Flows and symplectic structure on the higher Teichmuller space

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Teichmuller theory

Let S be a connected closed surface of genus $g \geq 2$.

定理

Riemann Uniformization Theorem: *Complex structure J on S $\Longleftrightarrow$ a discrete faithful representation $\pi_1(S) \to \mathrm{PSL}(2, \mathbb{R})$ such that (S, J) is conformal to $\mathbb{H}^2/\rho(\pi_1(S))$.*

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- the space of complex structures on S up to equivalence;
- or the space of hyperbolic structures on S up to equivalence;
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Teichmuller theory is a crossroad of complex analysis, algebraic/hyperbolic geometry, dynamical system etc.

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- $\mathrm{Hom}(\pi_1(\mathrm{S}), \mathrm{G})/\mathrm{G}$ can be viewed as the space of G-flat connections on principal G-bundle over S up to gauge transformations, the space of flat connections on certain vector bundle of S, or the space of certain geometric structures up to equivalence depending on G.

- *n*-**Fuchsian representation**
  $\rho : \pi_1(S) \xrightarrow{d.f.} \mathrm{PSL}(2,\mathbb{R}) \xrightarrow{irr.\iota} \mathrm{PGL}(n,\mathbb{R})$.
  $\iota : \mathrm{PSL}(2,\mathbb{R}) \to \mathrm{PGL}(n,\mathbb{R})$ is defined by

$$v(M \cdot [x,y]^T) = \iota(M) \cdot v([x,y]^T)$$

  where $v : \mathbb{RP}^1 \to \mathbb{RP}^{n-1}$ is the **Veronese curve**:
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Using Higgs bundle technique

定理

(Hitchin 92') The Hitchin component $\mathrm{Hit}_n(S)$ *is topologically a unit ball.*

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*(Fock–Goncharov 06'(*positive*), Labourie 06'(*Anosov*)) Any Hitchin representation is discrete and faithful, $\pi_1(S)$ into G is a quasi-isometric embedding. There is a lift $\widetilde{\rho}$ of ρ into* $\mathsf{SL}(n, \mathbb{R})$*, such that any non-peripheral $\gamma \in \pi_1(S)$ is* loxodromic*: the eigenvalues are $\lambda_1 > \cdots > \lambda_n > 0$*

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- n in general, Guichard-Wienhard, domain of discontinuity.

Atiyah–Bott–Goldman symplectic structure

- $\mathbb{T}_\rho \mathrm{Hom}(\pi_1(\mathrm{S}), \mathrm{G})/\mathrm{G} = \mathrm{H}^1_\rho(\pi_1(\mathrm{S}), \mathfrak{g})$.
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定义

(Atiyah–Bott 83') The **Atiyah–Bott–Goldman symplectic structure** on $\mathrm{Hom}(\pi_1(\mathrm{S}), \mathrm{G})/\mathrm{G}$ is defined to be:

$$H^1_\rho(S, \mathfrak{g}) \times H^1_\rho(S, \mathfrak{g}) \to \mathbb{R}$$

$$\omega(\alpha, \beta) = \int_S \alpha \wedge \beta.$$

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(Goldman 84') View from group cohomology

$$\omega : H^1_\rho(S,\mathfrak{g}) \times H^1_\rho(S,\mathfrak{g}) \xrightarrow{\cup} H^2_\rho(S,\mathfrak{g}\otimes\mathfrak{g}) \xrightarrow{B} H^2_\rho(S,\mathbb{R}) \xrightarrow{[S]} \mathbb{R}.$$

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- (Goldman 86') Poisson bracket
  $\{Tr_\alpha, Tr_\beta\} = \sum_{p\in\alpha\cap\beta} \epsilon(p,\alpha,\beta)\left(Tr_{\alpha_p\beta_p} - \frac{1}{n} Tr_\alpha Tr_\beta\right).$

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猜想

ω is Kählerian for $\mathrm{Hit}_n(S)$ when $n \geq 3$.

Wolpert formula

- Given a pants decomposition and transverse arcs to the pants curves, Fenchel-Nielsen coordinates $\{\ell_i, \theta_i\}_{i=1}^{3g-3}$ are the lengths and twists around the pants curves.
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- We generalize Wolpert formula for $\mathrm{Hit}_n(S)$.

Limit curve

- The space of (complete) flags
  $\mathcal{B} = \{F = (0 \subset F_1 \subset \cdots \subset F_{n-1}) | \dim F_i = i\} \simeq G/B$.
  $\mathcal{FR}_n = \{\textbf{continous } \xi : S^1 \to \mathcal{B} \textit{ Frenet}\} / \mathrm{PGL}(n, \mathbb{R})$.
- **Frenet**: Firstly $\xi(x_1)_{n_1} \oplus \cdots \oplus \xi(x_k)_{n_k}$, $n_1 + \cdots + n_k \leq n$.
- Secondly $x \in S^1$, $\{(x_{i,1}, \ldots, x_{i,k})\}_{i=1}^{\infty}$ pairwise distinct, $\forall j$, $\lim_{i \to \infty} x_{i,j} = x$, we have

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定理

(Labourie 06', Guichard 08')
$\rho \in \mathrm{Hit}_n(S) \Leftrightarrow \exists! \, \xi_\rho : \partial_\infty \pi_1(S) \cong S^1 \to \mathcal{B}$ ρ-equivariant Frenet up to $\mathrm{PGL}(n, \mathbb{R})$, we call ξ_ρ a limit curve.

$\mathrm{Hit}_n(S)$"$\subset$" $\mathcal{FR}_n$, we study the deformation of $\mathrm{Hit}_n(S)$ via deforming $\mathcal{FR}_n$.

