = 1$$

Idea of the proof:

- On $M \setminus \psi^{-1}(0)$, by using the conformal change of the Dirac operator, we obtain a solution φ_1 to

$$D_{g_1}\varphi_1 = H(\xi)\varphi_1 \quad \text{and} \quad |\varphi_1|_{g_1} = 1$$

where $g_1 = |\psi|^{\frac{4}{m-1}} g$. Then by Schrödinger-Lichnerowicz formula, we get

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- In case $m = 2$, let $\psi^{-1}(0) = \{\xi_1, \cdots, \xi_l\}$ for some $l \ge 1$ and n_j be the order of the first non-vanishing term in the Taylor expansion of ψ near each zero point, it can be calculated that the integral of the geodesic curvature κ_{g_1} over a small circle around the ξ_j is close to $-2\pi(2n_j + 1)$.

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- Gauß-Bonnet $\Rightarrow 4\pi \sum_{j=1}^{l} n_j \le 2\pi(2\gamma - 2) + \int_{M\setminus\psi^{-1}(0)} H(\xi)^2 |\psi|^4 d\mathrm{vol}_g$

- Prescribed mean curvature problem

  Let $M^2 = S^2$, then solutions of

$$D_{g_{S^2}}\psi = H(\xi)|\psi|^2_{g_{S^2}}\psi \tag{4}$$

  satisfying

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  * Recalling the strict inequality formulated by S. Raulot (2009):

$$\lambda_{min,H} \cdot \big(\max_M H\big)^{\frac{m-1}{m}} < \frac{m}{2}\omega_m^{\frac{1}{m}}.$$

  which is no longer valid on spheres. We have

$$\lambda_{min,H} \cdot \big(\max_{S^m} H\big)^{\frac{m-1}{m}} \geq \frac{m}{2}\omega_m^{\frac{1}{m}} \quad \text{on } S^m.$$

  In dimension 2, this is to say that the solution of Eq. (4) satisfies

$$\int_{S^2} H(\xi)|\psi|^4 d\mathrm{vol}_{g_{S^2}} \geq \frac{4\pi}{H_{max}},$$

- Variational structure

On (S^m, g_{S^m}), $m \geq 2$,

$$D\psi = H(\xi)|\psi|^{\frac{2}{m-1}}\psi \tag{5}$$

Eq. (5) is the Euler-Lagrange equation for the functional

$$\mathcal{L}(\psi) = \frac{1}{2}\int_{S^m}(D\psi, \psi)d\mathrm{vol}_{g_{S^m}} - \frac{1}{m^*}\int_{S^m} H(\xi)|\psi|^{m^*} d\mathrm{vol}_{g_{S^m}}$$

where $m^* = \frac{2m}{m-1}$.

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where $m^* = \frac{2m}{m-1}$. For $\varepsilon > 0$, we also consider the perturbed problem

$$D\psi = H(\xi)|\psi|^{\frac{2}{m-1}-\varepsilon}\psi \tag{6}$$

and the associated functional

$$\mathcal{L}_\varepsilon(\psi) = \frac{1}{2}\int_{S^m}(D\psi,\psi)d\mathrm{vol}_{g_{S^m}} - \frac{1}{m^*-\varepsilon}\int_{S^m} H(\xi)|\psi|^{m^*-\varepsilon}d\mathrm{vol}_{g_{S^m}}$$

- Variational structure

The L^2 spectrum of D is

$$Spec(D) = \Big\{ \pm(\frac{m}{2} + j) : \ j = 0, 1, 2, \dots \Big\}$$

– $Spec(D) = \{\lambda_k\}_{k\in\mathbb{Z}\setminus\{0\}}$

(1) By classical spectral theory

$$L^2(S^m, \mathbb{S}(S^m)) = \overline{\bigoplus_{\lambda\in Spec(D)} \ker(D - \lambda I)}.$$

(2) Eigenspinors $\{\eta_k\}_{k\in\mathbb{Z}\setminus\{0\}}$ be the complete orthonormal basis of $L^2(S^m, \mathbb{S}(S^m))$.

(3) Define the operator $|D|^{\frac{1}{2}} : L^2(S^m, \mathbb{S}(S^m)) \to L^2(S^m, \mathbb{S}(S^m))$ by

$$|D|^{\frac{1}{2}}\psi = \sum_{k\in\mathbb{Z}\setminus\{0\}} |\lambda_k|^{\frac{1}{2}} a_k \eta_k,$$

where $\psi = \sum_{k\in\mathbb{Z}\setminus\{0\}} a_k \eta_k \in L^2(S^m, \mathbb{S}(S^m))$.

