Flow polytopes in algebra and combinatorics

Karola Mészáros

Cornell University

Based on joint works with Laura Escobar, Alex Fink, June Huh, Kabir Kapoor, Jacob Matherne, Alejandro Maris, Alejandro Morales, Brendon Rhoades, Linus Setiabrata, Avery St. Dizier

Volume and discrete volume

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$\mathrm{vol}(P)$ is normalized volume with respect to underlying lattice

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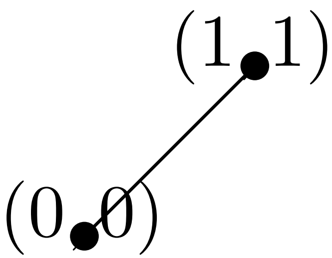

$$\mathrm{vol}(P) = 1$$

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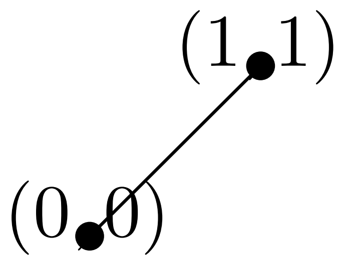

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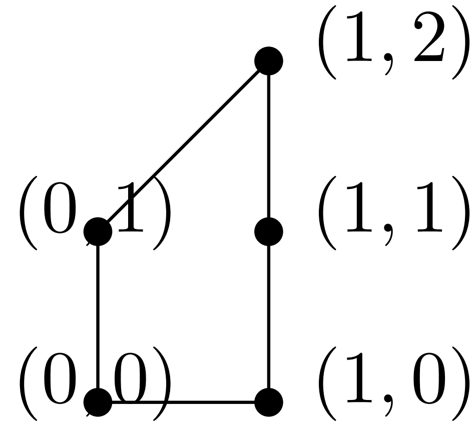

$\mathrm{vol}(P) = 3$

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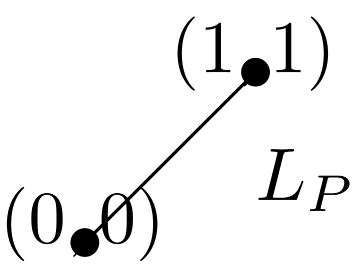

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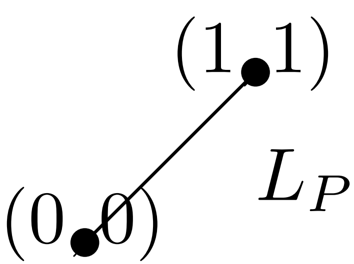

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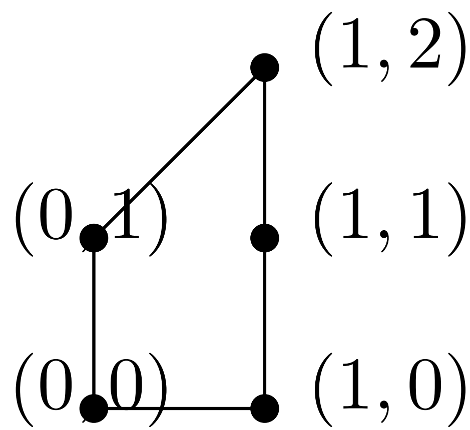

$L_P(t) = \frac{3}{2}t^2 + \frac{5}{2}t + 1$

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volume and number of lattice points of P are related:

$\mathrm{vol}(P)/\dim(P)! =$ leading coefficient $L_P(t)$

Flow polytopes

G directed graph on $n+1$ vertices

$\mathbf{a} = (a_1, a_2, \ldots, a_n) \in \mathbb{Z}^n_{\geq 0}$ netflow

$\mathcal{F}_G(\mathbf{a}) = \{$flows $x(\epsilon) \in \mathbb{R}_{\geq 0}$, $\epsilon \in E(G) \mid$ netflow$(i) = a_i\}$

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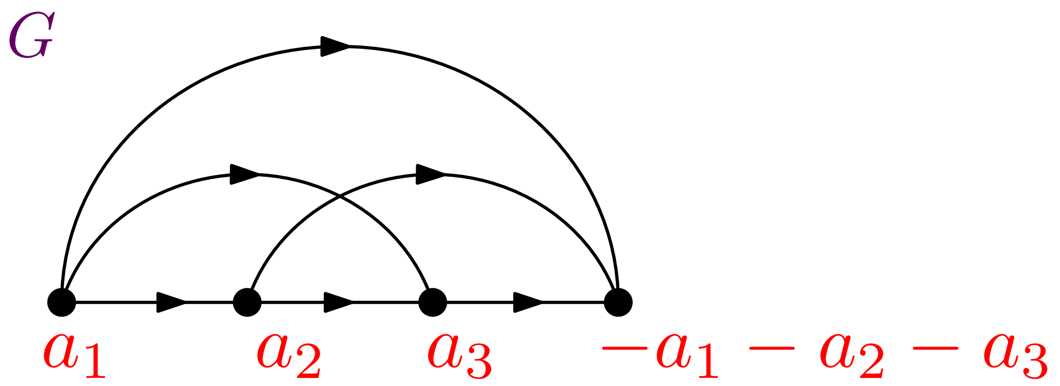

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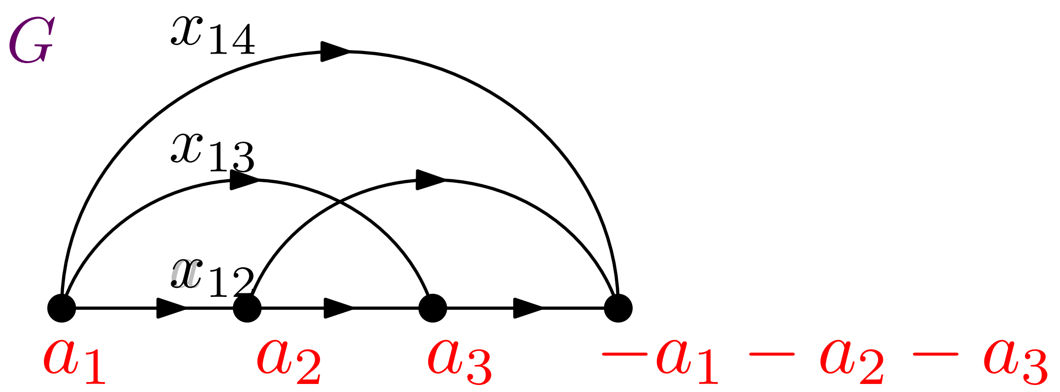

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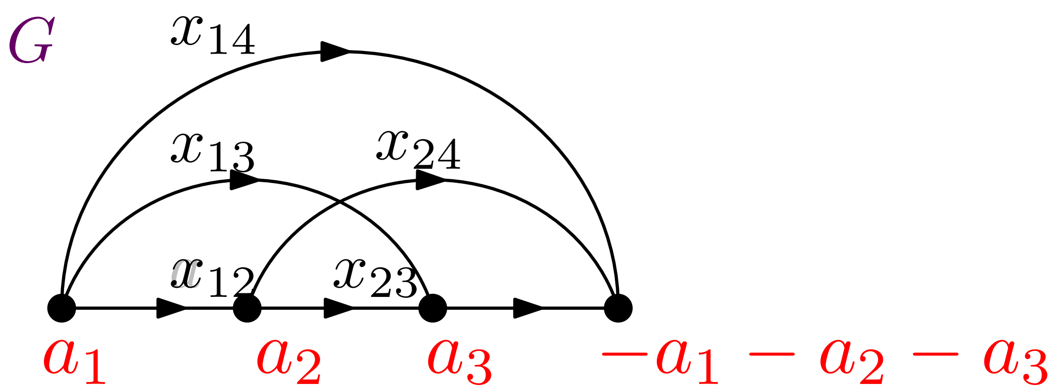
$$x_{23} + x_{24} - x_{12} = a_2$$



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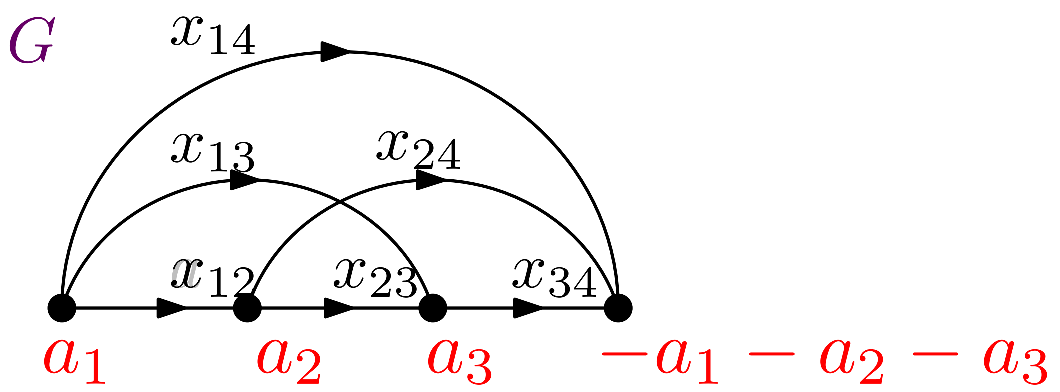

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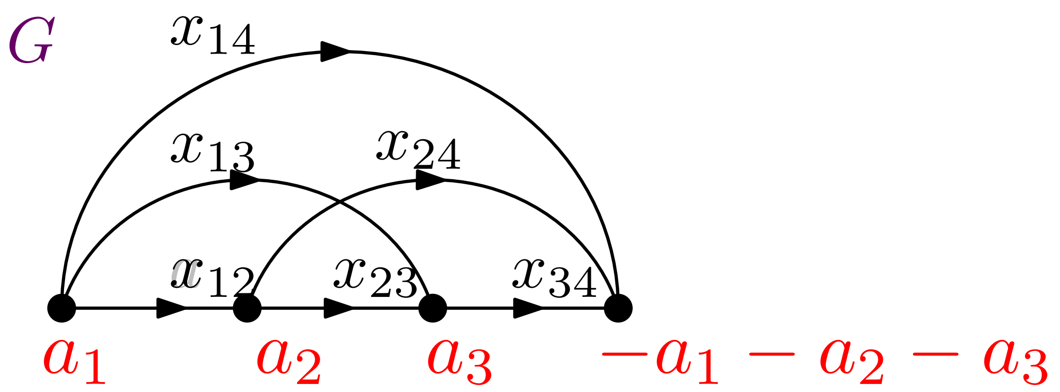

Lattice points of $\mathcal{F}_G(\mathbf{a})$ are integral flows on G with netflow $\mathbf{a}$.
Let $K_G(\mathbf{a}) := L_{\mathcal{F}_G(\mathbf{a})}(1)$.

Kostant partition function

When G is complete graph k_{n+1}, $K_{k_{n+1}}(\mathbf{a})$ is called the **Kostant partition function**.

$K_{k_{n+1}}(\mathbf{a}) = \#$ of ways of writing $\mathbf{a}$ as an $\mathbb{N}$-combination of vectors $e_i - e_j$, $1 \leq i < j \leq n+1$

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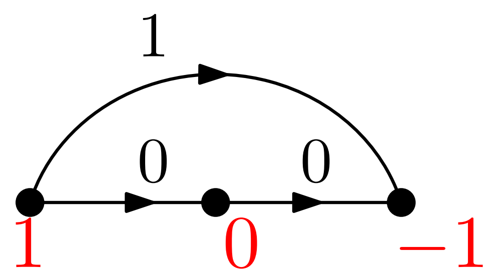
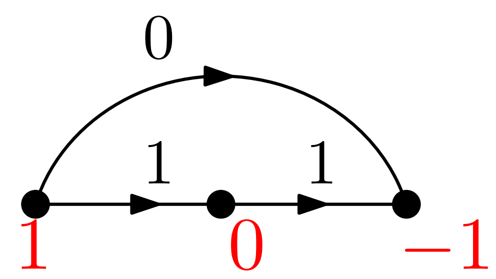


$(1, 0, -1) = e_1 - e_3$



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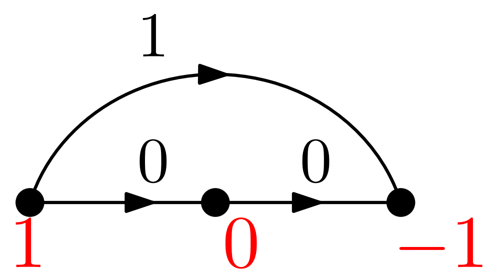

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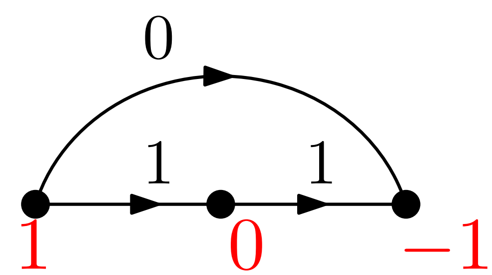

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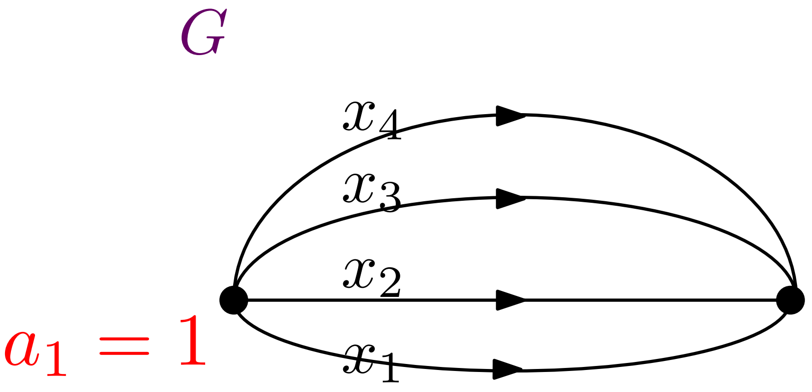
Formulas for **Kostka numbers** and **Littlewood-Richardson coefficients** in terms of $K_{k_{n+1}}(\mathbf{a})$.