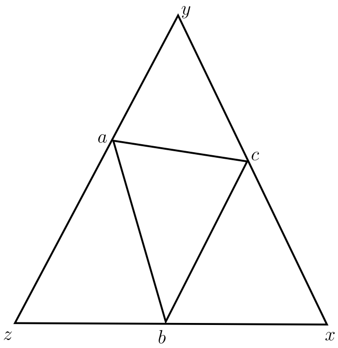

图: The flags are $F=(a,\overline{yz})$, $G=(b,\overline{zx})$, $H=(c,\overline{xy})$. Let $|\cdot|$ be the Euclidean norm. Then the **triple ratio** $T(F,G,H)=\frac{|ya|}{|az|}\frac{|zb|}{|bx|}\frac{|xc|}{|cy|}$. By Ceva theorem, $T(F,G,H)=1$ if and only if $\overline{ax}$, $\overline{by}$ and $\overline{cz}$ are colinear.

Toy model $\mathcal{B}^4 / \mathrm{PGL}_3$

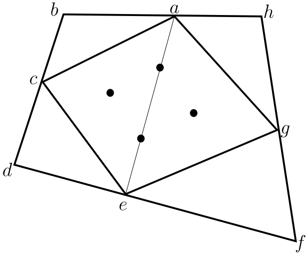

图: The flags are $X = (a, \overline{bh})$, $W = (c, \overline{bd})$, $Y = (e, \overline{df})$, $Z = (g, \overline{fh})$. Up to $\mathrm{PGL}(3, \mathbb{R})$, the position of (X, W, Y) is decided by the triple ratio $T(X, W, Y)$. The convention for cross ratio in $\mathbb{RP}^1$ is $CR(\alpha, \beta, \gamma, \delta) := \frac{\alpha-\gamma}{\alpha-\delta} \cdot \frac{\beta-\delta}{\beta-\gamma}$. The cross ratio $CR(\overline{ab}, \overline{ae}, \overline{ag}, \overline{ac})$ decides the line $\overline{ag}$ and $CR(\overline{ef}, \overline{ea}, \overline{ec}, \overline{eg})$ decides the line $\overline{eg}$, which fix the point G. In the end, the line $\overline{hf}$ is decided by the triple ratio $T(X, Y, Z)$.

Parameterizing $\mathcal{B}^N / \mathrm{PGL}_n(\mathbb{R}_{>0})$

- For $\mathcal{B}^N / \mathrm{PGL}_n(\mathbb{R}_{>0})$, fix a triangulation $\mathcal{T}$ and its n-triangulation $\mathcal{T}_n$ of a N-gon, fix quiver ϵ.
- $\{f_i\}_{i=1}^n$ a base of a flag F in $\mathbb{R}^n$. $f^0 := 1$,

$$f^k := f_1 \wedge \cdots \wedge f_k.$$

- Each vertex $I \in V(\mathcal{T}_n) \backslash V(\mathcal{T})$ associates with three flags X, Y, Z and $a, b, c \geq 0$ such that $a + b + c = n$, Ω volume form of $\mathbb{R}^n$

$$A_I = \pm\Omega\left(x^a \wedge y^b \wedge z^c\right).$$

- For $I \in V(\mathcal{T}_n) \backslash V(\mathcal{T})$ not on the boundary, **Fock–Goncharov $\mathcal{X}$ coordinate** at I is

$$X_I = \prod_J A_J^{\epsilon_{IJ}}.$$

Fock–Goncharov and ABG Poisson algebras

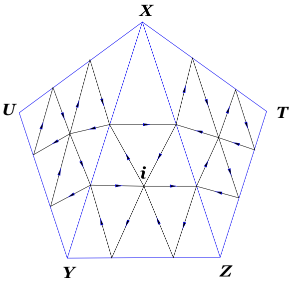

图: $\epsilon_{ij} = \#\{arrow\ i\ to\ j\} - \#\{arrow\ j\ to\ i\}$. $X_k = \prod_i A_i^{\epsilon_{ki}}$.
Fock–Goncharov Poisson bracket $\{X_i, X_j\} = \epsilon_{ij} X_i X_j$.

定理

(Labourie 18') ABG Poisson algebra is Poisson embedded into the rank n swapping multifraction algebra for $\tilde{S}$.
(S. 20') Fock–Goncharov Poisson algebra is Poisson embedded into Rank n swapping multifraction algebra.

Triple ratio and edge function

When I is in the interior of a triangle, the **triple ratio** is

$$T_{a,b,c}(X,Y,Z) := X_I$$

$$= \frac{\Omega\left(x^{a+1}\wedge y^{b}\wedge z^{c-1}\right)\Omega\left(x^{a-1}\wedge y^{b+1}\wedge z^{c}\right)\Omega\left(x^{a}\wedge y^{b-1}\wedge z^{c+1}\right)}{\Omega\left(x^{a+1}\wedge y^{b-1}\wedge z^{c}\right)\Omega\left(x^{a}\wedge y^{b+1}\wedge z^{c-1}\right)\Omega\left(x^{a-1}\wedge y^{b}\wedge z^{c+1}\right)} > 0.$$

When I is on an edge, suppose $x > y > z > t$, the **edge function** is

$$C_a(X,Y,T,Z) := -X_I$$

$$= \frac{\Omega\left(x^{a}\wedge z^{n-a-1}\wedge t^{1}\right)\cdot\Omega\left(x^{a-1}\wedge z^{n-a}\wedge y^{1}\right)}{\Omega\left(x^{a}\wedge z^{n-a-1}\wedge y^{1}\right)\cdot\Omega\left(x^{a-1}\wedge z^{n-a}\wedge t^{1}\right)} < 0.$$

Elementary eruption flow

- $$B_{F_1,F_2,F_3}^{i_1,i_2,i_3} := \{f_{1,1},\dots,f_{1,i_1},f_{2,1},\dots,f_{2,i_2},f_{3,1},\dots,f_{3,i_3}\}$$

$$b_{F_1,F_2,F_3}^{i_1,i_2,i_3}(t) := e^{\frac{(-i_2+i_3)t}{3n}} \cdot \begin{bmatrix} \mathrm{id}_{i_1} & 0 & 0 \\ 0 & e^{\frac{t}{3}} \cdot \mathrm{id}_{i_2} & 0 \\ 0 & 0 & e^{-\frac{t}{3}} \cdot \mathrm{id}_{i_3} \end{bmatrix},$$

$$b_{F_2,F_3,F_1}^{i_2,i_3,i_1}(t) := e^{\frac{(-i_3+i_1)t}{3n}} \cdot \begin{bmatrix} e^{-\frac{t}{3}} \cdot \mathrm{id}_{i_1} & 0 & 0 \\ 0 & \mathrm{id}_{i_2} & 0 \\ 0 & 0 & e^{\frac{t}{3}} \cdot \mathrm{id}_{i_3} \end{bmatrix},$$