- Variational structure

Set

$$E := \Big\{\psi = \sum_{k\in\mathbb{Z}} a_k\eta_k \in L^2(S^m, \mathbb{S}(S^m)) : \sum_{k\in\mathbb{Z}\setminus\{0\}} |\lambda_k||a_k|^2 < \infty\Big\}.$$

We have E coincides with the Sobolev space $W^{\frac{1}{2},2}(S^m, \mathbb{S}(S^m))$

- endow E with the inner product

$$\langle\psi, \varphi\rangle = \mathrm{Re}(|D|^{\frac{1}{2}}\psi, |D|^{\frac{1}{2}}\varphi)_2$$

  and the induced norm $\|\cdot\| = \langle\cdot,\cdot\rangle^{\frac{1}{2}}$.
- $E \hookrightarrow L^q(S^m, \mathbb{S}(S^m))$ for $2 \le q \le m^* := \frac{2m}{m-1}$.
- E has a splitting $E = E^+ \oplus E^-$ with

$$E^+ := \overline{\mathrm{span}\{\eta_k\}_{k>0}} \quad \text{and} \quad E^- := \overline{\mathrm{span}\{\eta_k\}_{k<0}},$$

  where the closure is taken in the $\|\cdot\|$-topology.

- Variational structure

$$D\psi = H(\xi)|\psi|^{p-2}\psi \quad \text{on } S^m,$$

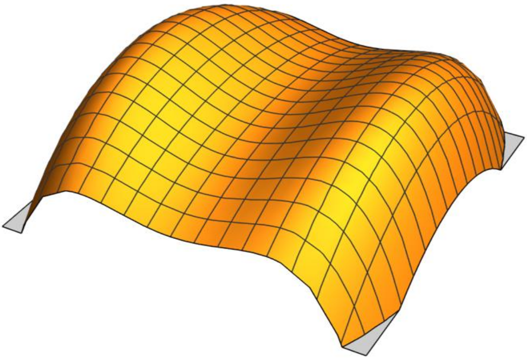
$p \in (2, m^*]$ and $m^* = \frac{2m}{m-1}$, has a variational structure: ψ is a solution if and only if ψ is a critical point of $\mathcal{L}_p$ defined by

$$\begin{aligned}\mathcal{L}_p(\psi) &= \frac{1}{2}\int_{S^m}(D\psi, \psi)d\mathrm{vol}_{g_{S^m}} - \frac{1}{p}\int_{S^m} H(\xi)|\psi|^p d\mathrm{vol}_{g_{S^m}} \\ &= \frac{1}{2}(\|\psi^+\|^2 - \|\psi^-\|^2) - \frac{1}{p}\int_{S^m} H(\xi)|\psi|^p d\mathrm{vol}_{g_{S^m}}\end{aligned}$$

for $\psi = \psi^+ + \psi^- \in E = E^+ \oplus E^-$.

Our perturbed equations are just taking the values $p = m^* - \varepsilon$.

$$\mathcal{L}_p(\psi) = \frac{1}{2}(\|\psi^+\|^2 - \|\psi^-\|^2) - \frac{1}{p}\int_{S^m} H(\xi)|\psi|^p d\mathrm{vol}_{g_{S^m}}$$



$$\mathcal{L}_p(tu + \psi^-) = \frac{1}{2}(t^2\|u\|^2 - \|\psi^-\|^2) - \frac{1}{p}\int_{S^m} H(\xi)|tu + \psi^-|^p d\mathrm{vol}_{g_{S^m}}$$

- Variational structure
  - Lyapunov-Schmidt type reduction

Proposition

There exists a C^1 *map* $h_p : E^+ \to E^-$ *such that*

$$\mathcal{L}_p(u + h_p(u)) = \max_{v \in E^-} \mathcal{L}_p(u + v).$$

Furthermore, if denoted by $I_p(u) = \mathcal{L}_p(u + h_p(u))$, *the function* $t \mapsto I_p(tu)$ *is* C^2 *and, for* $u \in E^+ \setminus \{0\}$ *and* $t > 0$,

$$I_p'(tu)[u] = 0 \;\Rightarrow\; I_p''(tu)[u, u] < 0.$$

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Furthermore, if denoted by $I_p(u) = \mathcal{L}_p(u + h_p(u))$, the function $t \mapsto I_p(tu)$ is C^2 and, for $u \in E^+ \setminus \{0\}$ and $t > 0$,

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- critical points of I_p and $\mathcal{L}_p$ are in one-to-one correspondence via the injective map $u \mapsto u + h_p(u)$ from E^+ to E;
- the set

$$\mathcal{N}_p = \{u \in E^+ \setminus \{0\} : \ I_p'(u)[u] = 0\}$$

  is a smooth manifold of codimension 1 in E^+ containing all critical points of I_p, moreover, $I_p\big|_{\mathcal{N}_p}$ is bounded from below.