Examples of flow polytopes

$\mathcal{F}_G(\mathbf{a}) = \{\text{flows } x(\epsilon) \in \mathbb{R}_{\geq 0},\ \epsilon \in E(G) \mid \text{netflow}(i) = a_i\}$

Example

$$x_1 + x_2 + x_3 + x_4 = 1$$



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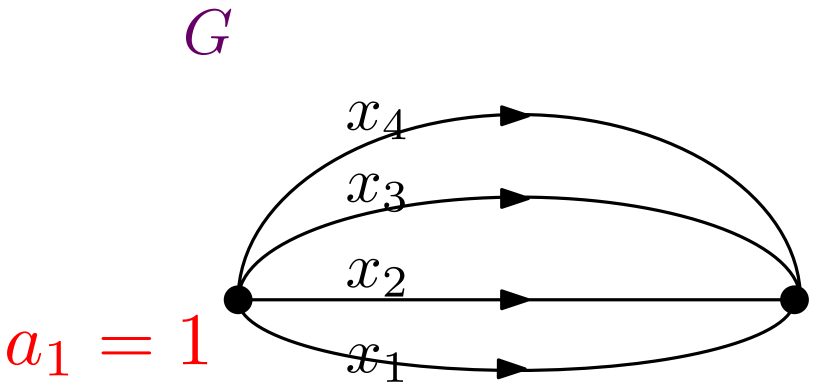
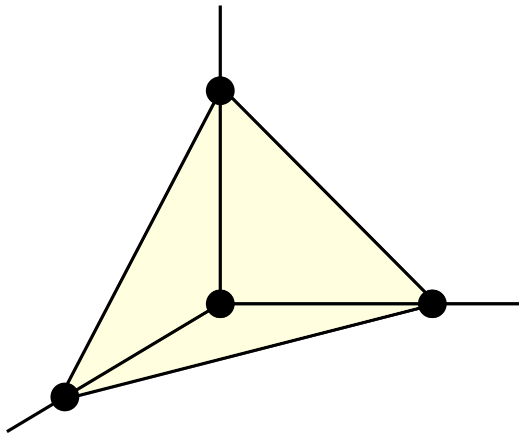


$\mathcal{F}_G(\mathbf{a})$ is a simplex



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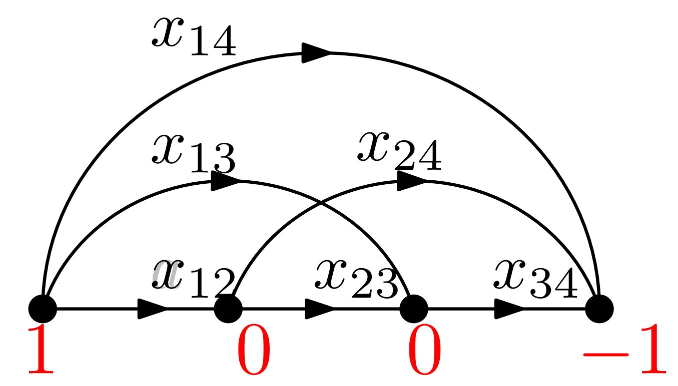
$\mathbf{a} = (1, 0, \ldots, 0, -1)$



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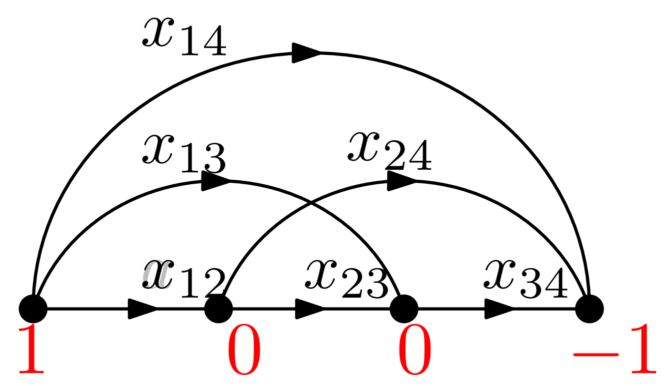

$\mathcal{F}_{k_{n+1}}(1, 0, \ldots, 0, -1)$ is called the **Chan-Robbins-Yuen** (CRY_n) polytope

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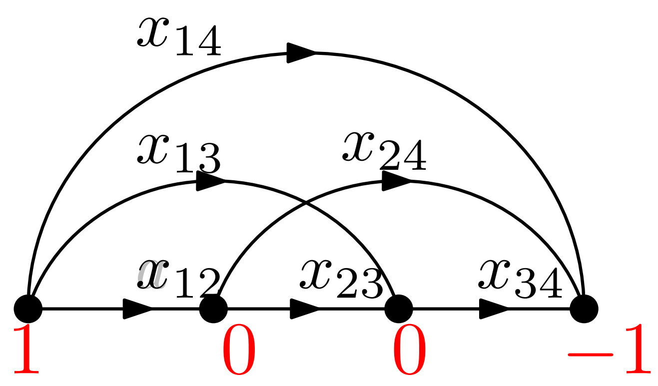

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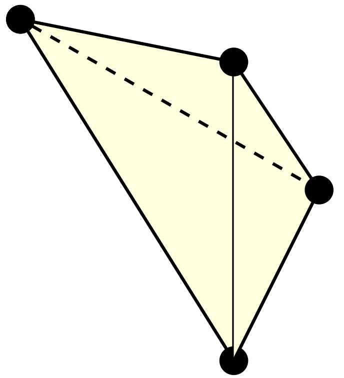
has 2^{n-1} vertices, dimension $\binom{n}{2}$



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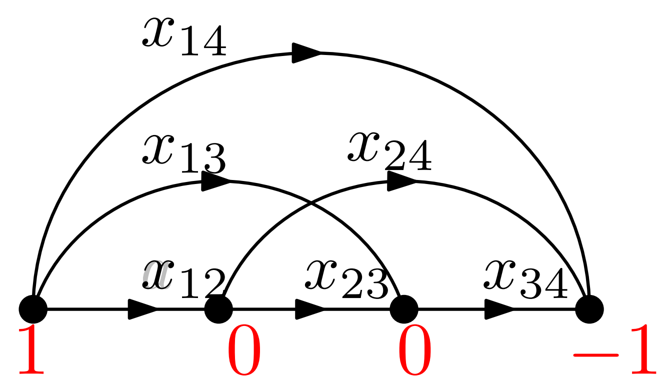

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(conjecture Chan-Robbins-Yuen 99)

- $\mathrm{vol}(CRY_n) = C_1 \cdots C_{n-2}$ (Zeilberger 99)

More examples of flow polytopes

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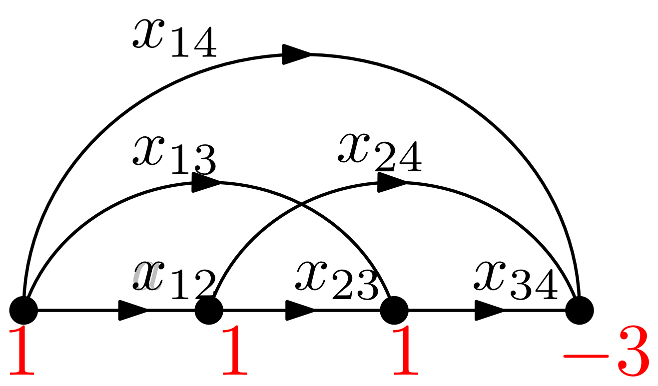

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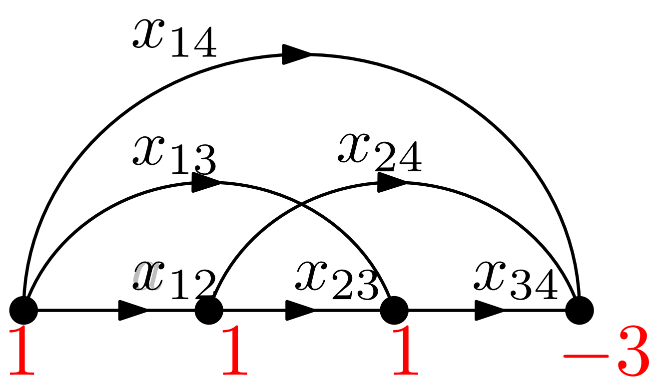

$\mathcal{F}_{k_{n+1}}(1, 1, \ldots, 1, -n)$ is called the **Tesler** polytope

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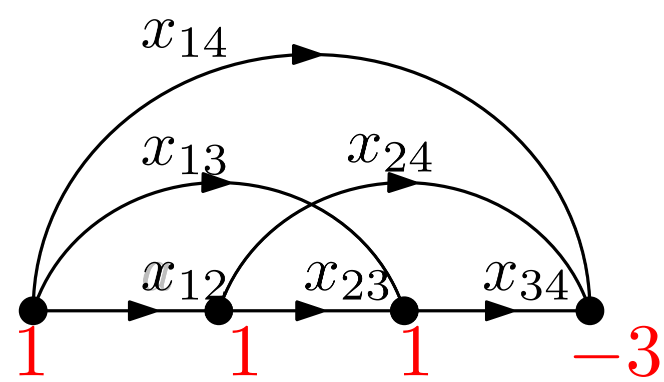

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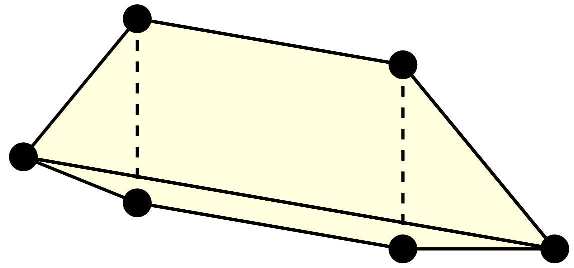
has $n!$ vertices, dimension $\binom{n}{2}$



Theorem (M, Morales, Rhoades 2014)

volume equals $\#\,\mathrm{SYT}(n-1, n-2, \ldots, 2, 1) \cdot C_1 C_2 \cdots C_{n-1}$

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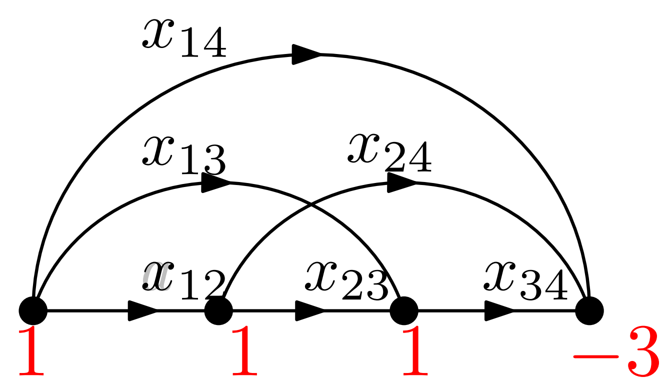

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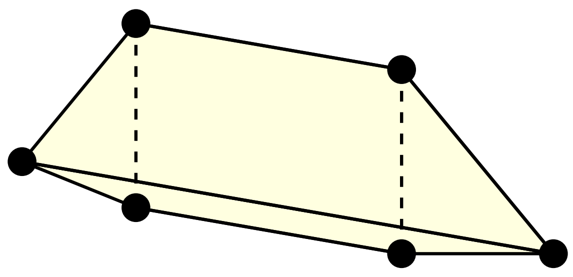
Combinatorial proof?



Relation to CRY?

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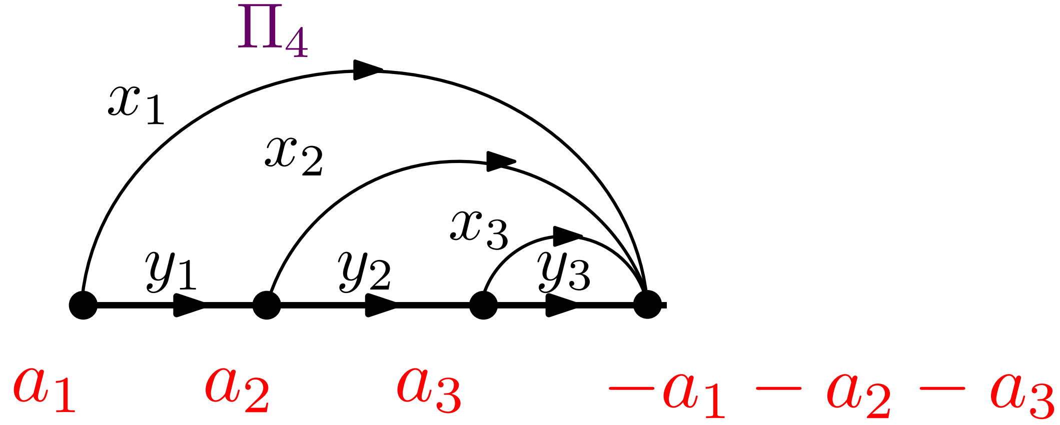
Example (Baldoni-Vergne 2008)



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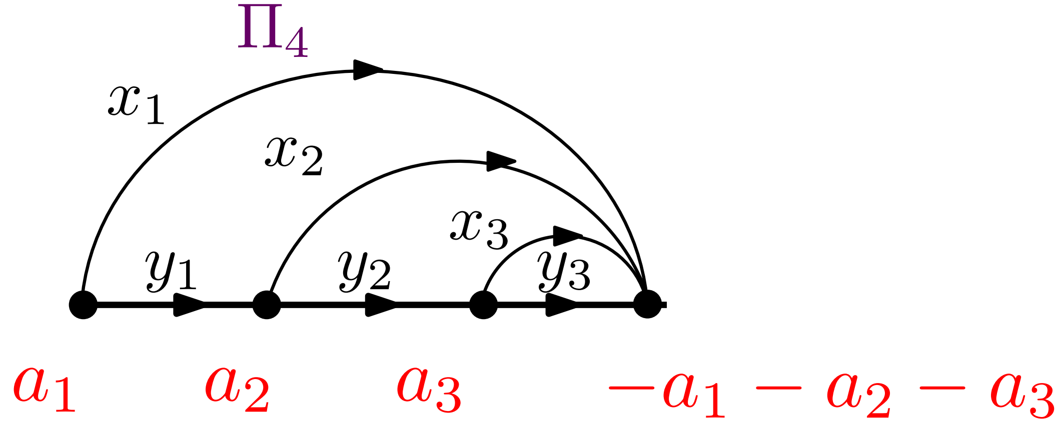

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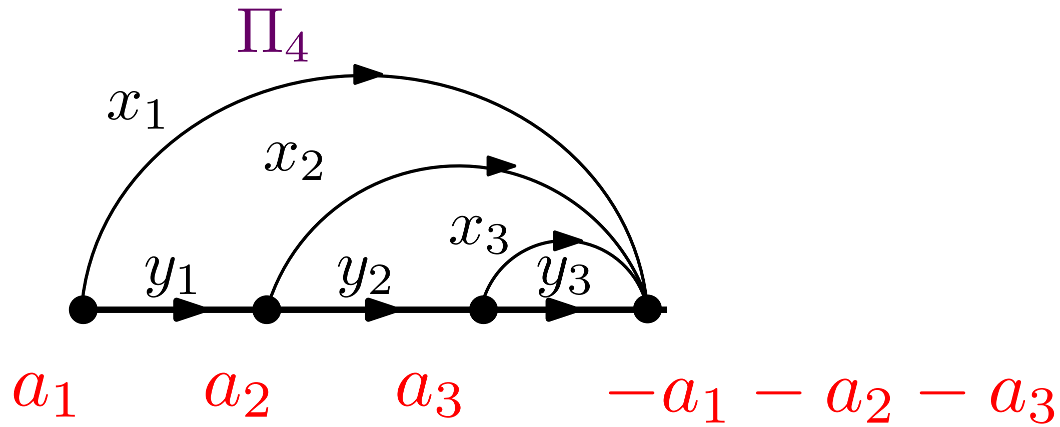

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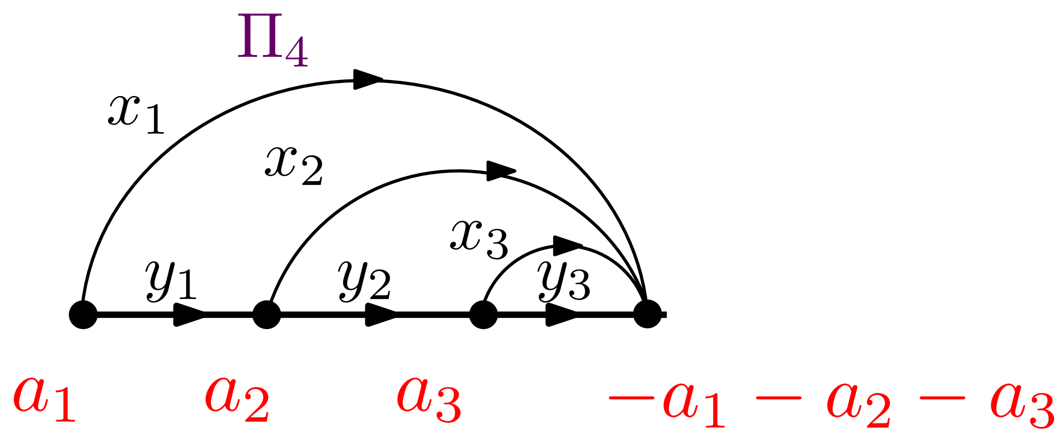