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Elementary eruption flow

- with respect to the basis $B^{i_1,i_2,i_3}_{F_1,F_2,F_3}$.
- $x_1 < x_2 < x_3 < x_1 \in S^1$, the (i_1, i_2, i_3)-**elementary eruption flow** is

$$\left(\epsilon^{i_1,i_2,i_3}_{x_1,x_2,x_3}\right)_t : \mathcal{FR}_n \to \mathcal{FR}_n$$

defined by

$$\left(\epsilon^{i_1,i_2,i_3}_{x_1,x_2,x_3}\right)_t (\xi) := \xi_t(p) = \begin{cases} b_1(t) \cdot \xi(p) & \text{if } p \in \overline{[x_2, x_3]} \\ b_2(t) \cdot \xi(p) & \text{if } p \in \overline{[x_3, x_1]} \\ b_3(t) \cdot \xi(p) & \text{if } p \in \overline{[x_1, x_2]} \end{cases}$$

where $b_m(t) := b^{i_m,i_{m+1},i_{m-1}}_{\xi(x_m),\xi(x_{m+1}),\xi(x_{m-1})}(t)$ for $m = 1, 2, 3$.

Elementary eruption flow

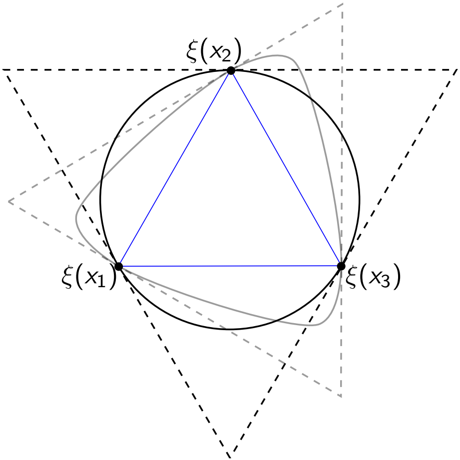


图: $n = 3$ case

Properties of elementary eruption flow

定理

(S.–Wienhard–Zhang 20') ξ_t is Frenet for any t.

Properties of elementary eruption flow

定理

(S.–Wienhard–Zhang 20') ξ_t is Frenet for any t.

引理

(SWZ) Let $\delta(j_1, j_2, j_3) = \begin{cases} 1 & \text{if } (i_1, i_2, i_3) = (j_1, j_2, j_3) \\ 0 & \text{otherwise} \end{cases}$ *Then*

$$T_{j_1,j_2,j_3}(\xi_t(x_1), \xi_t(x_2), \xi_t(x_3)) = e^{t\delta(j_1,j_2,j_3)} \cdot T_{j_1,j_2,j_3}(\xi(x_1), \xi(x_2), \xi(x_3)).$$

Elementary shearing flow

- $$b_{F_1,F_2}^{i,n-i}(t) := e^{\frac{(2n-3i)t}{6n}} \cdot \begin{bmatrix} e^{\frac{t}{6}}\,\mathrm{id}_i & 0 \\ 0 & e^{-\frac{2t}{6}} \cdot \mathrm{id}_{n-i} \end{bmatrix}$$

  with basis $B_{F_1,F_2}^{i,n-i}$.
- Let $x_1, x_2 \in S^1$, the $(i, n-i)$-**elementary shearing flow** is

$$(\psi_{x_1,x_2}^{i,n-i})_t : \mathcal{FR}(V) \to \mathcal{B}$$

defined by

$$\xi_t(p) = \begin{cases} b(-t) \cdot \xi(p) & \text{if } p \in \overline{[x_2, x_1]} \\ b(t) \cdot \xi(p) & \text{if } p \in \overline{[x_1, x_2]} \end{cases}$$

where $\xi_t := \left(\psi_{x_1,x_2}^{i,n-i}\right)_t(\xi)$, and $b(t) := b_{\xi(x_1),\xi(x_2)}^{i,n-i}(t)$.

Properties of elementary shearing flow

定理

(SWZ) ξ_t *is Frenet for any* t.

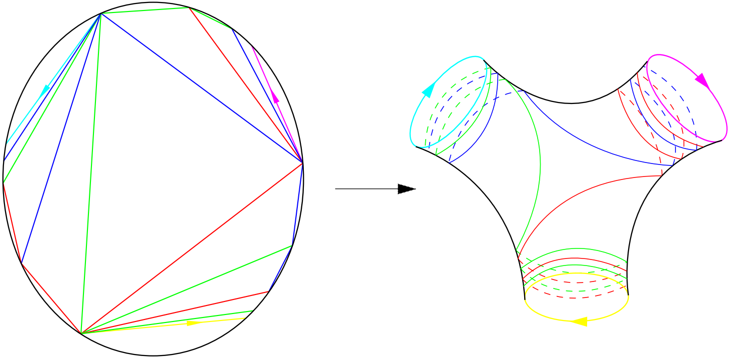
引理

(SWZ) $x_1 < x_2 < x_3 < x_4$ *in* S^1,

$$C_j\big(\xi_t(x_1), \xi_t(x_2), \xi_t(x_4), \xi_t(x_3)\big) = e^{t\delta(j)} \cdot C_j\big(\xi(x_1), \xi(x_2), \xi(x_4), \xi(x_3)\big).$$

We have commutativity for the flows with disjoint associated geometric figure (triangles, lines).