- Blow-up analysis

  Quantization of blowing up sequence of almost critical point for the functional

$$\mathcal{L}_n(\psi) = \frac{1}{2}\int_{S^m}(D\psi,\psi)d\mathrm{vol}_{g_{S^m}} - \frac{1}{p_n}\int_{S^m} H(\xi)|\psi|^{p_n} d\mathrm{vol}_{g_{S^m}}$$

  where $p_n \nearrow m^*$ as $n \to \infty$;

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Proposition (1)

Suppose $\{\psi_n\} \subset E$ *is a sequence such that*

$$\frac{1}{2m(H_{max})^{m-1}}\left(\frac{m}{2}\right)^m \omega_m \le \mathcal{L}_n(\psi_n) \le \frac{1}{m(H_{max})^{m-1}}\left(\frac{m}{2}\right)^m \omega_m - \theta \quad \text{and} \quad \mathcal{L}_n'(\psi_n) \to 0$$

for some constant $\theta > 0$*. Then, up to a subsequence, either* $\psi_n \rightharpoonup 0$ *or* $\psi_n \to \psi_0 \neq 0$ *in* E *where* ψ_0 *solves*

$$D\psi = H(\xi)|\psi|^{m^*-2}\psi.$$

- Blow-up analysis

Proposition (2)

Let $\{\psi_n\} \subset E$ fulfill the assumption of Proposition (1). If $\{\psi_n\}$ does not contain any compact subsequence. Then, up to a subsequence if necessary, there exist a convergent sequence $\{a_n\} \subset S^m$, $a_n \to a$ as $n \to \infty$, a sequence of radius $\{R_n\}$ converging to 0, a real number $\lambda \in (2^{-\frac{1}{m-1}}, 1]$ and a non-trivial solution ϕ_0 of

$$D_{g_{\mathbb{R}^m}} \phi_0 = \lambda H(a) |\phi_0|^{2^*-2} \phi_0 \quad \text{on } \mathbb{R}^m$$

such that $R_n^{\frac{m-1}{2}(2^*-p_n)} = \lambda + o_n(1)$ and

$$\psi_n = R_n^{-\frac{m-1}{2}} \eta(\cdot) \overline{(\mu_n)_*} \circ \phi_0 \circ \mu_n^{-1} + o_n(1) \quad \text{in } E$$

as $n \to \infty$, where $\mu_n(x) = \exp_{a_n}(R_n x)$ and $\eta \in C^\infty(S^m)$ is a cut-off function such that $\eta(\xi) = 1$ on $B_r(a)$ and $\operatorname{supp} \eta \subset B_{2r}(a)$, some $r > 0$. Moreover, we have

$$\mathcal{L}_n(\psi_n) \geq \frac{1}{2m(\lambda H(a))^{m-1}} \left(\frac{m}{2}\right)^m \omega_m + o_n(1)$$

as $n \to \infty$. If additionally $\mathcal{L}_n'(\psi_n) \equiv 0$ for all n, then $\nabla H(a) = 0$ and $\lambda = 1$.

- Criteria of H

  $H : S^m \to (0, +\infty)$ be a positive function of C^2 class

  - $H_{max} := \max_{\xi \in S^m} H(\xi)$ and $H_{min} := \min_{\xi \in S^m} H(\xi)$
  - $\mathcal{H} := \{\xi \in S^m : H(\xi) = H_{max}\}$ is the collection of maximum points
  - $\mathcal{H}_\delta := \{\xi \in S^m : \text{dist}_{g_{S^m}}(\xi, \mathcal{H}) < \delta\}$ is its δ-neighborhood for $\delta > 0$

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$(H*)$ *$\mathcal{H}$ is not contractible in its δ-neighborhood $\mathcal{H}_\delta$, for some small $\delta > 0$, but there exits $d \in \big(\max\{2^{-\frac{1}{m-1}} H_{max}, \ H_{min}\}, H_{max}\big)$ such that $\mathcal{H}$ is contractible in $\{\xi \in S^m : \ H(\xi) \geq d\}$.*