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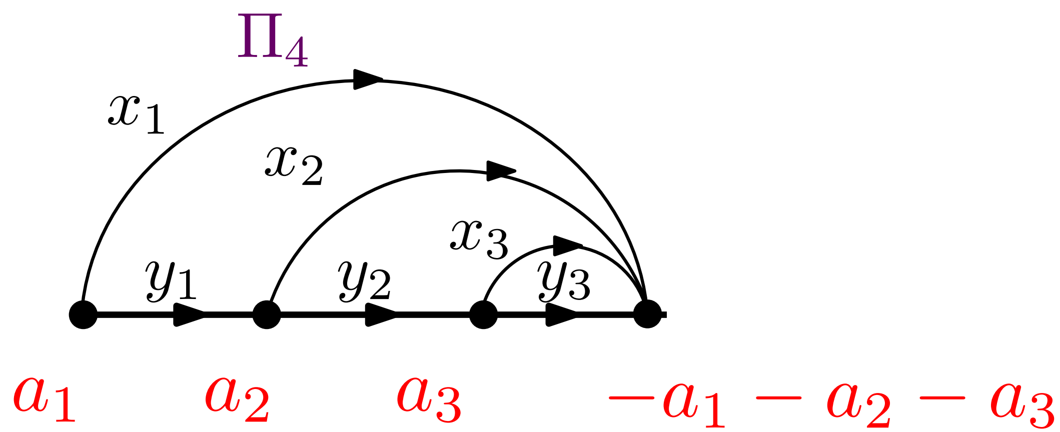

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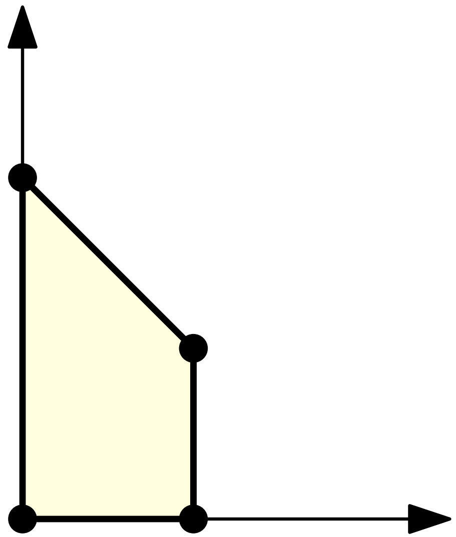
$\mathcal{F}_{\Pi_{n+1}}(\mathbf{a})$ is the Pitman-Stanley polytope

Pitman-Stanley polytope

$\mathbf{a} = (a_1, a_2, \ldots, a_n) \in \mathbb{Z}^n_{\geq 0}$

$$\mathrm{PS}_n(\mathbf{a}) = \left\{ (x_1, \ldots, x_n) \in \mathbb{R}^n_{\geq 0} \;\middle|\; \begin{array}{c} x_1 \leq a_1 \\ x_1 + x_2 \leq a_1 + a_2 \\ \vdots \\ x_1 + \cdots + x_n \leq a_1 + \cdots + a_n \end{array} \right\}$$

Example



$PS_2(1,1)$

Pitman-Stanley polytope

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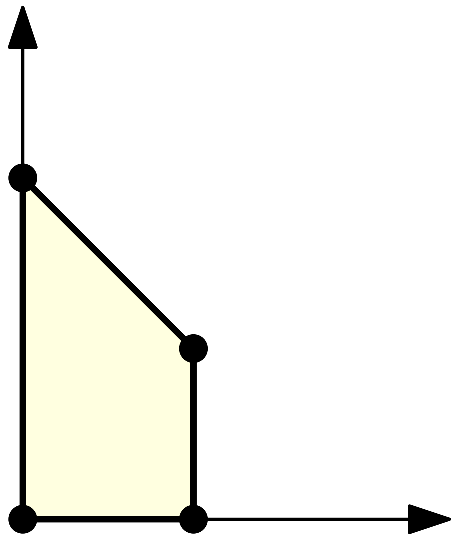
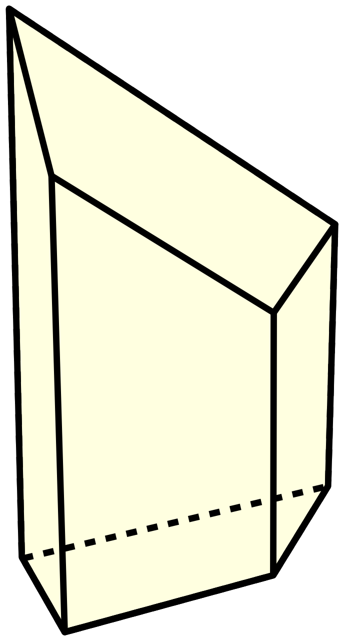


$PS_2(1,1)$



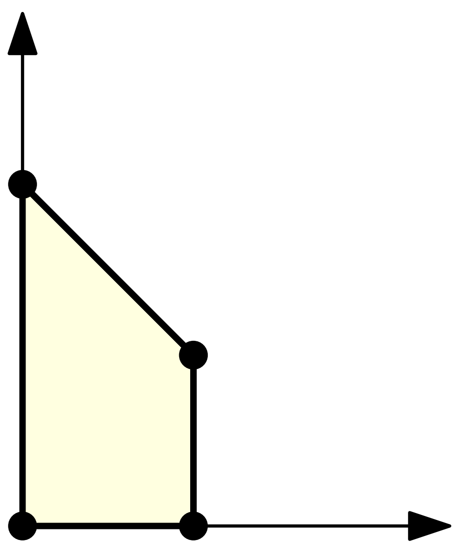
$PS_3(1,1,1)$

Pitman-Stanley polytope

$$\mathbf{a} = (a_1, a_2, \ldots, a_n) \in \mathbb{Z}^n_{\geq 0}$$

$$\mathrm{PS}_n(\mathbf{a}) = \left\{ (x_1, \ldots, x_n) \in \mathbb{R}^n_{\geq 0} \;\middle|\; \begin{array}{c} x_1 \leq a_1 \\ x_1 + x_2 \leq a_1 + a_2 \\ \vdots \\ x_1 + \cdots + x_n \leq a_1 + \cdots + a_n \end{array} \right\}$$

Example



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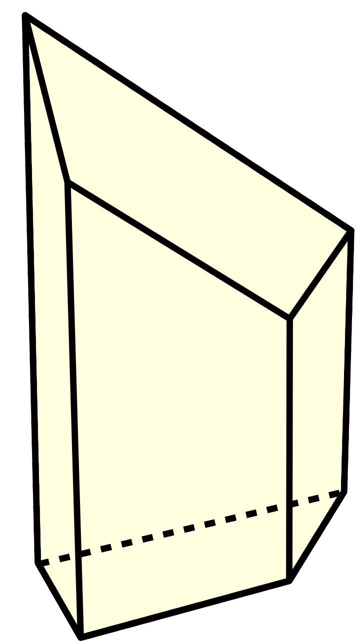
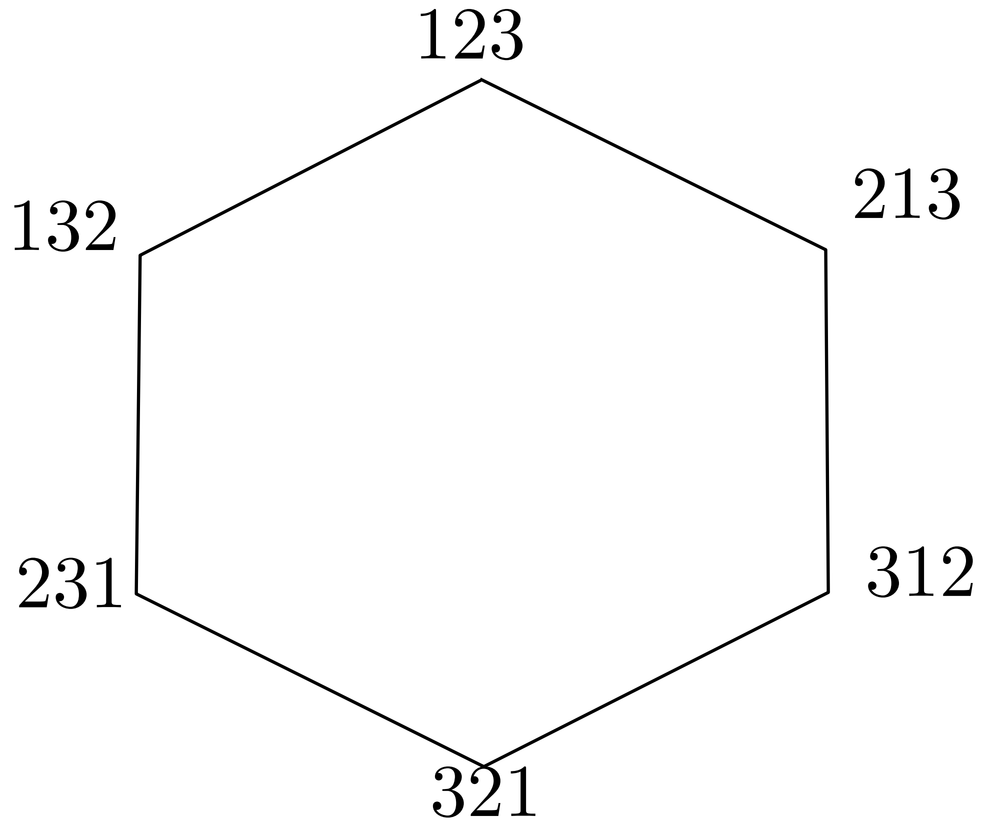


$PS_3(1,1,1)$

- 2^n vertices, n dimensional, is a generalized permutahedron

Generalized permutahedra



Volume of the Pitman-Stanley polytope

Theorem (Pitman-Stanley 01)

$$\mathrm{vol}\,\mathrm{PS}_n(\mathbf{a}) = \sum_{\mathbf{j} \succeq (1,\ldots,1)} \binom{n}{j_1, \ldots, j_n} a_1^{j_1} \cdots a_n^{j_n}$$

$$j_1 + \cdots + j_n = n, j_1, \ldots, j_n \geq 0$$

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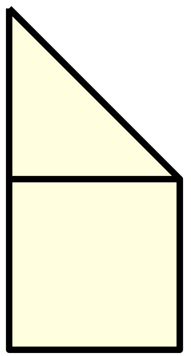

Example

$$\begin{aligned}\operatorname{vol} PS_2(a_1,a_2) &= 2a_1a_2 + a_1^2 \\ &= a_1a_2 + a_2a_1 + a_1^2\end{aligned}$$

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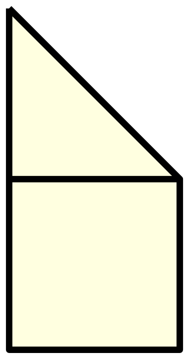

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$$j_1 + j_2 = 2, j_1, j_2 \geq 0, j_1 \geq 1, j_1 + j_2 \geq 2$$

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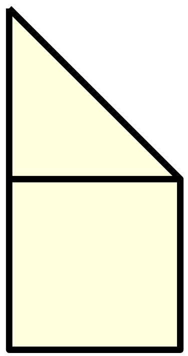

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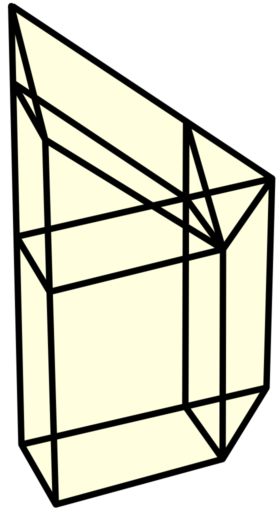

$$\mathrm{vol}PS_3(a_1,a_2,a_3) = 6a_1a_2a_3 + 3a_1^2a_2 + 3a_1a_2^2 + 3a_1^2a_3 + a_1^3$$

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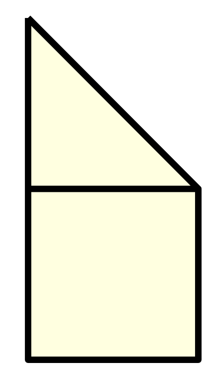

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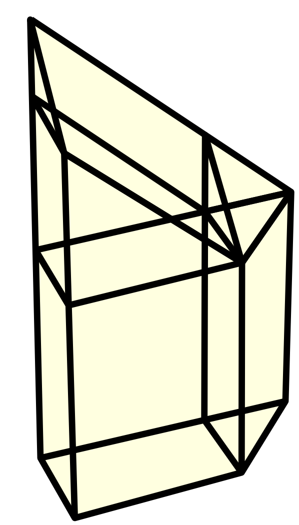

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Proof via a subdivision where each term corresponds to the volume of a cell in subdivision

Lattice points of the Pitman-Stanley polytope

Theorem (Pitman-Stanley, Gessel 01)

$$L_{\mathrm{PS}_n(\mathbf{a})}(t) = \sum_{\mathbf{j} \succeq (1,\ldots,1)} \left(\!\!\binom{a_1 t + 1}{j_1}\!\!\right) \left(\!\!\binom{a_2 t}{j_2}\!\!\right) \cdots \left(\!\!\binom{a_n t}{j_n}\!\!\right)$$

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$\left(\!\binom{3}{2}\!\right) = 6$, counting $\{1,1\}, \{1,2\}, \{1,3\}, \{2,2\}, \{2,3\}, \{3,3\}$

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Corollary

$$L_{\mathrm{PS}_n(\mathbf{a})}(t) \in \mathbb{N}[t]$$

Summary

$$\mathcal{F}_G(\mathbf{a}) = \{\text{flows } x(\epsilon) \in \mathbb{R}_{\geq 0},\ \epsilon \in E(G) \mid \text{netflow}(i) = a_i\}$$

Examples

- $\mathcal{F}_{k_{n+1}}(\mathbf{a})$: CRY polytope ($\mathbf{a} = (1, 0, \ldots, 0, -1)$),
  Tesler polytope ($\mathbf{a} = (1, 1, \ldots, 1, -n)$);
  volumes divisible by $C_1 \cdots C_{n-2}$

- $\mathcal{F}_{\Pi_{n+1}}(\mathbf{a})$: **Pitman-Stanley polytope**, explicit volume and lattice point formulas related to parking functions.

Question

- Is there a formula for volume and lattice points of $\mathcal{F}_G(\mathbf{a})$?

Lidskii volume formula

Theorem (Baldoni-Vergne 08, Postnikov-Stanley - unpublished)

G m edges, $n+1$ vertices, $a_i \geq 0$

$$\mathrm{vol}\mathcal{F}_G(a_1,\ldots,a_n) = \sum_{\mathbf{j}\succeq\mathbf{o}} \binom{m-n}{j_1,\ldots,j_n} a_1^{j_1}\cdots a_n^{j_n} \times K_G(j_1-o_1,\ldots,j_n-o_n,0)$$

where $\mathbf{o}=(o_1,\ldots,o_n)$, $o_v = outdeg(v)-1$ and $|\mathbf{j}| = m-n$.