Ideal triangulation



Given an ideal triangulation $\mathcal{T}$, $\mathcal{Q} = \{isolated \;\; edge\}$ where $\#\mathcal{Q} = 6g - 6$, $\mathcal{P} = \{closed \;\; edge\}$ where $1 \leq \#\mathcal{P} \leq 3g - 3$, $\Theta = \{ideal \;\; triangle\}$ where $\#\Theta = 4g - 4$.

Bridge system

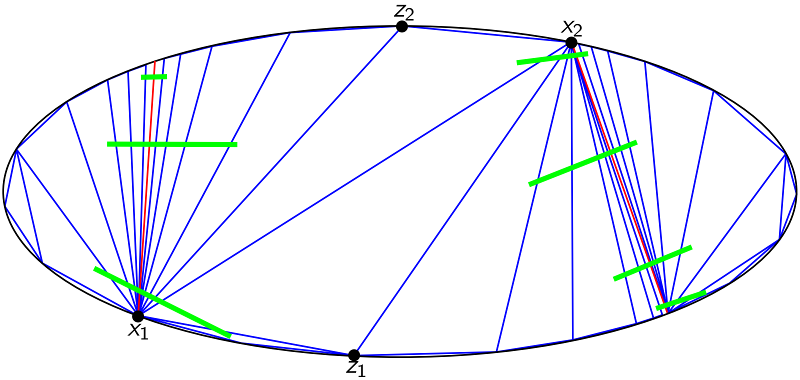


图: A **bridge** $\{T_1, T_2\} \in \widetilde{\mathcal{J}}$ is a pair of ideal triangles "across" a closed edge (green line). $\mathcal{J} = \widetilde{\mathcal{J}}/\pi_1(S)$ is a **bridge system** compatible with $\mathcal{T}$.

Bonahon-Dreyer parameterization

Given an ideal triangulation $\mathcal{T}$, fix a representative ξ of $[\xi] \in \mathrm{Hit}_n(S)$, one can associate ξ invariant flags to all the vertices of $\mathcal{T}$. Then one can define the Fock–Goncharov $\mathcal{X}$ coordinates in each polygon of the fundamental domain as before. The problem is along the closed edge of $\mathcal{T}$.

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定理

(Bonahon-Dreyer 14') Given an ideal triangulation $\mathcal{T}$ and a bridge system, adding the edge invariant along the closed edges with respect to the bridge system, there is a polytope $P_{\mathcal{T}}$ satisfying closed leaf equations and closed leaf inequalities and a homeomorphism

$$\Phi_{\mathcal{T},\mathcal{J}} : \mathrm{Hit}_n(S) \to P_{\mathcal{T}}.$$

Bonahon-Dreyer parameterization

- For $\xi \in \mathrm{Hit}_n(S)$ and closed edge $\gamma \in \mathcal{T}$, we have $\lambda_1(\xi(\gamma)) > \cdots > \lambda_n(\xi(\gamma)) > 0$, then $\ell_\xi^i(\gamma) := \log \left| \frac{\lambda_i(\xi(\gamma))}{\lambda_{i+1}(\xi(\gamma))} \right|$.

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- $\ell^i_\xi(\gamma)$ can be written as a linear combination of Fock–Goncharov $\mathcal{X}$ coordinates.
- Closed leaf equation: $\ell^i_\xi(\gamma)$ can be written in two ways for both sides of γ.

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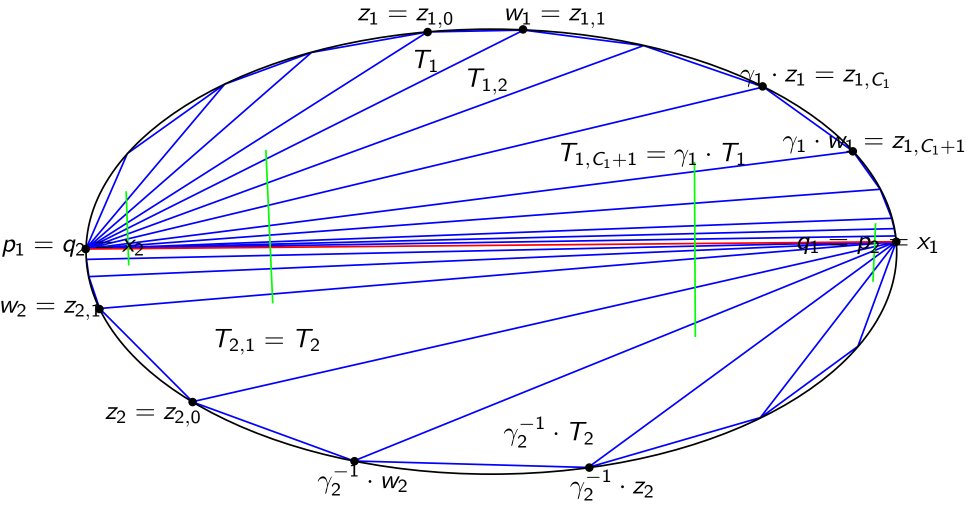
- Closed leaf equation: $\ell^i_\xi(\gamma)$ can be written in two ways for both sides of γ.
- Closed leaf inequalities: $\ell^i_\xi(\gamma) > 0$.

Bonahon-Dreyer parameterization



Symplectic closed-edge invariants

For a bridge J connecting two ideal triangles (p_1, w_1, z_1) and (p_2, w_2, z_2) on two sides of the closed edge (x_1, x_2), let $u_m \in \mathrm{PGL}(V)$ be the unique unipotent projective transformation that fixes the flag $\xi(p_m)$ and sends the flag $\xi(z_m)$ to $\xi(q_m)$ where $[p_m, q_m] = [x_1, x_2]$.

