Eigenvalues of Laplacian on Riemannian manifolds

Qing-Ming Cheng

Saga University

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1. Introduction
   - A Dirichlet eigenvalue problem of Laplacian

2. Universal inequalities for eigenvalues
   - The case of a Euclidean space
   - The case of a unit sphere
   - The case of a hyperbolic space
   - The case of a complete Riemannian manifold

3. Lower bounds for eigenvalues
   - A conjecture of Pólya
   - A generalized conjecture of Pólya
1 Introduction
   - A Dirichlet eigenvalue problem of Laplacian

2 Universal inequalities for eigenvalues
   - The case of a Euclidean space
   - The case of a unit sphere
   - The case of a hyperbolic space
   - The case of a complete Riemannian manifold

3 Lower bounds for eigenvalues
   - A conjecture of Pólya
   - A generalized conjecture of Pólya
Contents

1 Introduction
   - A Dirichlet eigenvalue problem of Laplacian

2 Universal inequalities for eigenvalues
   - The case of a Euclidean space
   - The case of a unit sphere
   - The case of a hyperbolic space
   - The case of a complete Riemannian manifold

3 Lower bounds for eigenvalues
   - A conjecture of Pólya
   - A generalized conjecture of Pólya
Contents

1. Introduction
   - A Dirichlet eigenvalue problem of Laplacian

2. Universal inequalities for eigenvalues
   - The case of a Euclidean space
   - The case of a unit sphere
   - The case of a hyperbolic space
   - The case of a complete Riemannian manifold

3. Lower bounds for eigenvalues
   - A conjecture of Pólya
   - A generalized conjecture of Pólya
Let $M$ be an $n$-dimensional complete Riemannian manifold, $\Omega$ a bounded domain with piecewise smooth boundary $\partial \Omega$ in $M$. We consider

\begin{align*}
\begin{cases}
\Delta u = -\lambda u, & \text{in } \Omega, \\
u = 0, & \text{on } \partial \Omega,
\end{cases}
\end{align*}

(1.1)

where $\Delta$ denotes the Laplacian on $M$.

The Dirichlet eigenvalue problem (1.1) is also called a fixed membrane problem.
It is well known that the spectrum of this eigenvalue problem (1.1) is real and discrete.

\[ 0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots \to \infty. \]

where each \( \lambda_i \) has finite multiplicity which is repeated according to its multiplicity.
Further, the following Weyl’s asymptotic formula holds (cf. [3]):

$$\lambda_k \sim \frac{4\pi^2}{(\omega_n \text{vol}\Omega)^{\frac{2}{n}}} k^{\frac{2}{n}}, \quad k \to \infty,$$

where $\omega_n$ is the volume of the unit ball in $\mathbb{R}^n$. From the formula, it is not difficult to infer

$$\frac{1}{k} \sum_{i=1}^{k} \lambda_i \sim \frac{n}{n+2} \frac{4\pi^2}{(\omega_n \text{vol}\Omega)^{\frac{2}{n}}} k^{\frac{2}{n}}, \quad k \to \infty.$$
Introduction

Universal inequalities for eigenvalues

Lower bounds for eigenvalues

The case of a Euclidean space

The case of a unit sphere

The case of a hyperbolic space

The case of a complete Riemannian manifold

Contents

1 Introduction
   A Dirichlet eigenvalue problem of Laplacian

2 Universal inequalities for eigenvalues
   The case of a Euclidean space
   The case of a unit sphere
   The case of a hyperbolic space
   The case of a complete Riemannian manifold

3 Lower bounds for eigenvalues
   A conjecture of Pólya
   A generalized conjecture of Pólya
In this subsection, we consider universal inequalities for eigenvalues of the Dirichlet eigenvalue problem (1.1) for a bounded domain $\Omega$ in an $n$-dimensional Euclidean space. The investigation of universal inequalities for eigenvalues of the Dirichlet eigenvalue problem (1.1) was initiated by Payne, Pólya and Weinberger [21] and [22]. They proved

$$\lambda_{k+1} - \lambda_k \leq \frac{4}{nk} \sum_{i=1}^{k} \lambda_i.$$  \hspace{1cm} (2.1)

Although this result of Payne, Pólya and Weinberger has been extended by many mathematicians in several way, there are two main contributions due to Hile and Protter [15] and Yang [27].
In fact, in 1980, Hile and Protter improved this result of Payne, Pólya and Weinberger to

$$\sum_{i=1}^{k} \frac{\lambda_i}{\lambda_{k+1} - \lambda_i} \geq \frac{nk}{4}. \quad (2.2)$$

Further, Yang [27] (cf. [8]) has obtained a very sharp inequality, that is, he has derived

$$\sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)^2 \leq \frac{4}{n} \sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)\lambda_i. \quad (2.3)$$

The inequality (2.3) is called the first inequality of Yang (cf. [1], [2]).
Remark

By making use of the Chebyshev’s inequality, it is not difficult to prove the following relation

\[(2.3) \implies (2.2) \implies (2.1)\].