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$(H*)$ *$\mathcal{H}$ is not contractible in its δ-neighborhood $\mathcal{H}_\delta$, for some small $\delta > 0$, but there exits $d \in \big(\max\{2^{-\frac{1}{m-1}} H_{max}, H_{min}\}, H_{max}\big)$ such that $\mathcal{H}$ is contractible in $\{\xi \in S^m : H(\xi) \geq d\}$. There is no critical value of H in the interval (d, H_{max}), and if $\xi \in S^m$ is a critical point of H with $H(\xi) = d$ then the Hessian of H at ξ is positive definite.*

- Criteria of H

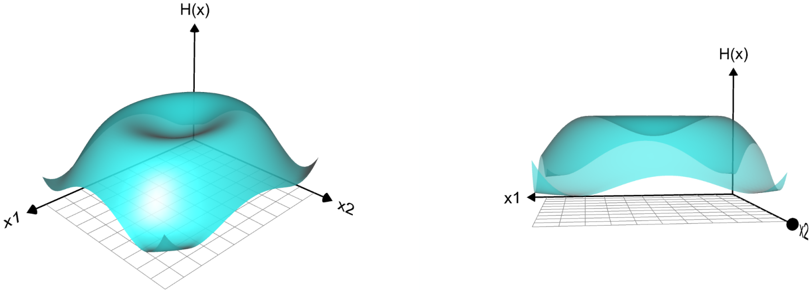
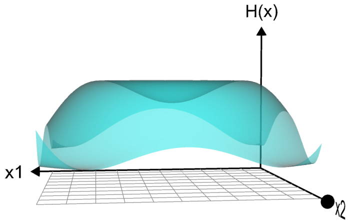


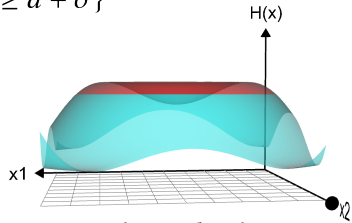
Figure: A possible shape of function H's, in local coordinates, which satisfies the criteria (H).

★ Remark that the class of H's which satisfy the hypothesis (H) is dense, in C^1-topology, in the class of positive smooth functions.

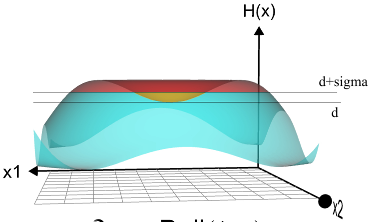
- The existence

- $\mathcal{H}$ is contractible in $\{H \geq d + \sigma\} \cup \mathcal{O}_\nu$
- $\mathcal{H}$ is not contractible in $\{H \geq d + \sigma\}$





$\{H \geq d+\sigma\}$



$\mathcal{O}_\nu :=$ Ball(ξ, ν)

- The existence

Recall that, for $p \in (2, m^*]$, we have reduced the functional $\mathcal{L}_p$ on E to a functional I_p on E^+

$$I_p(u) = \mathcal{L}_p(u + h_p(u))$$

and

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Lemma (the main estimate 1)

There exists $\theta > 0$ small (depending on the function H) and a embedding $\Phi_p : \{H \geq d + \sigma\} \cup \mathcal{O}_\nu \to \mathcal{N}_p$ such that

$$\max_{\xi \in \{H \geq d+\sigma\} \cup \mathcal{O}_\nu} I_p(\Phi_p(\xi)) \leq \hat{a} := \frac{1}{2m\, d^{m-1}} (\frac{m}{2})^m \omega_m - \theta$$

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$$d > 2^{-\frac{1}{m-1}} H_{max} \Rightarrow \hat{a} < \frac{1}{m(H_{max})^{m-1}} (\frac{m}{2})^m \omega_m - \theta$$

- The existence

Lemma (the main estimate 2)

Fix $\theta > 0$ small in the previous lemma, for $2 < p < m^*$ sufficiently close to m^*, there exists a continuous map $\Psi_p : \mathcal{N}_p \to S^m$ such that

$$I_p(u) \leq \hat{b} := \frac{1}{2m(H_{max})^{m-1}}(\frac{m}{2})^m \omega_m + \theta \Longrightarrow \Psi_p(u) \in \mathcal{H}_\delta.$$

- The existence

Lemma (the main estimate 2)