定义

(SWZ) The **symplectic closed-edge invariants** of $\{x_1, x_2\} \in \widetilde{\mathcal{P}}$ is

$$\alpha^{i,n-i}_{x_1,x_2,J}[\xi] := \log\big(-C_i\big(\xi(x_1), u_2 \cdot \xi(w_2), u_1 \cdot \xi(w_1), \xi(x_2)\big)\big).$$

Symplectic closed-edge invariants

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定理

(SWZ) Given $\mathcal{T}$ and $\mathcal{J}$, by replacing the edge invariants along the closed edges of Bonahon–Dreyer by the symplectic closed-edge invariants, we have a homeomorphism

$$\Omega_{\mathcal{T},\mathcal{J}} : \mathrm{Hit}_n(S) \to P_{\mathcal{T}}.$$

Symplectic closed-edge invariants

引理

(SWZ) Let $\xi_t := \left(\psi_{x_1,x_2}^{i,n-i}\right)_t(\xi)$*. Then for all* $i = 1, \ldots, n-1$*,*

$$\alpha_{x_1,x_2,J}^{i,n-i}[\xi_t] = \alpha_{x_1,x_2,J}^{i,n-i}[\xi] + t.$$

Symplectic closed-edge invariants

引理

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$$\alpha^{i,n-i}_{x_1,x_2,J}[\xi_t] = \alpha^{i,n-i}_{x_1,x_2,J}[\xi] + t.$$

Let $W_{\mathcal{T}} \subset P_{\mathcal{T}}$ be the vector space satisfying the closed leaf equations. The above theorem provides $W_{\mathcal{T}} \cong \mathbb{T}_\xi \operatorname{Hit}_n(S)$.

Symplectic closed-edge invariants

引理

(SWZ) Let $\xi_t := \left(\psi^{i,n-i}_{x_1,x_2}\right)_t(\xi)$. *Then for all* $i = 1, \ldots, n-1$,

$$\alpha^{i,n-i}_{x_1,x_2,J}[\xi_t] = \alpha^{i,n-i}_{x_1,x_2,J}[\xi] + t.$$

Let $W_{\mathcal{T}} \subset P_{\mathcal{T}}$ be the vector space satisfying the closed leaf equations. The above theorem provides $W_{\mathcal{T}} \cong \mathbb{T}_\xi \operatorname{Hit}_n(S)$.

定义

(SWZ) The $(\mathcal{T}, \mathcal{J})$-**parallel flow** associated to μ is $\phi^\mu_t : \operatorname{Hit}_n(S) \to \operatorname{Hit}_n(S)$

$$\phi^\mu_t := \left(\prod_{c \in \mathcal{P}} (\phi^\mu_c)_t\right) \circ \left(\phi^\mu_{\mathcal{Q},\Theta}\right)_t .$$

Main Theorem

定理

(SWZ) Given $\mathcal{T}$ and $\mathcal{J}$, for any $\xi \in \mathrm{Hit}_n(S)$ and $\mu \in W_{\mathcal{T}}$, ϕ_t^{μ} is well-defined. Let

$$I_{[\xi],\mu} := \{t \in \mathbb{R} : \Omega[\xi] + t \cdot \mu \textit{ satisfy the closed leaf inequalities}\}$$

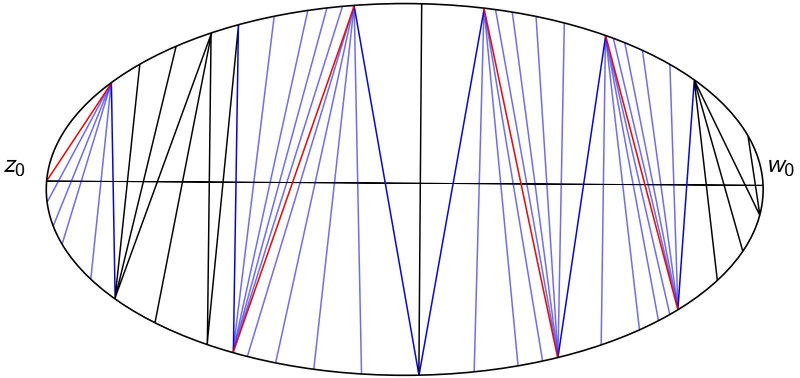
For any $t \in I_{[\xi],\mu}$, let

$$[\xi_t] := \Omega^{-1}(\Omega[\xi] + t\mu).$$

Then $\phi_t^{\mu}[\xi] = [\xi_t]$.

Consequence: Every pair of $(\mathcal{T}, \mathcal{J})$-parallel flows on $\mathrm{Hit}_n(S)$ commute, and the space of $(\mathcal{T}, \mathcal{J})$-parallel flows on $\mathrm{Hit}_n(S)$ is naturally in bijection with $\mathbb{T}_{[\xi]}\mathrm{Hit}_n(S)$. In particular, the pair $(\mathcal{T}, \mathcal{J})$ determines a trivialization of $\mathbb{T}\mathrm{Hit}_n(S)$.

Reason of convergence



Tangent cocycle

- View the cohomology classes in $[\nu] \in H^1_\xi(S, \mathfrak{sl}(n, \mathbb{R})_{\mathrm{Ad} \circ \xi})$ as describing infinitesimal deformations of Frenet curves instead of representations.

Tangent cocycle

- View the cohomology classes in $[\nu] \in H^1_\xi(S, \mathfrak{sl}(n,\mathbb{R})_{\mathrm{Ad}\circ\xi})$ as describing infinitesimal deformations of Frenet curves instead of representations.
- $\xi_t(x_{h,0}) = \xi_0(x_{h,0})$, $\xi_t(y_{h,0}) = \xi_0(y_{h,0})$, $\xi_t^{(1)}(z_{h,0}) = \xi_0^{(1)}(z_{h,0})$, $\exists! g_{h,t} \in \mathrm{PGL}(n,\mathbb{R})$ so that

$$g_{h,t} \cdot \xi_0(x_{h,1}) = \xi_t(x_{h,1}), \quad g_{h,t} \cdot \xi_0(y_{h,1}) = \xi_t(y_{h,1}),$$