The first inequality of Yang is optimal in the meaning of the order of \(k\).
Remark

By making use of the Chebyshev’s inequality, it is not difficult to prove the following relation

\[(2.3) \implies (2.2) \implies (2.1).\]

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Introduction

Universal inequalities for eigenvalues

Lower bounds for eigenvalues

Contents

1 Introduction
   - A Dirichlet eigenvalue problem of Laplacian

2 Universal inequalities for eigenvalues
   - The case of a Euclidean space
   - The case of a unit sphere
   - The case of a hyperbolic space
   - The case of a complete Riemannian manifold

3 Lower bounds for eigenvalues
   - A conjecture of Pólya
   - A generalized conjecture of Pólya
For a domain $\Omega$ in an $n$-dimensional unit sphere, universal inequalities for eigenvalues of the Dirichlet eigenvalue problem (1.1) has been studied by Cheng and Yang. We have proved

**Theorem 2.1**

(Cheng and Yang, Math Ann. 2005). Let $\Omega$ be a domain in an $n$-dimensional unit sphere. Eigenvalue $\lambda_i$’s of the Dirichlet eigenvalue problem (1.1) satisfy

$$
\sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)^2 \leq \frac{4}{n} \sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)(\lambda_i + \frac{n^2}{4}).
$$
Remark

The above inequality is best possible since this inequality does not depend on the domain $\Omega$ and when $\Omega$ tends to the unit sphere, this inequality becomes an equality for all of $k$. 
When $M$ is $H^n(-1)$, although many mathematicians want to derive a universal inequality for eigenvalues, there are no any results on universal inequalities for eigenvalues of the Dirichlet eigenvalue problem (1.1) excepting $n = 2$.

If $n = 2$, by making use of estimates for eigenvalues of the eigenvalue problem of the Schrödinger like operator with a weight, Harrell and Michel [14] and Ashbaugh [2] have obtained several results. In fact, if $n = 2$, the Laplacian on $H^2(-1)$ is like to the Laplacian on $\mathbb{R}^2$ with a weight. But, when $n > 2$, this property does not hold again.

For a bounded domain in $H^n(-1)$, main reason, that one could not derive a universal inequality, is that one can not find an appropriate trial function.
Recently, Cheng and Yang find an appropriate trial function for $M = H^n(-1)$. Hence, we can derive a universal inequality for eigenvalues of the Dirichlet eigenvalue problem (1.1), that is, we prove the following:

**Theorem 2.2**

(Cheng and Yang). For a bounded domain $\Omega$ in $H^n(-1)$, eigenvalue $\lambda_i$’s of the Dirichlet eigenvalue problem (1.1) satisfy

$$\sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)^2 \leq 4 \sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)(\lambda_i - \frac{(n-1)^2}{4}).$$
Next, we will consider an application of our universal inequality. Let $\Omega$ be an $n$-disk of radius $r > 0$ in $H^n(-1)$. McKean [20] has proved that the first eigenvalue $\lambda_1(r)$ of the Dirichlet eigenvalue problem (1.1) satisfies

$$\lambda_1(r) \geq \frac{(n - 1)^2}{4},$$

$$\lim_{r \to \infty} \lambda_1(r) = \frac{(n - 1)^2}{4}.$$
From domain monotonicity of eigenvalues, we have, for any bounded domain \( \Omega \) in \( H^n(-1) \),

\[
\lambda_1(\Omega) \geq \frac{(n - 1)^2}{4},
\]

\[
\lim_{\Omega \to H^n(-1)} \lambda_1(\Omega) = \frac{(n - 1)^2}{4}.
\]

It is obvious that, for any \( k > 1 \),

\[
\lambda_k(\Omega) > \lambda_1(\Omega) \geq \frac{(n - 1)^2}{4}.
\]

It would be interesting to study behaviors of \( \lambda_k(\Omega) \), for \( k \geq 2 \), when \( \Omega \) tends to \( H^n(-1) \). As an application of our universal inequality, we can prove the following:
Theorem 2.3

(Cheng and Yang) Let $\Omega$ be a bounded domain in $H^n(-1)$. Then, eigenvalue $\lambda_k(\Omega)$, for any $k$, of the Dirichlet eigenvalue problem (1.1) satisfies

$$\lim_{\Omega \to H^n(-1)} \lambda_k(\Omega) = \frac{(n - 1)^2}{4}.$$ 

In order to prove the theorem, we need to investigate upper bounds for eigenvalues.
Introduction

Universal inequalities for eigenvalues

Lower bounds for eigenvalues

A Dirichlet eigenvalue problem of Laplacian

2

Universal inequalities for eigenvalues

The case of a Euclidean space

The case of a unit sphere

The case of a hyperbolic space

The case of a complete Riemannian manifold

3

Lower bounds for eigenvalues

A conjecture of Pólya

A generalized conjecture of Pólya
When Ω is a bounded domain in an $n$-dimensional complete Riemannian manifold, in my joint work with Chen, we have studied universal inequalities for eigenvalues of the Dirichlet eigenvalue problem (1.1).