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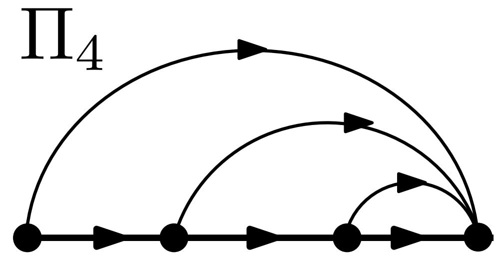
Pitman-Stanley polytope:



$$\mathrm{vol}\mathcal{F}_{\Pi_{n+1}}(\mathbf{a}) = \sum_{\mathbf{j}\succeq(1,\ldots,1)} \binom{n}{j_1,\ldots,j_n} a_1^{j_1}\cdots a_n^{j_n}\cdot 1$$

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Corollary:

$$\mathrm{vol}\mathcal{F}_G(1,0,\ldots,0,-1) = 1 \cdot K_G(m-n-o_1,-o_2,\ldots,-o_n,0).$$

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$$\times K_G(j_1-o_1,\ldots,j_n-o_n,0)$$

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Example: (CRY polytope)

$$\mathrm{vol}\mathcal{F}_{k_{n+1}}(1,0,\ldots,0,-1) = K_{k_{n+1}}(\tbinom{n-1}{2},-n+2,\ldots,-2,-1,0)$$

Lidskii lattice point formula

Theorem (Baldoni-Vergne 08, Postnikov-Stanley – unpublished)
G m edges, $n+1$ vertices, $a_i \geq 0$

$$K_G(a_1,\ldots,a_n) = \sum_{\mathbf{j}\succeq\mathbf{o}} \left(\!\!\binom{a_1-i_1}{j_1}\!\!\right)\cdots\left(\!\!\binom{a_n-i_n}{j_n}\!\!\right) \times K_G(j_1-o_1,\ldots,j_n-o_n,0)$$

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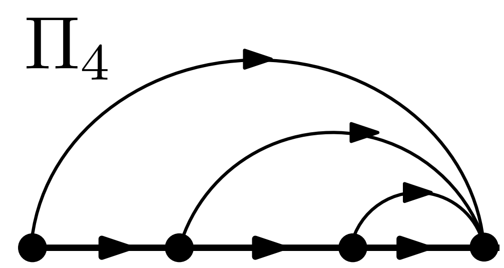
Pitman-Stanley polytope:



$$\mathcal{F}_{\Pi_{n+1}}(\mathbf{a}) = \sum_{\mathbf{j}\succeq(1,\ldots,1)} \left(\!\!\binom{a_1+1}{j_1}\!\!\right) \left(\!\!\binom{a_2}{j_2}\!\!\right) \cdots \left(\!\!\binom{a_n}{j_n}\!\!\right)$$

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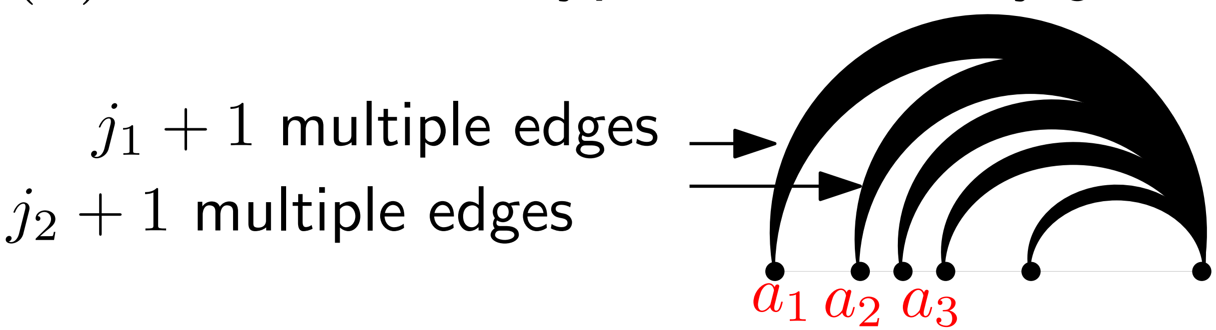
- type D analogue by Maris-M (2023+) generalizing both of the above approaches

Subdivision proof of Lidskii formulas

$$\mathrm{vol}\mathcal{F}_G(a_1,\ldots,a_n) = \sum_{\mathbf{j}\succeq\mathbf{o}} \binom{m-n}{j_1,\ldots,j_n} a_1^{j_1}\cdots a_n^{j_n} \times K_G(j_1-o_1,\ldots,j_n-o_n,0)$$

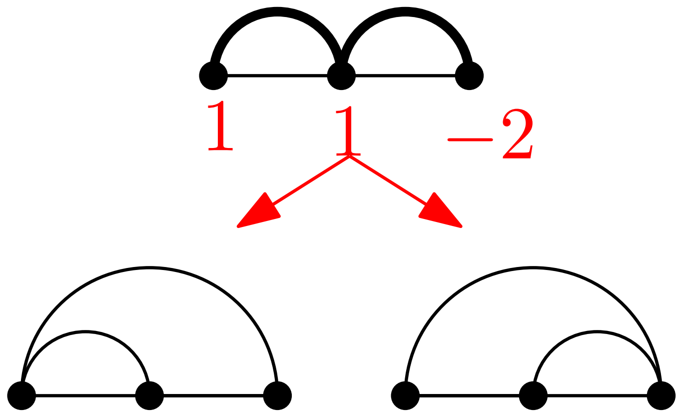
Subdivide $\mathcal{F}_G(\mathbf{a})$ into **cells** of types indexed by $\mathbf{j}$.



volume of each type **j** cell : $\binom{m-n}{j_1,\ldots,j_n} a_1^{j_1}\cdots a_n^{j_n}$

times type **j** cell appears: $K_G(j_1-o_1,\ldots,j_n-o_n,0)$

Example subdivision



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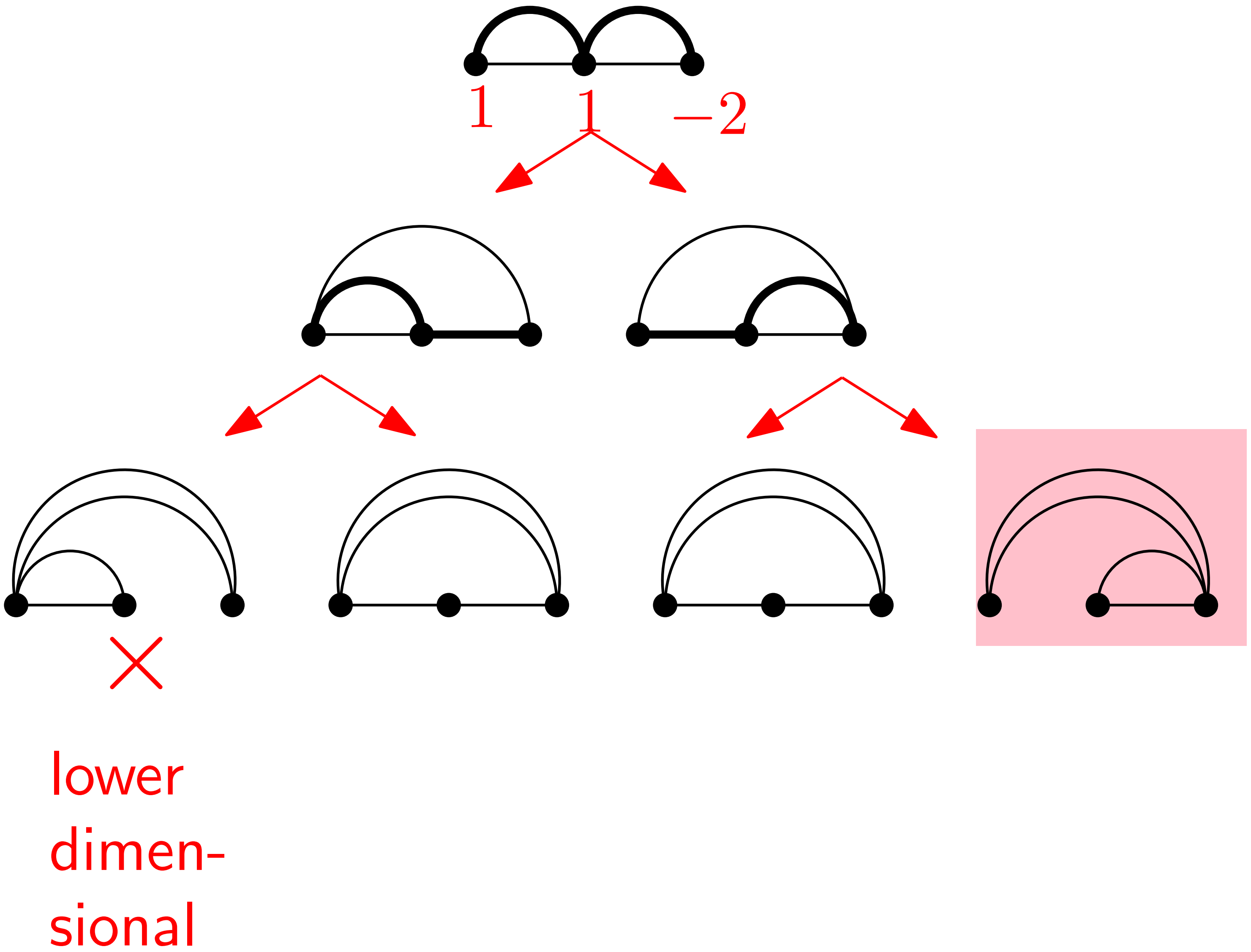

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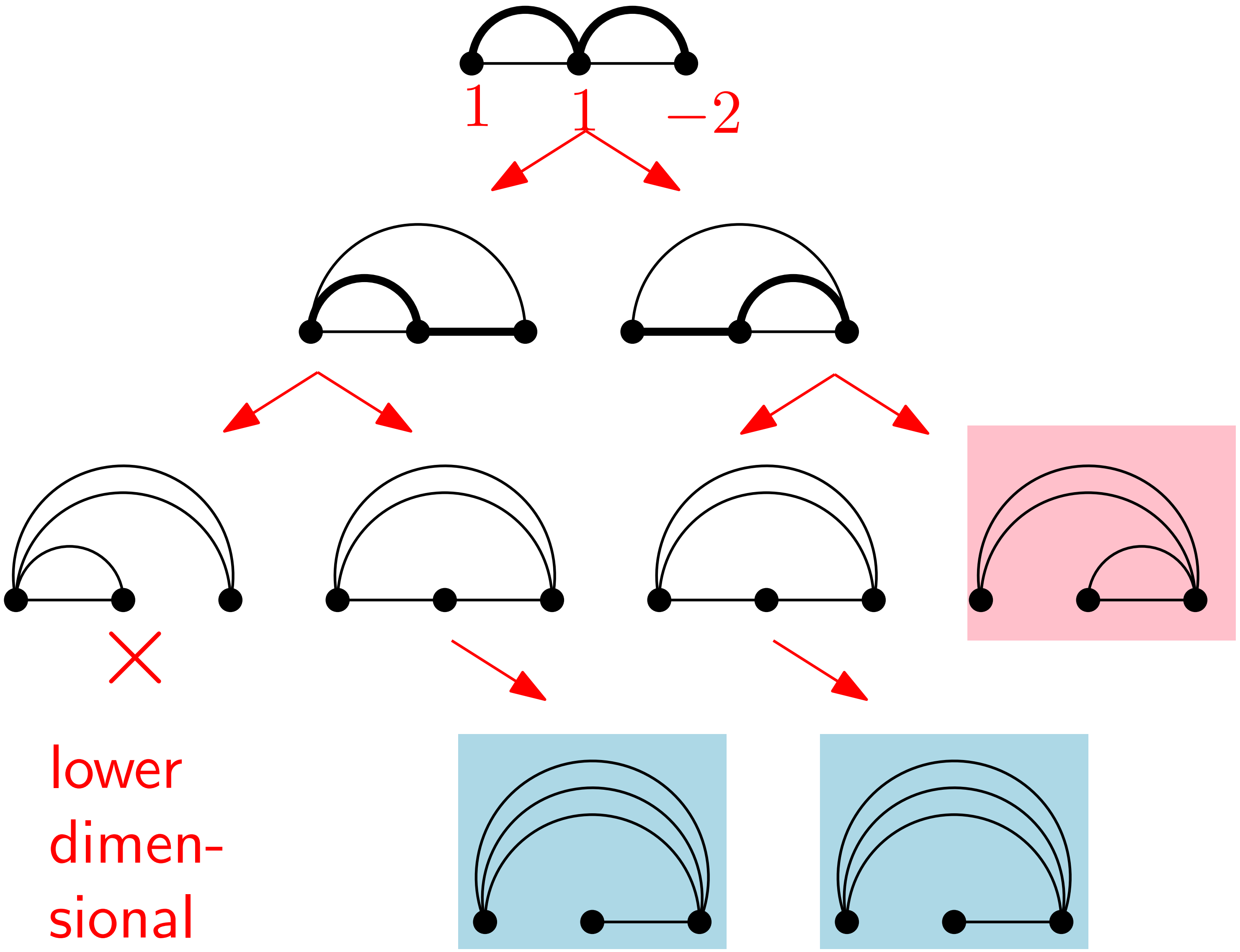

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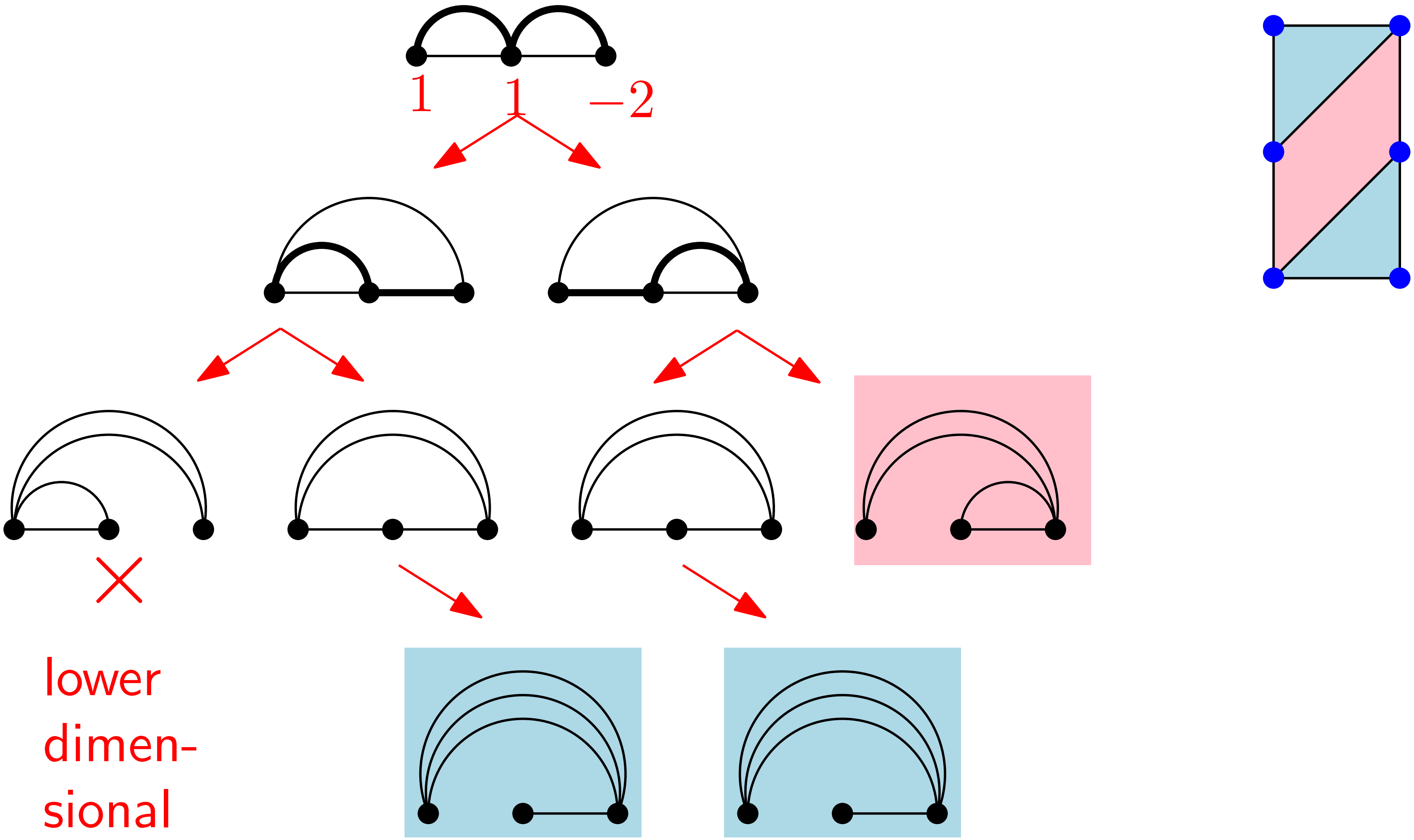