$$g_{h,t} \cdot \xi_0^{(1)}(z_{h,1}) = \xi_t^{(1)}(z_{h,1}).$$

$t \mapsto g_{h,t}$ with $g_{h,0} = \mathrm{id}$. We define

$$\widetilde{\mu}_{\xi,[\nu]}(h) := \left.\frac{d}{dt}\right|_{t=0} g_{h,t}.$$

Tangent cocycle

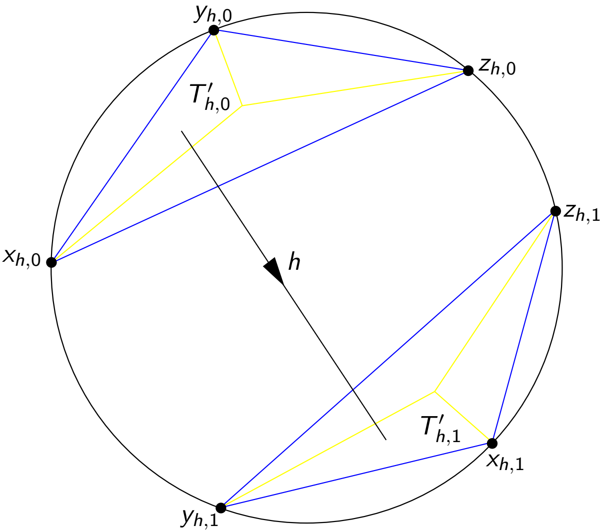
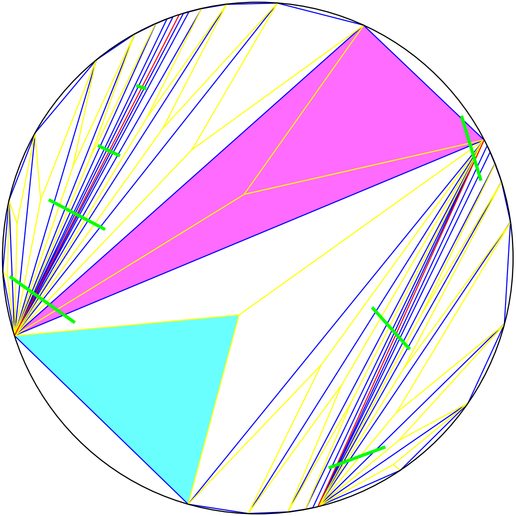


图: 1-simplices considered in Step 1.

Barrier system $\tilde{\mathcal{B}}$



Admissible labellings

- For $F, G, H \in \mathcal{B}$,

$$A^{i,j,k}_{F,G,H} = \begin{bmatrix} \frac{n-i}{n} \cdot \mathrm{id}_i & 0 \\ 0 & -\frac{i}{n} \cdot \mathrm{id}_{n-i} \end{bmatrix}$$

  with basis $B^{i,j,k}_{F,G,H}$.

- An **admissible labelling** is a $Ad \circ \xi$-equivariant map $L : \tilde{\mathcal{B}} \to \mathfrak{sl}(n, \mathbb{R})$ satisfying certain symmetries w.r.t. triple ratios and edge functions and the "tangent version of closed leaf equation". The images are linear combinations of $A^{i,j,k}_{F,G,H}$.
- Denote the set of admissible labellings at ξ by $\mathcal{A}(\xi, \mathcal{T})$.

Admissible labellings and tangent cocycles

- Define $Ad \circ \xi$-equivariant $\widetilde{\mu}_L$ on piecewise $h : [0,1] \to \widetilde{S}$

$$\widetilde{\mu}_L(h) := \sum_{b \in \widetilde{\mathcal{B}}} \hat{i}(h,b) L(b)$$

  where h cross closed edge $e \in \widetilde{\mathcal{P}}$ via a bridge $J \in \widetilde{\mathcal{J}}$.
- $Ad \circ \xi$-equivariant $\widetilde{\mu}_L$ induce $\mu_L \in C^1(S, \mathfrak{sl}(n,\mathbb{R})_{\mathrm{Ad}\circ\xi})$.

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定理

(S.–Zhang) $\Phi_{\xi,\mathcal{T},\mathcal{J}} : \mathcal{A}(\xi,\mathcal{T}) \to H^1(S, \mathfrak{sl}(n,\mathbb{R})_{\mathrm{Ad} \circ \xi})$*:* $L \mapsto [\mu_L]$ *is an isomorphism.*

Note that we identify both $\mathcal{A}(\xi,\mathcal{T})$ and the space of $(\mathcal{T},\mathcal{J})$-vector fields at ξ with $\mathbb{T}_\xi \operatorname{Hit}_n(S)$.

Simplicial complex

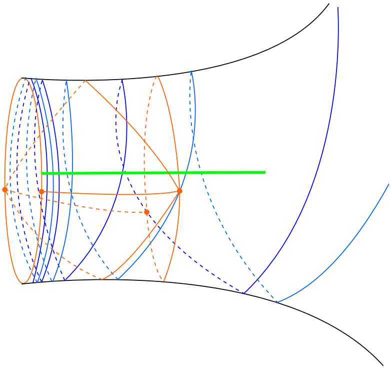
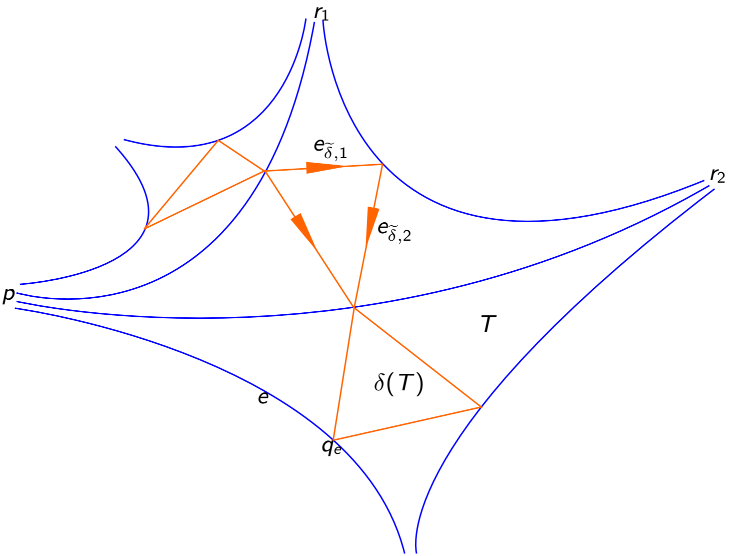
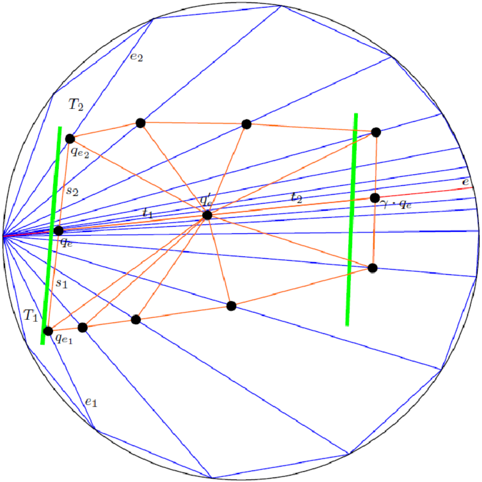