In the cases of a Euclidean space and a unit sphere, one can make use of the coordinate functions to construct the trial functions. Thus, one can derive universal inequalities for eigenvalues according to the Rayleigh-Ritz inequality. But for a complete Riemannian manifold, it is very difficult to construct an appropriate trial function. By making use of Nash’s theorem, we successfully construct trial functions which satisfy good properties.
Nash’s Theorem

Each complete Riemannian manifold can be isometrically immersed in a Euclidean space.

According to the Nash’s theorem, we can construct appropriate trial functions and we can prove, by making use of the Rayleigh-Ritz inequality,
Theorem 2.4

(Chen and Cheng, J. Math. Soc. Japan, 2008). Let $\Omega$ be a bounded domain in an $n$-dimensional complete Riemannian manifold $M$. Then, there exists a constant $H_0^2$ such that eigenvalues of the Dirichlet eigenvalue problem (1.1) satisfy

$$\sum_{i=1}^{k}(\lambda_{k+1} - \lambda_i)^2 \leq \frac{4}{n} \sum_{i=1}^{k}(\lambda_{k+1} - \lambda_i)(\lambda_i + \frac{n^2}{4}H_0^2),$$

where $\lambda_i$ denotes the $i^{th}$ eigenvalue and $H_0^2 = \inf_{\varphi \in \Phi} \sup_{\Omega} |H|^2$. Here $\Phi$ denotes a set of all isometric immersions with the mean curvature vector $H$ from $M$ into a Euclidean space.
In particular, when $M$ is a complete minimal submanifold in $\mathbb{R}^N$, we have

**Corollary 2.1**

Let $\Omega$ be a bounded domain in an $n$-dimensional complete minimal submanifold $M^n$ in $\mathbb{R}^N$. Then, we have

$$\sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)^2 \leq \frac{4}{n} \sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)\lambda_i.$$
Remark

Since $\mathbb{R}^n$ can be seen as a totally geodesic minimal hypersurface in $\mathbb{R}^{n+1}$, we know that the result of Yang is included in the corollary 2.1. Further, since there exist many complete minimal submanifolds in $\mathbb{R}^N$, we know that the first inequality of Yang for eigenvalues also hold for any bounded domain in any complete minimal submanifold in $\mathbb{R}^N$. 
Since the $n$-dimensional unit sphere $S^n(1)$ can be seen as a totally umbilical hypersurface with constant mean curvature 1 in $\mathbb{R}^{n+1}$, we know

**Corollary 2.2**

Let $\Omega$ be a domain in an $n$-dimensional unit sphere. Then, we have

$$\sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)^2 \leq \frac{4}{n} \sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)(\lambda_i + \frac{n^2}{4}),$$

which is the universal inequality of Cheng and Yang [5]. Thus, we know that our results in theorem 2.4 is also optimal.

In order to prove the theorem 2.4, the following lemma plays a key role.
Lemma

(Chen and Cheng). Let $M$ be an $n$-dimensional complete Riemannian manifold with metric $g$ isometrically immersed in a Euclidean space $\mathbb{R}^N$. For any point $P$ in $M$, assuming that $y$ with components $y^\alpha$ defined by $y^\alpha = y^\alpha(x^1, x^2, \ldots, x^n)$ is the position vector of $P$ in $\mathbb{R}^N$, we have,

$$\sum_{\alpha=1}^{N} g(\nabla y^\alpha, \nabla y^\alpha) = n, \quad \sum_{\alpha=1}^{N} (\Delta y^\alpha)^2 = n^2 |H|^2$$

$$\sum_{\alpha=1}^{N} \Delta y^\alpha \nabla y^\alpha = 0, \quad \sum_{\alpha=1}^{N} g(\nabla y^\alpha, \nabla u) = |\nabla u|^2,$$

for any function $u \in C^\infty(M)$, where $H$ is the mean curvature vector of $M$. 