Fix $\theta > 0$ small in the previous lemma, for $2 < p < m^*$ sufficiently close to m^*, there exists a continuous map $\Psi_p : \mathcal{N}_p \to S^m$ such that

$$I_p(u) \le \hat{b} := \frac{1}{2m(H_{max})^{m-1}}\left(\frac{m}{2}\right)^m \omega_m + \theta \Longrightarrow \Psi_p(u) \in \mathcal{H}_\delta.$$

* For each $2 < p < m^*$ sufficiently close to m^*, suppose there is no critical value of $I_p\big|_{\mathcal{N}_p}$ in the interval $[\hat{b}, \hat{a}]$. Then by the Deformation Lemma, $\{I_p\big|_{\mathcal{N}_p} \le \hat{a}\}$ can be continuously deformed into $\{I_p\big|_{\mathcal{N}_p} \le \hat{b}\}$. And hence

$$\mathcal{H} \xrightarrow{\Phi_p} \{I_p\big|_{\mathcal{N}_p} \le \hat{a}\} \xrightarrow{\text{deform}} \{I_p\big|_{\mathcal{N}_p} \le \hat{b}\} \xrightarrow{\Psi_p} \mathcal{H}_\delta$$

is contractible, a contradiction!

- The existence

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$\Downarrow$ via blow-up

if $\{\psi_n\}$ is not compact, then there exist a blow-up point $a \in S^m$ such that

$$\hat{a} \geq \mathcal{L}_n(\psi_n) \geq \frac{1}{2m(H(a))^{m-1}}(\frac{m}{2})^m \omega_m + o_n(1)$$

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$\Downarrow$ no critical value in (d, H_{max})

$H(a) = H_{max}$ i.e. $a \in \mathcal{H}$

- The existence

In dimension 2, if $\{\psi_n\}$ blows up, there exists a non-trivial solution ϕ_0 of

$$D_{g_{\mathbb{R}^2}} \phi_0 = H_{max} |\phi_0|^2 \phi_0 \quad \text{on } \mathbb{R}^2$$

such that

$$H_{max} \int_{\mathbb{R}^2} |\phi_0|^4 d\mathrm{vol}_{g_{\mathbb{R}^2}} < \frac{8\pi}{H_{max}}.$$

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Via stereographic projection, it corresponds to a solution $\bar{\phi}_0$ of

$$D_{g_{S^2}}\bar{\phi}_0 = H_{max}|\bar{\phi}_0|^2\bar{\phi}_0 \quad \text{on } S^2$$

with

$$H_{max}\int_{S^2}|\bar{\phi}_0|^4 d\mathrm{vol}_{g_{S^2}} = H_{max}\int_{\mathbb{R}^2}|\phi_0|^4 d\mathrm{vol}_{g_{\mathbb{R}^2}} < \frac{8\pi}{H_{max}}.$$

Spinorial Yamabe problem and related topics

- The existence

  * $\bar{\phi}_0$ has no zero
  * $(S^2, |\bar{\phi}_0|^4 g_{S^2})$ is isometrically immersed into $\mathbb{R}^3$ with constant mean curvature $H_{max} > 0$
  * the Willmore energy

$$\int_{S^2} H_{max}^2 d\mathrm{vol}_{|\bar{\phi}_0|^4 g_{S^2}} = H_{max}^2 \int_{S^2} |\bar{\phi}_0|^4 d\mathrm{vol}_{g_{S^2}} < 8\pi.$$

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Li-Yau's inequality implies $(S^2, |\bar{\phi}_0|^4 g_{S^2})$ is embedded into $\mathbb{R}^3$.

$$\Downarrow$$

Alexandrov's theorem suggests $(S^2, |\bar{\phi}_0|^4 g_{S^2})$ must be a round sphere

$$\Downarrow$$

$$H_{max}^2 \int_{S^2} |\bar{\phi}_0|^4 d\mathrm{vol}_{g_{S^2}} = 4\pi$$

- The existence

  Back to the blow-up result, we see that

$$\mathcal{L}_n(\psi_n) \to \frac{H_{max}}{4} \int_{\mathbb{R}^2} |\phi_0|^4 d\mathrm{vol}_{g_{\mathbb{R}^2}} = \frac{H_{max}}{4} \int_{S^2} |\bar{\phi}_0|^4 d\mathrm{vol}_{g_{S^2}} = \frac{\pi}{H_{max}} < \hat{b}$$

  Contradicts to the fact $\hat{b} \le \mathcal{L}_n(\psi_n) \le \hat{a}$.