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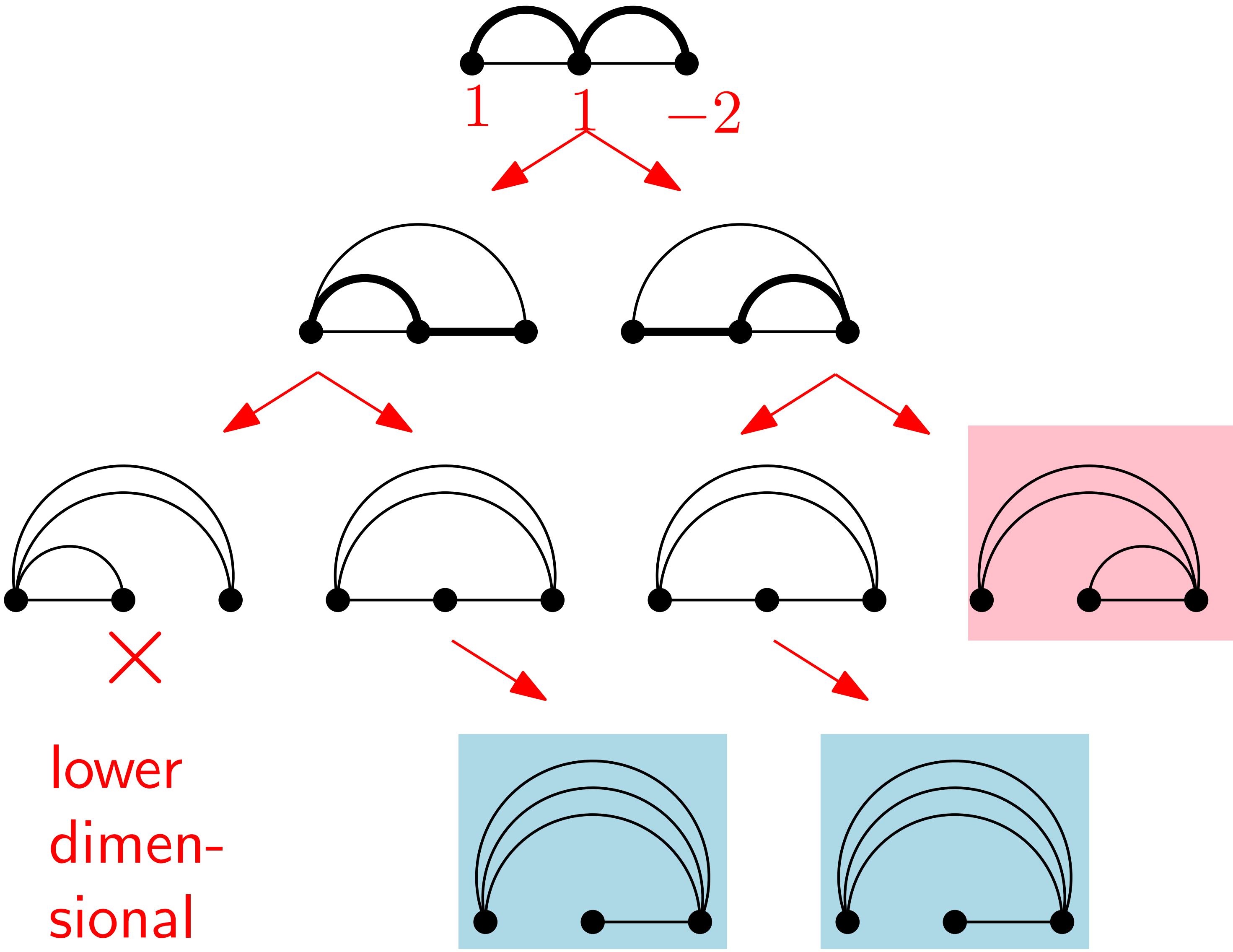
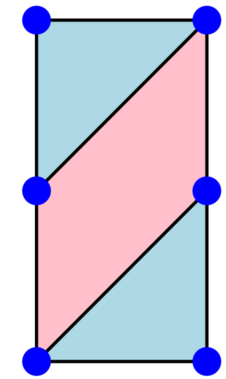




volume:
$2 \cdot 1 + 1 \cdot 2 = 4.$

lattice points:
$0 + 3 \cdot 2 = 6.$

Flow Polytopes - again

Start with a graph G.

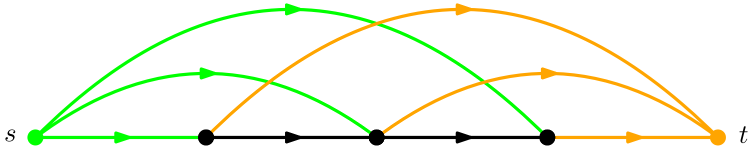
Flow Polytopes - again

Start with a graph G.

Fix an acyclic orientation of G.

Flow Polytopes

Add a source s and a sink t connected to all the original vertices of G. Call the new graph $\widetilde{G}$.



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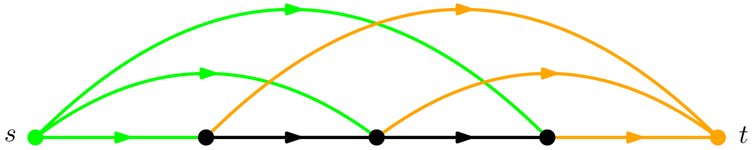
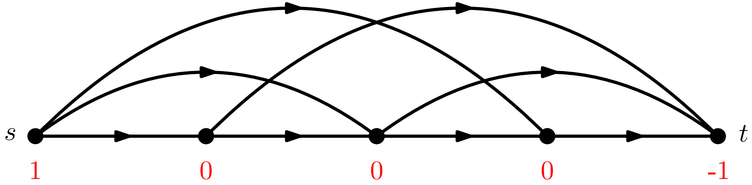
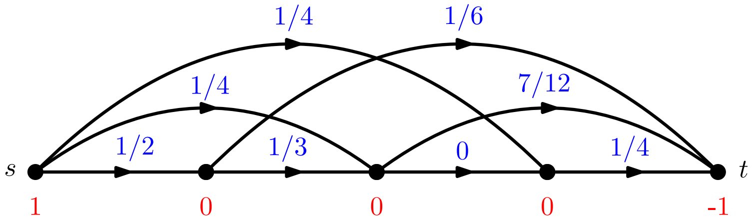


Assign the source s netflow 1, the sink t netflow -1, and all other vertices netflow 0.



Flow Polytopes

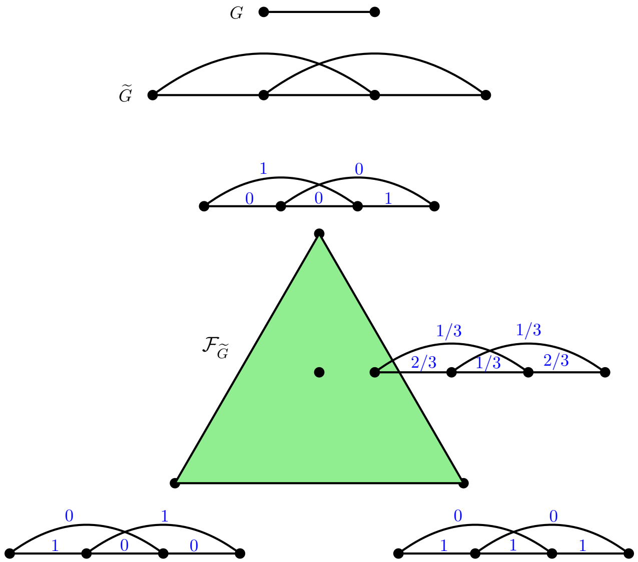
A **flow** on $\widetilde{G}$ is an assignment of nonnegative real numbers to each edge of $\widetilde{G}$ so that at every vertex, outflow minus inflow equals netflow.



The **flow polytope** $\mathcal{F}_{\widetilde{G}}$ is the convex hull in $\mathbb{R}^{E(\widetilde{G})}$ of all flows on $\widetilde{G}$.

$$\left(\frac{1}{2}, \frac{1}{4}, \frac{1}{4}, \frac{1}{3}, \frac{1}{6}, 0, \frac{7}{12}, \frac{1}{4}\right) \in \mathcal{F}_{\widetilde{G}}$$

An Example Flow Polytope



Recall: Volumes of Flow Polytopes

Theorem (Baldoni-Vergne 2008, Postnikov-Stanley unpublished)

If G is a graph on vertices $[0, n+1]$,

$$\mathrm{Vol}\ \mathcal{F}_G(1, 0, \ldots, 0, -1) = K_G\left(0,\ d_1,\ \ldots,\ d_n,\ -\sum_{i=1}^{n} d_i\right)$$

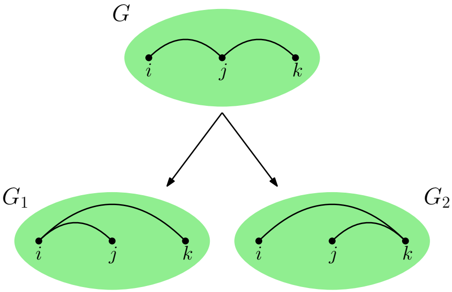
where $d_i = \mathrm{indeg}_G(i) - 1$ for each vertex i.

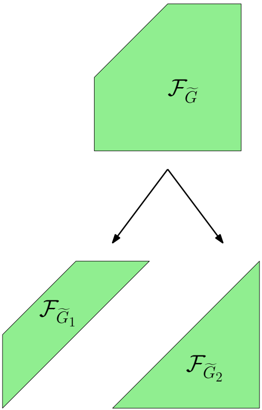
$K_G(\alpha_1, \ldots, \alpha_n)$ is the Kostant partition function from representation theory. It equals the number of ways to write α as a sum of the positive roots $\{e_i - e_j : (i, j) \in G\}$.

Subdividing Flow Polytopes

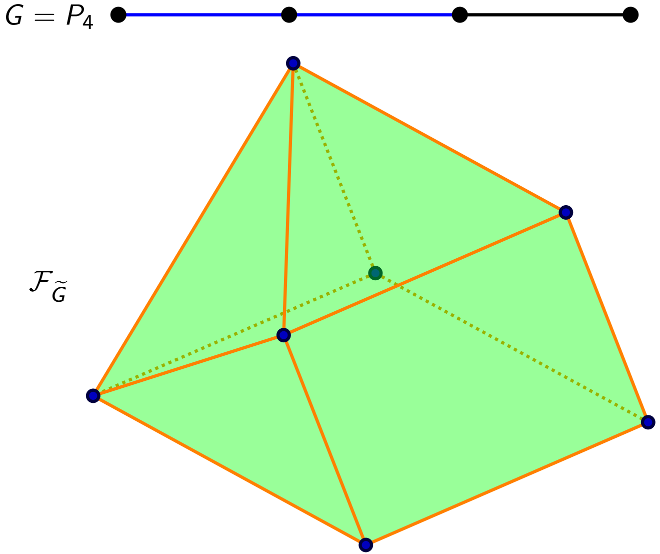
Flow polytopes can be subdivided combinatorially by performing a sequence of changes to the original graph.

A **reduction** on a graph G is a construction of two new graphs G_1 and G_2 from a choice of two adjacent edges $(i, j), (j, k) \in G$:



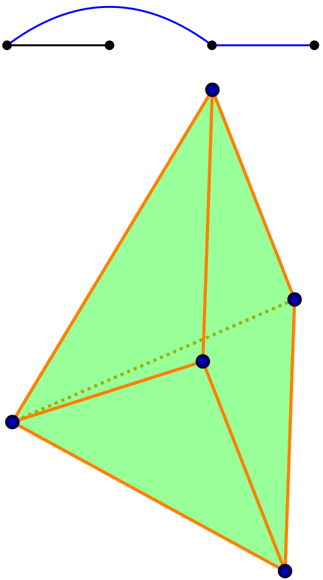


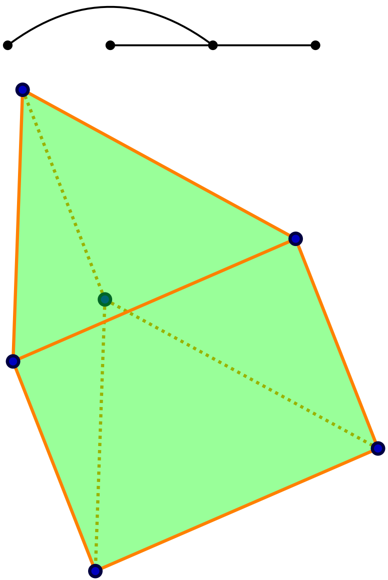
Subdividing Flow Polytopes



*Not technically a picture of $\mathcal{F}_{\widetilde{G}}$, but the root polytope of G.