图: Pick one point in each isolated edge, two points in each closed edge.
$\mathbb{T}$ cuts S into triangles and cylinders.

Triangle



Cylinder



Simplicial complex

- Choose triangulation $\mathbb{T}$ based on $\mathcal{T}$
- Label on all the vertices $\mathbb{T}$.
- The Goldman symplectic pairings:

$$\omega\big([\mu_{L_1}],[\mu_{L_2}]\big) = \sum_{\delta\in\mathbb{T}} \mathrm{sgn}(\delta)\,\mathrm{tr}\left(\widetilde{\mu}_{L_1}(e_{\widetilde{\delta},1})\cdot\widetilde{\mu}_{L_2}(e_{\widetilde{\delta},2})\right).$$

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定理

(S.–Zhang) Given $\mathcal{T},\mathcal{J}$. Let X_1 and X_2 be a pair of $(\mathcal{T},\mathcal{J})$-parallel vector fields on $\mathrm{Hit}_n(S)$, then the map $\mathrm{Hit}_n(S)\to\mathbb{R}$ given by

$$[\xi]\mapsto\omega(X_1[\xi],X_2[\xi])$$

is constant.

$\mathcal{T}$ subordinate to a pants decomposition

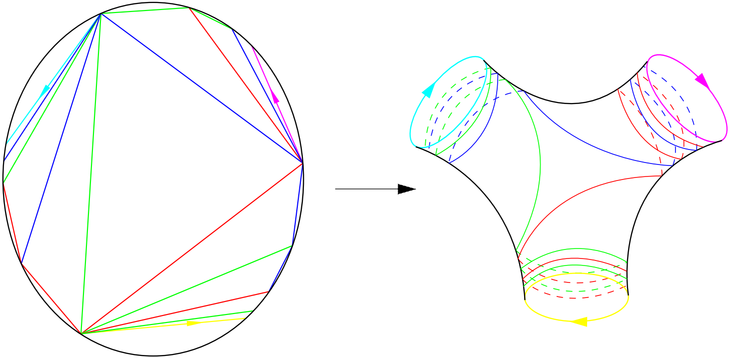
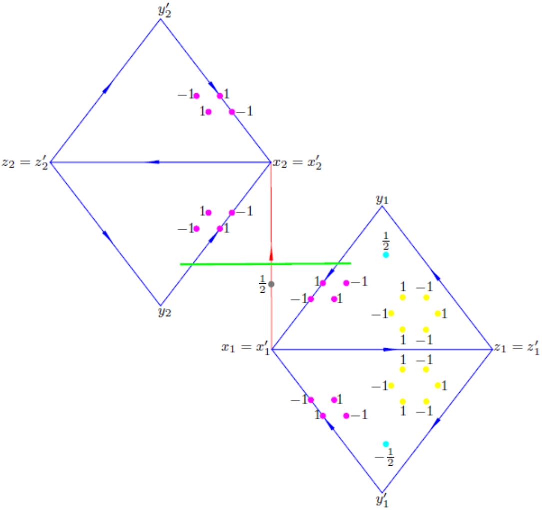


图: $\#\mathcal{P} = 3g - 3$.

Special admissible labellings



- The (i,j,k)-**eruption labelling** associated to P is

$$E_{x,y,z}^{i,j,k}=E_{y,z,x}^{j,k,i}=E_{z,x,y}^{k,i,j}:=\frac{1}{2}\left(L_{x,y,z}^{i,j,k}-L_{x',z',y'}^{i,k,j}\right).$$

- The (i,j,k)-**hexagon labelling** associated to P is

$$\begin{aligned}&H_{x,y,z}^{i,j,k}=H_{y,z,x}^{j,k,i}=H_{z,x,y}^{k,i,j}:=\\&L_{x,y,z}^{i,j+1,k-1}-L_{x,y,z}^{i-1,j+1,k}+L_{x,y,z}^{i-1,j,k+1}-L_{x,y,z}^{i,j-1,k+1}+L_{x,y,z}^{i+1,j-1,k}-L_{x,y,z}^{i+1,j,k-1}\\&+L_{x',z',y'}^{i,k-1,j+1}-L_{x',z',y'}^{i-1,k,j+1}+L_{x',z',y'}^{i-1,k+1,j}-L_{x',z',y'}^{i,k+1,j-1}+L_{x',z',y'}^{i+1,k,j-1}-L_{x',z',y'}^{i+1,k-1,j}.\end{aligned}$$

Special admissible labellings

- The i-**twist labelling** associated to $\hat{e}$ is

$$S^i_{x_1,x_2} = -\frac{1}{2} L^{i,n-i}_{x_1,x_2}.$$

- The i-**length labelling** associated to $\hat{e}$ is

$$Y^i_{x_1,x_2} := Z^i_{x_1,x_2} + E^{i,n-i,1}_{x_1,y_1,z_1} - E^{i-1,n-i+1,1}_{x_1,y_1,z_1} - E^{n-i,i,1}_{x_2,y_2,z_2} + E^{n-i-1,i+1,1}_{x_2,y_2,z_2},$$