  Hence $\{\psi_n\}$ must be compact.

- The existence

Theorem

If $H : S^2 \to (0, \infty)$ is a C^2 function satisfying condition (H), then there is a solution ψ to

$$D_{g_{S^2}}\psi = H(\xi)|\psi|^2\psi \quad \text{on } S^2$$

with the energy estimate

$$\frac{4\pi}{H_{max}} < \int_{S^2} H(\xi)|\psi|^4_{g_{S^2}} d\mathrm{vol}_{g_{S^2}} < \frac{8\pi}{H_{max}}.$$

Particularly, if H is smooth, then the solution ψ has no zero at all, i.e. the nodal set $\psi^{-1}(0)$ is empty.

- The existence

Corollary

Let C_+^∞ denote the class of positive smooth functions on S^2 and let

$$\mathscr{H} = \{H \in C_+^\infty : H \equiv constant \text{ or } H \text{ satisfies hypothesis } (H)\}.$$

Then $\mathscr{H}$ has the following properties:

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(2) *for any $H \in \mathscr{H}$, there exists a conformal metric $g_H \in [g_{S^2}]$ and an isometric immersion $F : (S^2, g_H) \to (\mathbb{R}^3, g_{\mathbb{R}^3})$ with H being its mean curvature,*

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(3) *the Willmore energy of F satisfies*

$$\mathcal{W}(F) = \int_{S^2} H(\xi)^2 d\mathrm{vol}_{g_H} < 8\pi,$$

and thus F is an isometric embedding.

Some remarks

(1) One may simply expect that all smooth function can be a mean curvature of some conformal immersion. However, it is easy to see that this cannot be true. Indeed, there is a necessary condition on H for Eq. (5) to be solvable. This follows from the obstruction found by Ammann, Humbert and Ahmedou: if ψ solves (5), then

$$\int_{S^m} (\nabla_X H)|\psi|^{m^*} d\mathrm{vol}_{g_{S^m}} = 0$$

for any conformal vector field X on S^m.

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Let us consider $x_3 : S^2 \to \mathbb{R}^3$ be the third component of the standard inclusion. The vector field $X := \mathrm{grad}\, x_3$ is a conformal vector field on S^2. It can be calculated that $\nabla_X(1 + \varepsilon x_3) = \varepsilon g_{S^2}(\mathrm{grad}\, x_3, \mathrm{grad}\, x_3)$, which is of constant sign. Hence, $1 + \varepsilon x_3$ is not a mean curvature of any conformal immersion on S^2.

Some remarks

(2) In 2013, T. Isobe obtained a existence result. He proved that when the function H is very close to a constant, that is $H(\xi) = 1 + \varepsilon Q(\xi)$ on S^2 with Q being some Morse function, then there exists a branched immersion $F_{\varepsilon,H} : S^2 \to \mathbb{R}^3$ whose mean curvature is $H = 1 + \varepsilon Q$. Here, by branched immersion we mean that $F_{\varepsilon,H}$ is an immersion except at a discrete set of points. The approach of Isobe to this result is also based on the Spinorial Weierstraß representation and the nonlinear Dirac equation.

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Another result concerning this problem is due to M. Anderson 2015. He proved that for any positive function $H : S^2 \to (0, \infty)$ in the class $C^{k,\alpha}$, $k \geq 0$ and $\alpha \in (0, 1)$, there exists a branched immersion $F : S^2 \to \mathbb{R}^3$ and an positive affine function ℓ on S^2 such that the mean curvature of F is prescribed by $H + \ell$. In this setting the function $H > 0$ lies arbitrarily in the class $C^{k,\alpha}$, and thus it can be far away from a constant. However, the affine function ℓ can not be prescribed.

Some remarks

(3) The hypersurfaces with constant mean curvature are much more complicated in higher dimensions, surprising examples have been shown: there exist infinitely many distinct differentiable immersions of the 3-sphere into Euclidean 4-space having a given positive constant mean curvature, moreover, the total "area" as well as the total integral of the norm of the second fundamental form of such examples can be as large as one wants. This makes the picture of blow-up unclear to us when the dimension $m \geq 3$, particularly we do not know whether or not $\bar{\phi}_0$ has the minimal energy. Our approach in this regard up to now have failed.

Reference:

W-Y.Hsiang, Z-H.Teng, W-C.Yu, New examples of constant mean curvature immersions of $(2k-1)$-spheres into Euclidean $2k$-Space, Ann. of Math., 117 (1983), 609-625.

Thanks for your attention!