Subdividing Flow Polytopes





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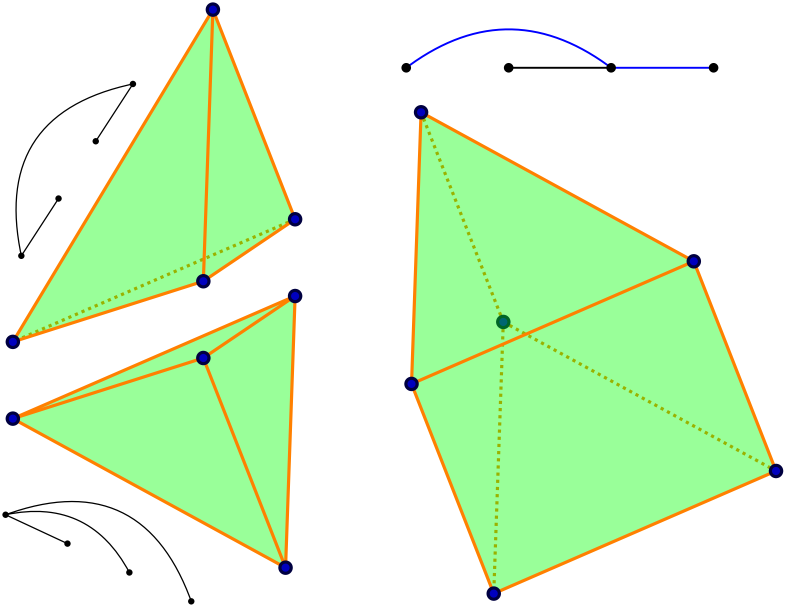

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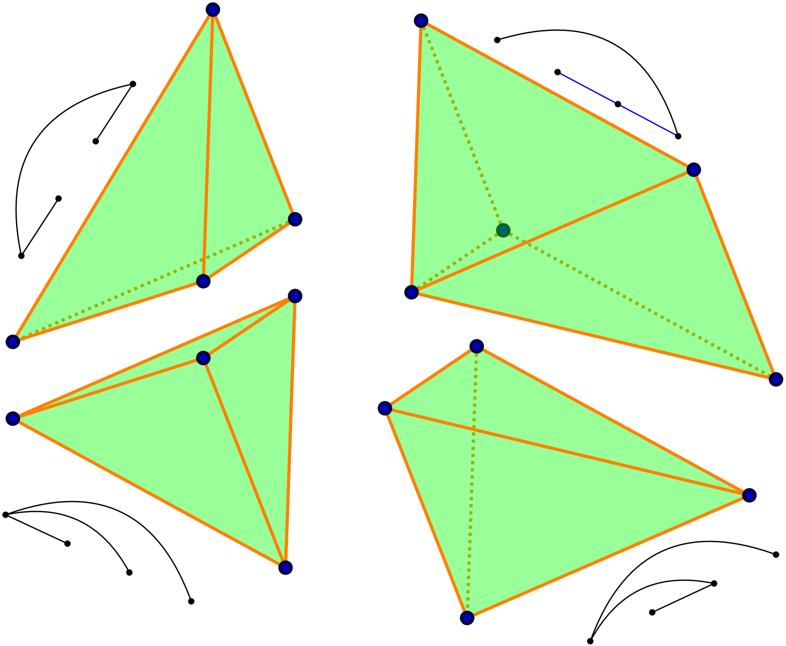
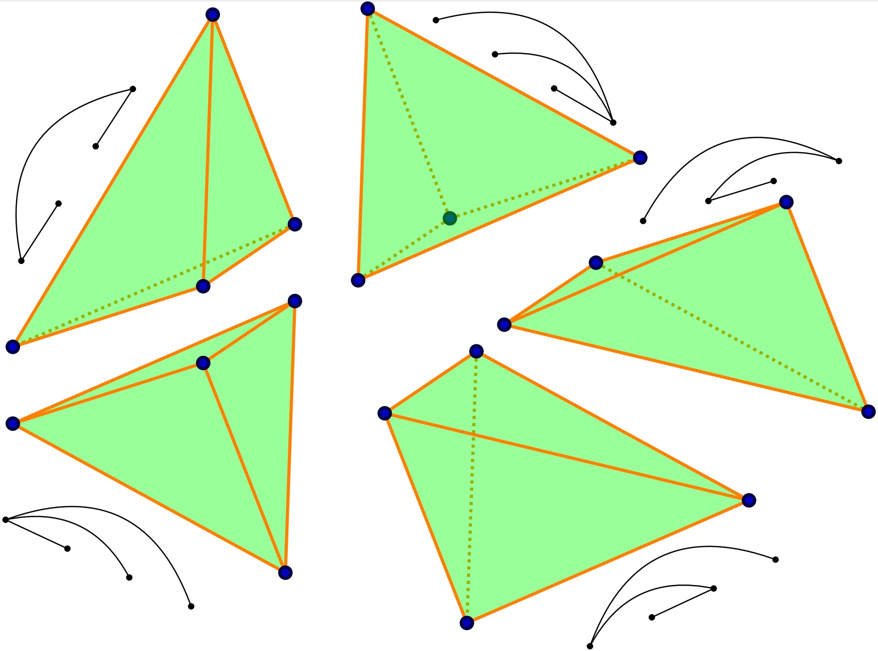


Subdividing Flow Polytopes



Subdividing Flow Polytopes

More compactly, this subdivision procedure can be represented by a **reduction tree**.

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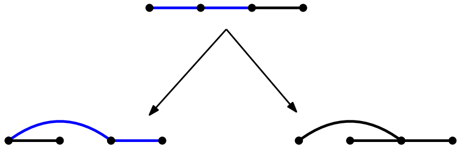

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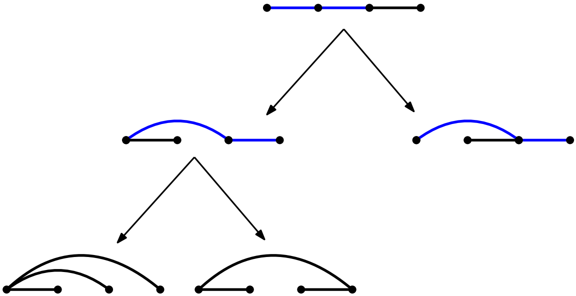

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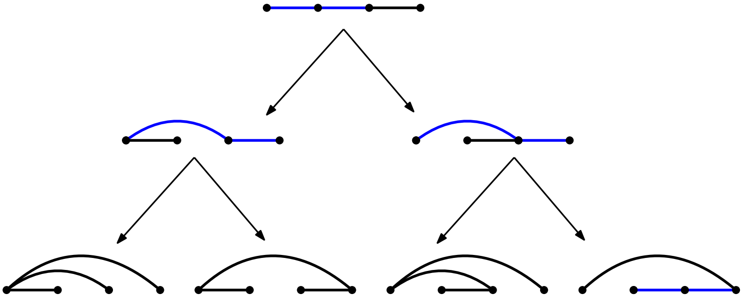

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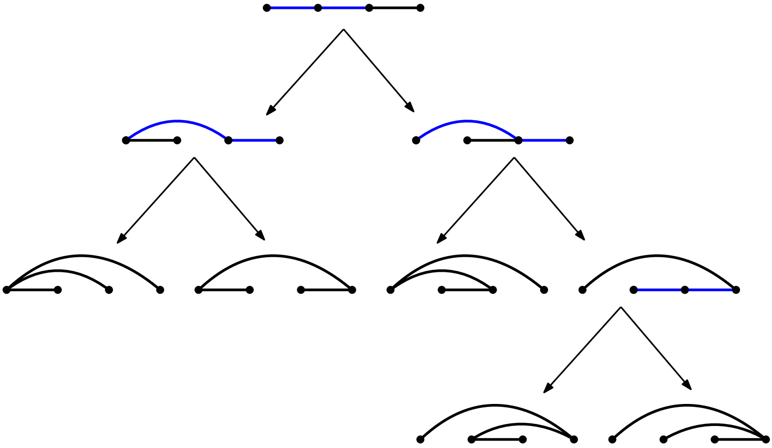
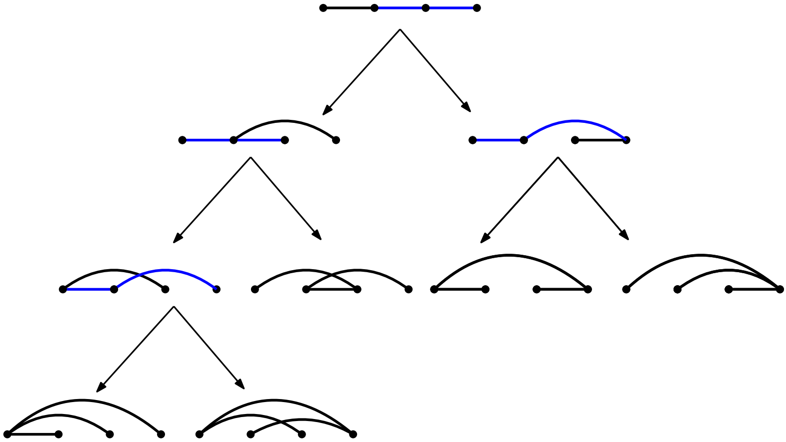


Subdividing Flow Polytopes

The individual graphs appearing in a reduction tree depend on the choice of cuts used to subdivide the flow polytope.



Are there subdivision invariants?

On the one hand, we have seen the leaves of a reduction tree are dependent on choices made.

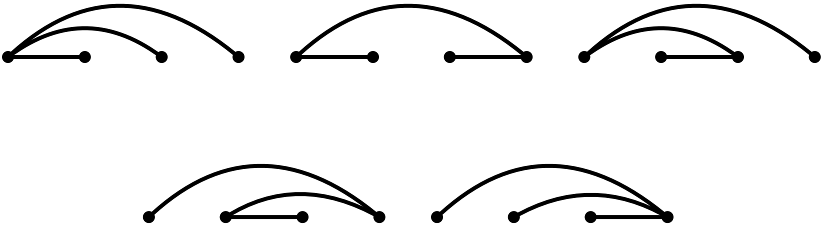
On the other hand, the simplices produced by the reduction process are always unimodular, so the number of leaves in any reduction tree is always the normalized volume of the flow polytope regardless of any choices.

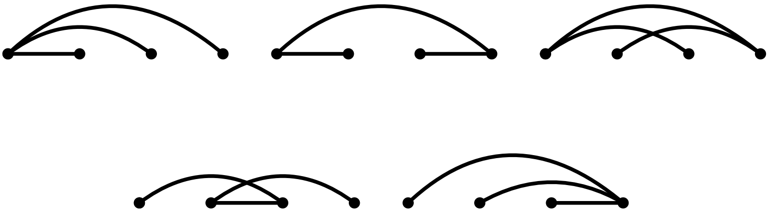
Question

Is there any stronger invariant across all the different ways to fully subdivide a flow polytope using reductions?

Subdivisions to Degree Sequences

Is there an invariant of different subdivisions of a flow polytope?





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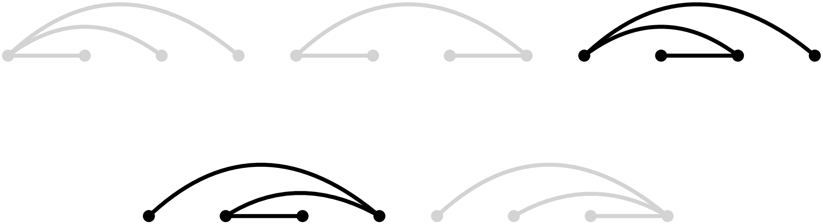

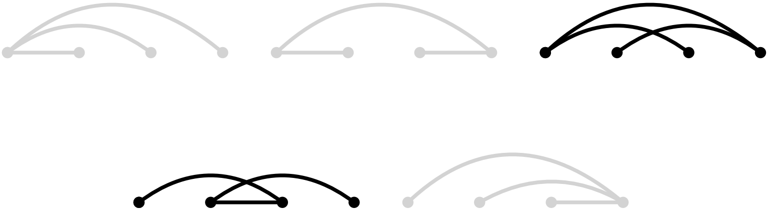

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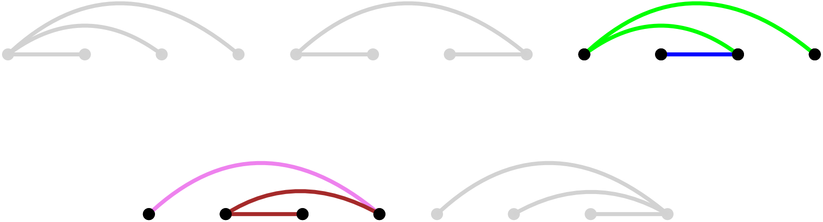
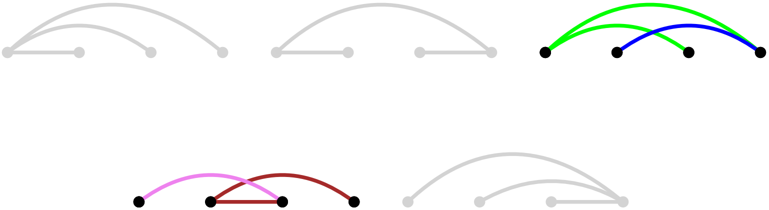
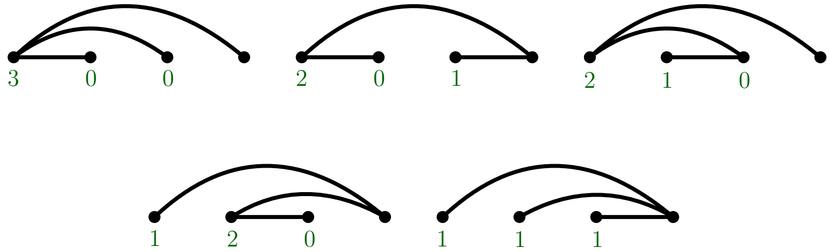
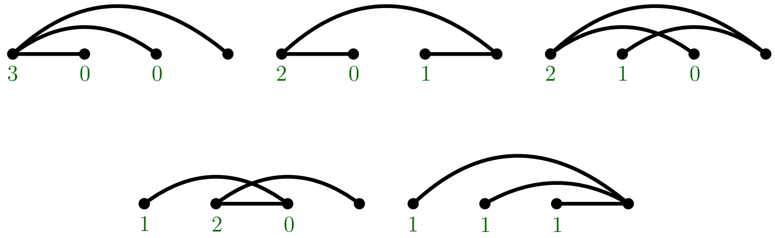


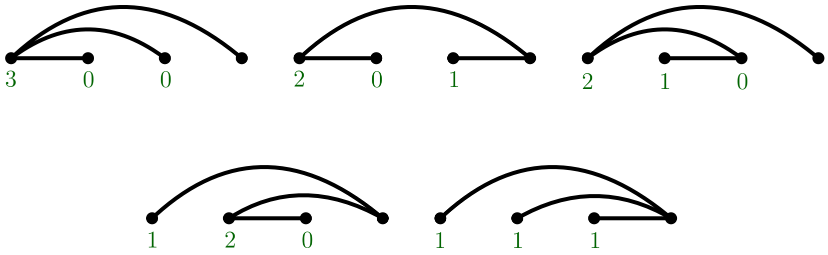


Subdivisions to Degree Sequences



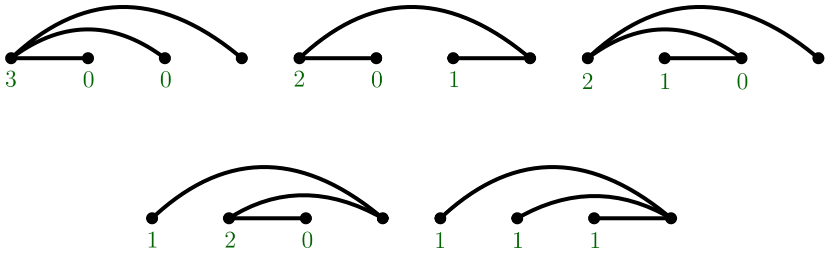


Subdivisions to Degree Sequences



Is $\{(3,0,0), (2,0,1), (2,1,0), (1,2,0), (1,1,1)\}$ dependent only on the original graph?

Subdivisions to Degree Sequences



Is $\{(3,0,0), (2,0,1), (2,1,0), (1,2,0), (1,1,1)\}$ dependent only on the original graph?