- where the i-**lozenge labelling** $Z^i_{x_1,x_2}$ is

$$\begin{aligned} Z^i_{x_1,x_2} := & - L^{i+1,n-i-1,0}_{x_1,y_1,z_1} + L^{i,n-i,0}_{x_1,y_1,z_1} + L^{i,n-i-1,1}_{x_1,y_1,z_1} - L^{i-1,n-i,1}_{x_1,y_1,z_1} \\ & - L^{i+1,0,n-i-1}_{x'_1,z'_1,y'_1} + L^{i,0,n-i}_{x'_1,z'_1,y'_1} + L^{i,1,n-i-1}_{x'_1,z'_1,y'_1} - L^{i-1,1,n-i}_{x'_1,z'_1,y'_1} \\ & - L^{n-i+1,i-1,0}_{x_2,y_2,z_2} + L^{n-i,i,0}_{x_2,y_2,z_2} + L^{n-i,i-1,1}_{x_2,y_2,z_2} - L^{n-i-1,i,1}_{x_2,y_2,z_2} \\ & - L^{n-i+1,0,i-1}_{x'_2,z'_2,y'_2} + L^{n-i,0,i}_{x'_2,z'_2,y'_2} + L^{n-i,1,i-1}_{x'_2,z'_2,y'_2} - L^{n-i-1,1,i}_{x'_2,z'_2,y'_2}. \end{aligned}$$

定理

(S.–Zhang) Fix an ideal triangulation $\mathcal{T}$ subordinate to a pants decomposition and a bridge system.
If $L_1 = S^i_{x_1,x_2}$, then

$$\omega([\mu_{L_1}],[\mu_{L_2}]) = \begin{cases} 1 & \text{if } L_2 = Y^i_{x_1,x_2}; \\ 0 & \text{otherwise.} \end{cases}$$

If $L_1 = H^{i,j,k}_{x,y,z}$, then

$$\omega([\mu_{L_1}],[\mu_{L_2}]) = \begin{cases} 1 & \text{if } L_2 = E^{i,j,k}_{x,y,z}; \\ 0 & \text{otherwise.} \end{cases}$$

Global Darboux vector fields

- If $L_1 = Y^i_{x_1,x_2}$, then

$$\omega([\mu_{L_1}],[\mu_{L_2}]) = \begin{cases} -1 & \text{if } L_2 = S^i_{x_1,x_2}; \\ 0 & \text{otherwise.} \end{cases}$$

- If $L_1 = E^{i,j,k}_{x,y,z}$, then

$$\omega([\mu_{L_1}],[\mu_{L_2}]) = \begin{cases} -1 & \text{if } L_2 = H^{i,j,k}_{x,y,z}; \\ 0 & \text{otherwise.} \end{cases}$$

定理

(SWZ)

$$H(\mathcal{S}^i_{x_1,x_2}) = \sum_{k=1}^{i} \frac{(i-n)k}{2n} \cdot \ell^k_{[\gamma]} + \sum_{k=i+1}^{n-1} \frac{i(k-n)}{2n} \cdot \ell^k_{[\gamma]}.$$

$$\begin{aligned} H(\mathcal{H}^{i,j,k}_{x,y,z}) &= \tau^{i,j,k}_{x,y,z} - \tau^{i,k,j}_{x',z',y'} + \delta_{k,1}\big(H(\mathcal{S}^{i-1}_{x,x_0}) - H(\mathcal{S}^{i}_{x,x_0})\big) \\ &\quad + \delta_{i,1}\big(H(\mathcal{S}^{j-1}_{y,y_0}) - H(\mathcal{S}^{j}_{y,y_0})\big) + \delta_{j,1}\big(H(\mathcal{S}^{k-1}_{z,z_0}) - H(\mathcal{S}^{k}_{z,z_0})\big) \end{aligned}$$

$$H(\mathcal{Y}^i_{x_1,x_2}) = -2\alpha^{i,n-i}_{x_1,x_2}.$$

$$H(\mathcal{E}^{i,j,k}_{x,y,z}) = \sum_{(p,q,r)\in\mathbb{T}_n} c^{p,q,r}_{i,j,k} \cdot \big(\tau^{p,q,r}_{x,y,z} + \tau^{p,r,q}_{x',z',y'}\big),$$

Global Darboux basis

- where

$$\begin{aligned} T_x &:= \{(p,q,r) \in \mathbb{T}^n : p \geq i \text{ and } q \leq j\}, \\ T_y &:= \{(p,q,r) \in \mathbb{T}^n : q \geq j \text{ and } r \leq k\}, \\ T_z &:= \{(p,q,r) \in \mathbb{T}^n : r \geq k \text{ and } p \leq i\}, \end{aligned}$$

- and

$$c_{i,j,k}^{p,q,r} := \begin{cases} \dfrac{ir+iq+kq}{2n} & \text{if } (p,q,r) \in T_x; \\ \dfrac{jp+jr+ir}{2n} & \text{if } (p,q,r) \in T_y; \\ \dfrac{kq+kp+jp}{2n} & \text{if } (p,q,r) \in T_z. \end{cases}$$

Global Darboux basis

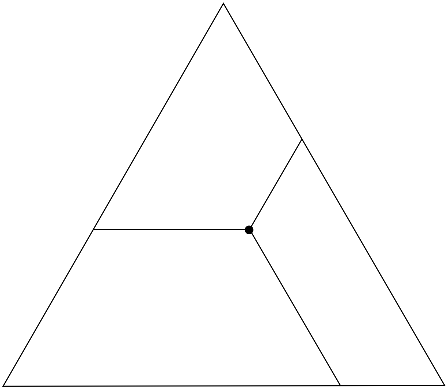


图: $T_x \cup T_y \cup T_z$

Thanks!