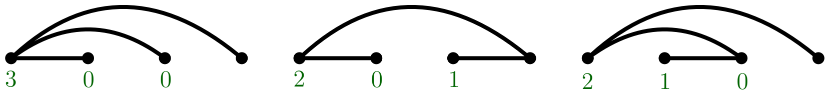
Theorem (Grinberg 2017, M-St. Dizier 2017)

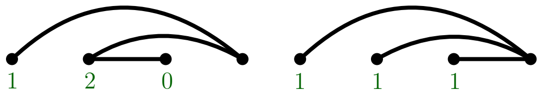
Yes!

Right-Degree Sequences

Definition

For a graph G, let $RD(G)$ denote the multiset of right-degree sequences of the leaves in any reduction tree of G.





$RD(G) = \{(3,0,0), (2,0,1), (2,1,0), (1,2,0), (1,1,1)\}.$

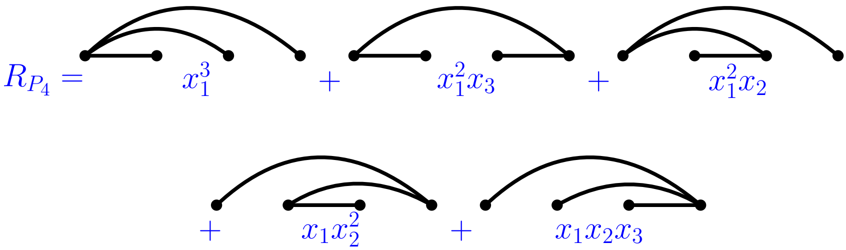
Right-Degree Polynomial

Definition

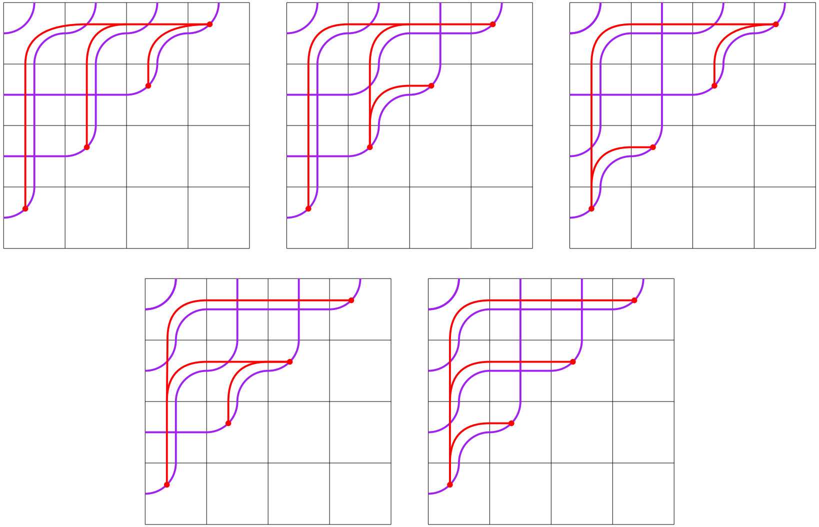
Define the **right-degree polynomial** of G by

$$R_G(x) = \sum_{\alpha \in RD(G)} x^{\alpha}.$$

$$RD(P_4) = \{(3,0,0),\ (2,0,1),\ (2,1,0),\ (1,2,0),\ (1,1,1)\}$$



Trees & Pipe Dreams



Schubert Polynomials (geometrically)

Geometrically, Schubert polynomials arise as distinguished representatives of the cohomology classes of the Schubert varieties in the flag variety of $\mathbb{C}^n$.

Pipe Dreams

A **pipe dream** for $w \in S_n$ is a tiling of an $n \times n$ matrix with crosses ┼ and elbows ╯╭ such that

- All tiles in the weak south-east triangle are elbows, and
- If you write $1, 2, \ldots, n$ on the top and follow the strands (ignoring second crossings among the same strands), they come out on the left and read w from top to bottom.

A pipe dream is **reduced** if no two strands cross twice.

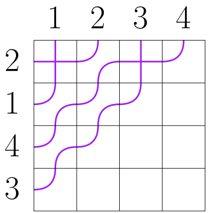
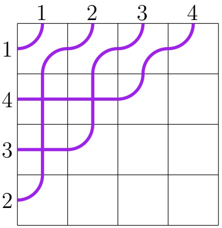
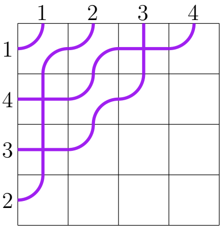


Figure: A reduced pipe dream for $w = 2143$.

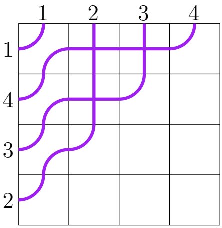
Schubert Polynomial of 1432



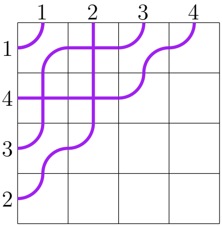
$x_2^2 x_3$



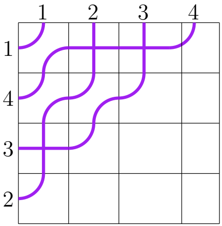
$x_1 x_2 x_3$



$x_1^2 x_2$



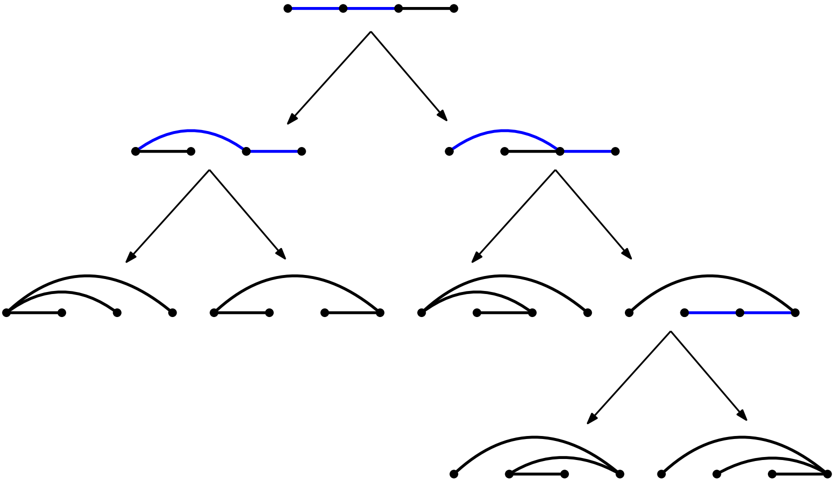
$x_1 x_2^2$



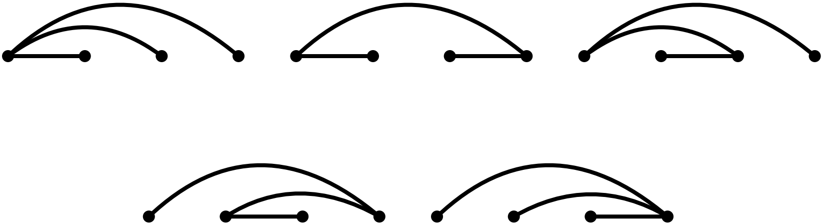
$x_1^2 x_3$

$$\mathfrak{S}_{1432} = x_2^2 x_3 + x_1 x_2 x_3 + x_1^2 x_2 + x_1 x_2^2 + x_1^2 x_3$$

The canonical reduction tree

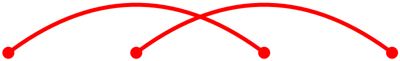


Noncrossing and Alternating Trees

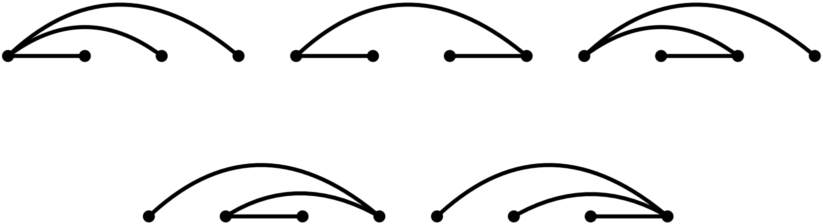


A tree is **alternating** if it has no pair of edges

A tree is **noncrossing** if it has no pair of edges



Noncrossing and Alternating Trees



Theorem (M 2009)

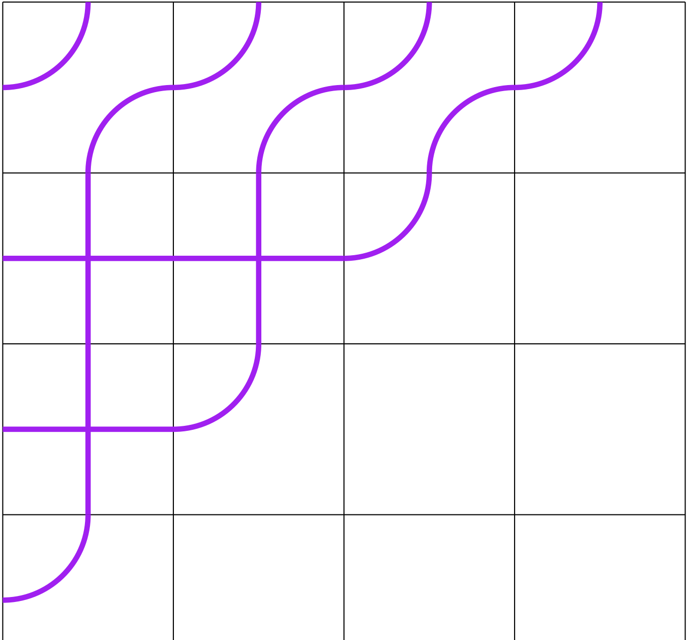
Every tree T has a canonical reduction tree whose leaves are exactly the alternating noncrossing spanning trees of the directed transitive closure $\overline{T}$ of T.

Noncrossing Alternating Spanning Trees to Permutations

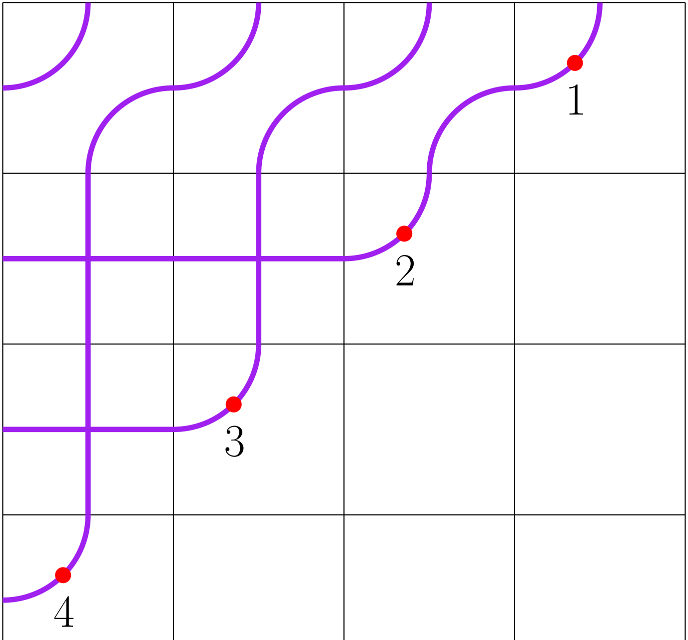
Theorem (Escobar-M 2015)

For permutations of the form $w = 1w'$ where w' is dominant (132-avoiding), there is an tree T_w such that the reduced pipe dreams of w are in bijection with the noncrossing alternating spanning trees of the directed transitive closure $\overline{T}_w$.

Pipe dreams to noncrossing alternating spanning trees



Pipe Dreams to Trees



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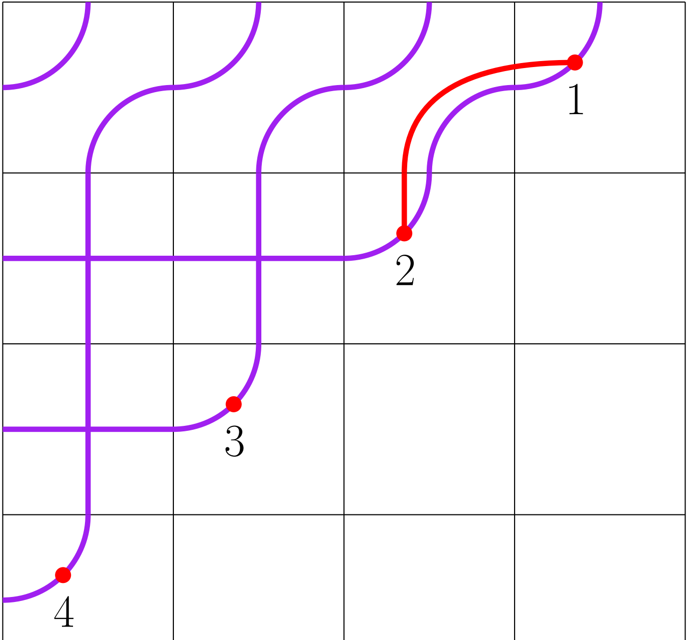
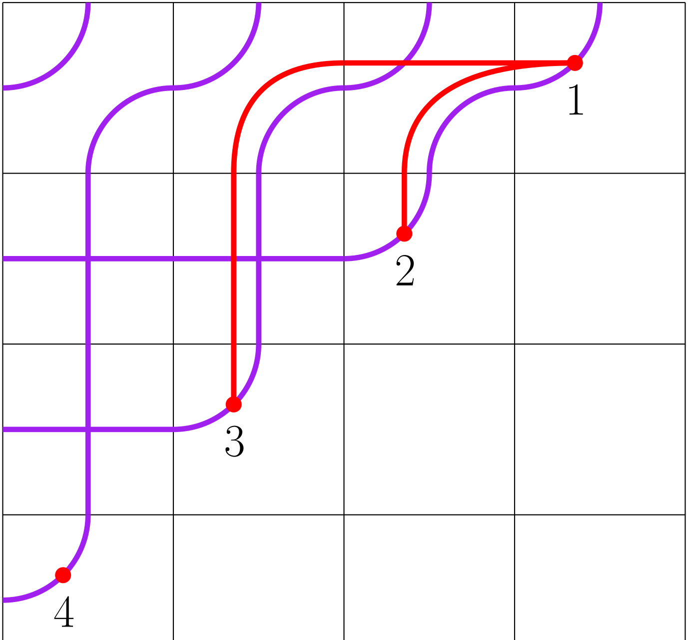


Pipe Dreams to Trees



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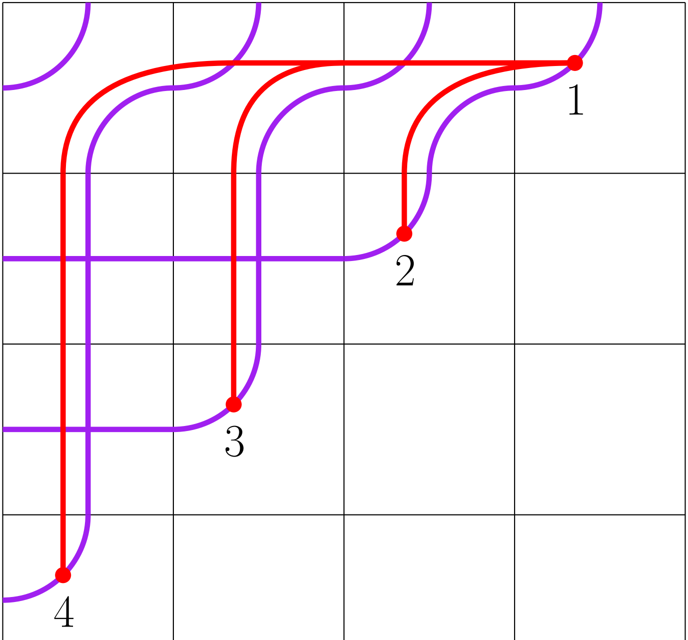

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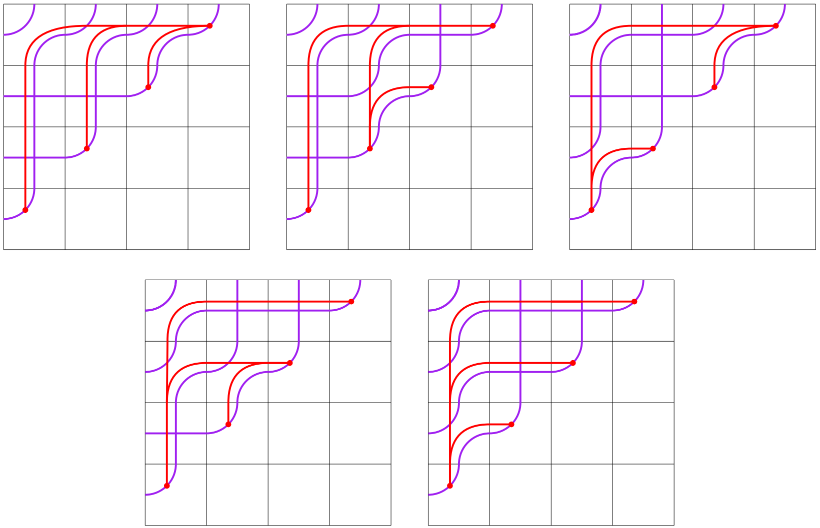


Right-Degree and Schubert Polynomials

Theorem (Escobar-M 2015)

For permutations of the form $w = 1w'$, where w' is 132-avoiding, there is a tree T_w such that the right-degree polynomial R_{T_w} is a reparameterization of $\mathfrak{S}_w$.

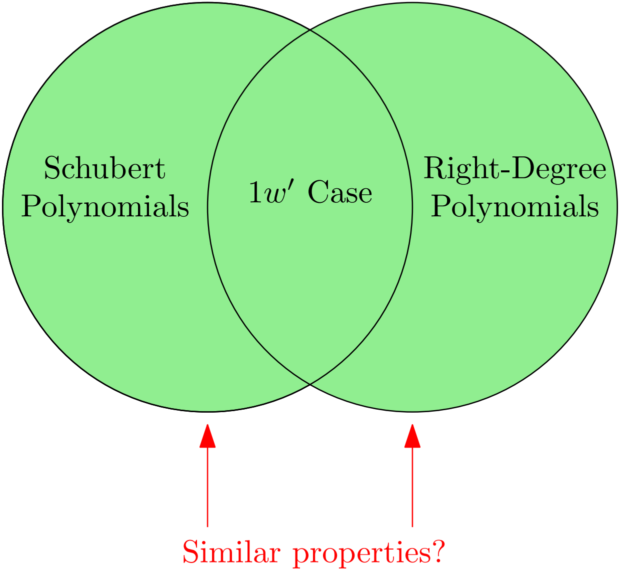
T_{1432} 

$$\mathfrak{S}_{1432} = x_2^2x_3 + x_1x_2x_3 + x_1x_2^2 + x_1^2x_3 + x_1^2x_2$$

$$R_{T_{1432}} = x_1^3 + x_1^2x_2 + x_1^2x_3 + x_1x_2^2 + x_1x_2x_3$$

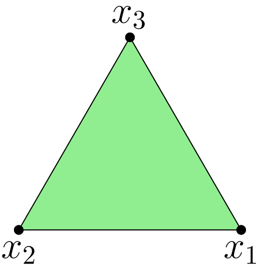
$$\mathfrak{S}_{1432}(x_1, x_2, x_3) = x_1^3x_2^2x_3R_{T_{1432}}(x_1^{-1}, x_2^{-1}, x_3^{-1})$$

Right-Degree and Schubert Polynomials

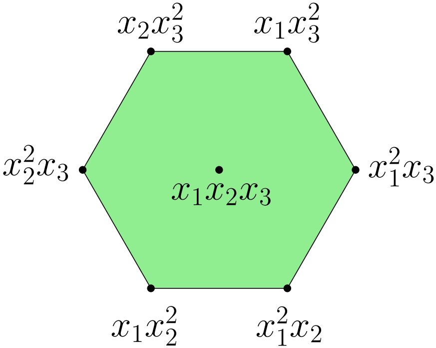


Schubert Polynomial Newton Polytopes

What kind of polytopes are the Newton polytopes of Schubert polynomials?

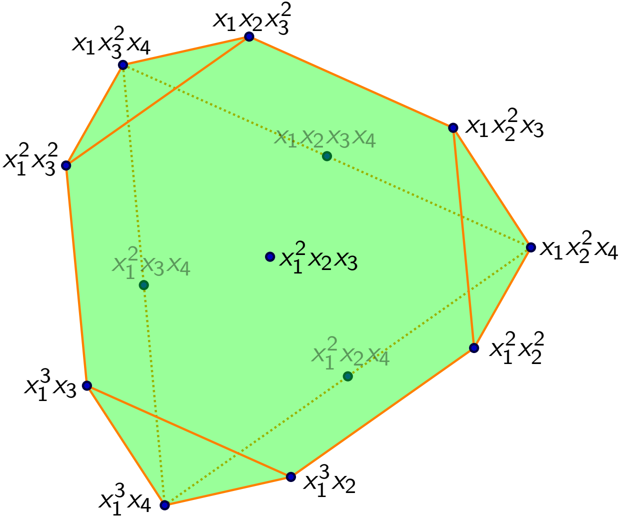


$\mathfrak{S}_{1243} = x_1 + x_2 + x_3$



$\mathfrak{S}_{13524} = x_2x_3^2 + x_1x_3^2 + x_1^2x_3 + x_1^2x_2 + x_1x_2^2 + x_2^2x_3 + 2x_1x_2x_3$

Schubert Polynomial Newton Polytopes



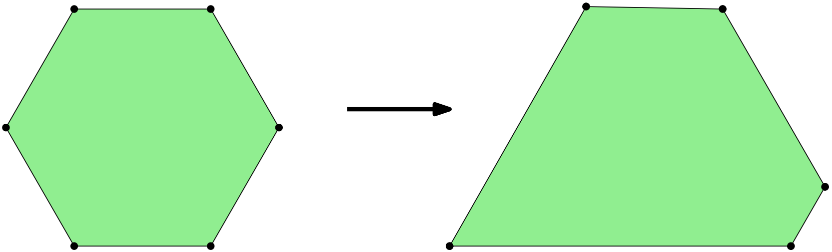
$$\mathfrak{S}_{21543} = x_1^3x_2 + x_1^3x_3 + x_1^3x_4 + x_1^2x_2^2 + x_1^2x_3^2 + 2x_1^2x_2x_3 + x_1^2x_2x_4 + x_1^2x_3x_4 + x_1x_2x_3^2 + x_1x_2^2x_3 + x_1x_2^2x_4 + x_1x_3^2x_4 + x_1x_2x_3x_4$$

Generalized Permutahedra

- The standard **permutahedron** in $\mathbb{R}^n$ is the convex hull of all rearrangements of the vector $(1, 2, \ldots, n)$.

Definition (Postnikov 2005, Edmonds 1970)

A **generalized permutahedron** is any polytope obtained by deforming the standard permutahedron by moving the vertices in any way so that all edge directions are preserved.

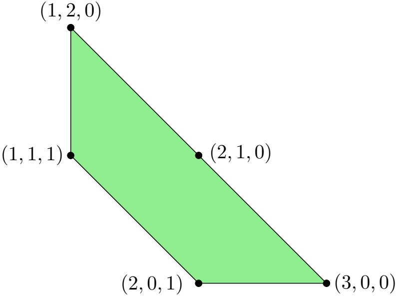


An Answer For R_G

Theorem (M-St. Dizier 2017)

For any graph G, $\mathrm{Newton}(R_G)$ is a generalized permutahedron.

T_{1432} 

$\mathrm{Newton}(R_{T_{1432}}) =$ 

Schubert Newton Polytopes

Conjecture (Monical-Tokcan-Yong 2017)

For any $w \in S_n$, $\mathrm{Newton}(\mathfrak{S}_w)$ *is a generalized permutahedron.*

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- $\mathrm{Newton}(\mathfrak{S}_w)$ should be a generalized permutahedron

Theorem (Fink-M-St. Dizier 2017)

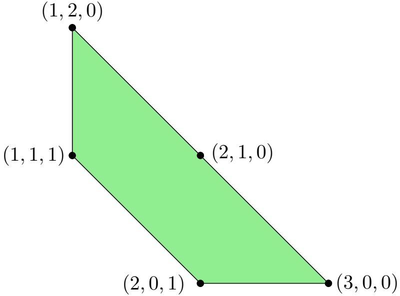
The Newton polytopes of the Schubert polynomials are generalized permutahedra.

A Saturation Property of R_G

What does $RD(G)$ look like? Specifically, how do the points in $RD(G)$ sit inside the Newton polytope of R_G?

T_{1432}

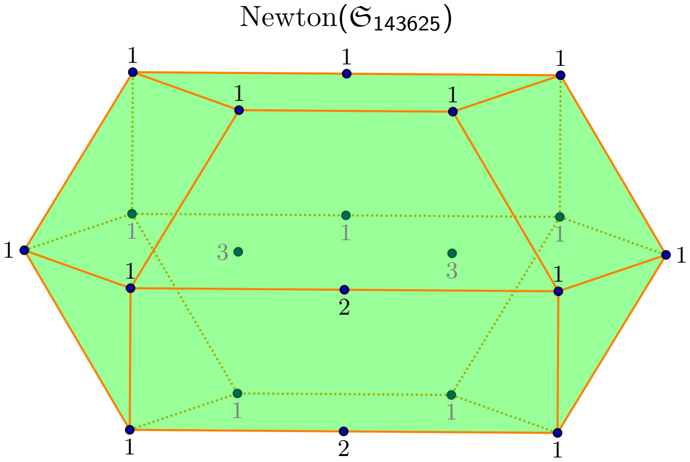
$\mathrm{Newton}(R_{T_{1432}}) =$



Theorem (M-St. Dizier 2017)

$\mathrm{Newton}(R_G)$ *is a generalized permutahedron whose integral points are exactly* $RD(G)$.

Coefficients



Question

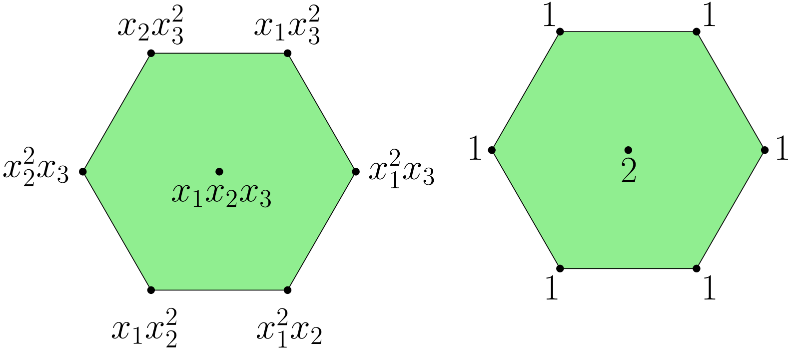
Can there be zeros in the polytope?

Saturated Newton Polytopes

Definition (Monical-Tokcan-Yong 2017)

A polynomial f is said to have **saturated Newton polytope** (SNP) if every integer point in the Newton polytope corresponds to a monomial with nonzero coefficient in f.

$$\mathfrak{S}_{13524} = x_2x_3^2 + x_1x_3^2 + x_1^2x_3 + x_1^2x_2 + x_1x_2^2 + x_2^2x_3 + 2x_1x_2x_3$$



SNP in Algebraic Combinatorics

Theorem (Monical-Tokcan-Yong 2017)

The following all have SNP:

- *Schur polynomials*
- *Skew-Schur polynomials*
- *Stanley symmetric functions*
- (q, t) *evaluations of symmetric Macdonald polynomials*

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Conjecture (Monical-Tokcan-Yong 2017)

The following all have SNP:

- *Schubert polynomials*
- *Key polynomials*
- *Double Schubert polynomials*
- *Grothendieck polynomials*

SNP in Algebraic Combinatorics

Theorem

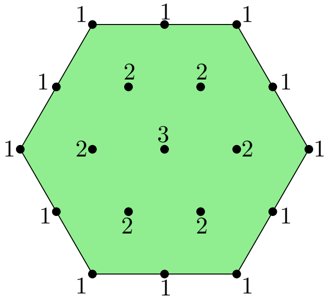
The following all have SNP:

- *Schubert polynomials (Fink-M-St. Dizier 2017)*
- *Key polynomials (Fink-M-St. Dizier 2017)*
- $1w$ *Grothendieck polynomials (M-St. Dizier 2017)*
- *Symmetric Grothendieck polynomials (Escobar-Yong 2017)*
- *Double Schubert, some K-polynomials (Castillo- Cid-Ruiz -Mohammadi-Montaño 2021, 2022)*

More About Coefficients

SNP says there are no zeros within the Newton polytope. How are the nonzero coefficients distributed?

Newton($s_{(4,2,0)}$)



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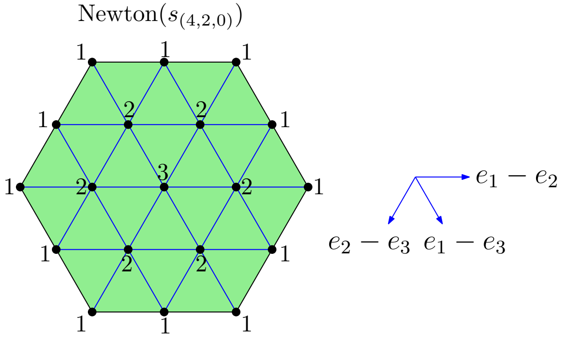
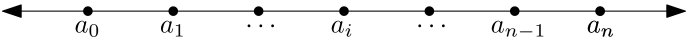


Idea: look along lines in root directions!

Unimodal and Log-Concave Sequences



Unimodal: $a_0 \leq a_1 \leq \cdots \leq a_j$ and $a_j \geq a_{j+1} \geq \cdots \geq a_n$ for some j

Log-concave: $a_i^2 \geq a_{i-1}a_{i+1}$ for all i.

(Positive and log-concave implies unimodal)

Question

Do the coefficients form unimodal sequences along lines in root directions? Even better, are they log-concave?

On the coefficients of Schubert Polynomials

Let $\mathfrak{S}_w = \sum_\alpha C_{w\alpha} x^\alpha$.

Theorem (Huh–Matherne–M–St. Dizier 2019)

For any $w \in S_n$ and $i, j \in [n]$,

$$C_{w\alpha}^2 \geq C_{w,\alpha+e_i-e_j} C_{w,\alpha-e_i+e_j}.$$

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Question

What other polynomials from algebraic combinatorics have Lorentzian normalizations?

Happy Birthday Michèle!