Qing-Ming Cheng
Theorem 2.5

(Cheng and Yang, J. Math. Soc. Japan, 2006). Let $\lambda_i$ be the $i^{th}$ eigenvalue of the Dirichlet eigenvalue problem $(1.1)$ on an $n$-dimensional compact Riemannian manifold $\bar{\Omega} = \Omega \cup \partial \Omega$ with boundary $\partial \Omega$ and $u_i$ be an orthonormal eigenfunction corresponding to $\lambda_i$. Then, for any function $f \in C^3(\Omega) \cap C^2(\partial \Omega)$ and any integer $k$, we have

$$\sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)^2 \|u_i \nabla f\|^2 \leq \sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i) \|2 \nabla f \cdot \nabla u_i + u_i \Delta f\|^2,$$

where $\|f\|^2 = \int_M f^2$ and $\nabla f \cdot \nabla u_i = g(\nabla f, \nabla u_i)$. 

Qing-Ming Cheng
Eigenvalues of Laplacian on Riemannian manifolds
A proof of the theorem 2.4

Outline of the proof of theorem 2.4

From the above lemma of Chen and Cheng and the above theorem of Cheng and Yang with $f = y^\alpha$, we can finish proof of the theorem 2.4.
Introduction

Universal inequalities for eigenvalues

Lower bounds for eigenvalues

Contents

1 Introduction
   - A Dirichlet eigenvalue problem of Laplacian

2 Universal inequalities for eigenvalues
   - The case of a Euclidean space
   - The case of a unit sphere
   - The case of a hyperbolic space
   - The case of a complete Riemannian manifold

3 Lower bounds for eigenvalues
   - A conjecture of Pólya
   - A generalized conjecture of Pólya

Qing-Ming Cheng

Eigenvalues of Laplacian on Riemannian manifolds
As an application of our universal inequalities for eigenvalues of
the Dirichlet eigenvalue problem (1.1), which have been
considered in the section 2, we will give lower bounds for
eigenvalues. First of all, we consider the case of Euclidean
space.
When \( M = \mathbb{R}^n \), for the Dirichlet eigenvalue problem (1.1), Pólya
[23] proved

\[
\lambda_k \geq \frac{4\pi^2}{(\omega_n \text{vol}\Omega)^2} k^2\left(\frac{n}{n}\right)^n, \quad \text{for} \quad k = 1, 2, \ldots,
\]

if \( \Omega \) is a tiling domain in \( \mathbb{R}^n \).
Further, he conjectured, for a bounded domain,

**Conjecture of Pólya**

If $\Omega$ is a bounded domain in $\mathbb{R}^n$, then eigenvalue $\lambda_k$ of the Dirichlet eigenvalue problem (1.1) satisfies

$$
\lambda_k \geq \frac{4\pi^2}{(\omega_n \text{vol}\Omega)^\frac{2}{n}} k^{\frac{2}{n}}, \text{ for } k = 1, 2, \ldots .
$$
On the conjecture of Pólya, Li and Yau [19] (cf. Lieb [16]) attacked it and obtained

**Theorem 3.1**

(Li and Yau, Comm. Math. Phys. 1983). *If \( \Omega \) is a bounded domain in \( \mathbb{R}^n \), then eigenvalue \( \lambda_k \) of the Dirichlet eigenvalue problem (1.1) satisfies*

\[
\frac{1}{k} \sum_{i=1}^{k} \lambda_i \geq \frac{n}{n+2} \frac{4\pi^2}{(\omega_n \text{vol} \Omega)^{\frac{2}{n}}} k^{\frac{2}{n}}, \text{ for } k = 1, 2, \cdots ,
\]
Remark

According to the Weyl’s asymptotic formula, we know that the result of Li and Yau is optimal in the meaning of average. From this formula, we have

\[ \lambda_k \geq \frac{n}{n+2} \frac{4\pi^2}{(\omega_n \text{vol}(\Omega))^{2/n}} k^{2/n}, \text{ for } k = 1, 2, \ldots, \]

which gives a partial solution for the conjecture of Pólya with a factor \( \frac{n}{n+2} \).
Main tools which are used in proof of the theorem of Li and Yau are the Fourier transform and a lemma of Hörmander

**Lemma of Hörmander**

If $f$ is a function defined on $\mathbb{R}^n$ satisfying

$$0 \leq f \leq a_1, \quad \int_{\mathbb{R}^n} |z|^2 f(z)dz \leq a_2$$

then, we have

$$\int_{\mathbb{R}^n} f(z)dz \leq \left( a_1 \omega_n \right)^{\frac{2}{n+2}} a_2^{\frac{n}{n+2}} \left( \frac{n + 2}{n} \right)^{\frac{n}{n+2}}.$$

where $a_1$ and $a_2$ are constant.
Outline of proof of the theorem of Li and Yau.

Let $u_i$ be an eigenfunction corresponding to eigenvalue $\lambda_i$ such that $\{u_i\}$ becomes an orthonormal basis of $L^2(\Omega)$. By defining a function

$$\varphi(x, y) = \begin{cases} 
\sum_{i=1}^{k} u_i(x)u_i(y), & (x, y) \in \Omega \times \Omega \\
0, & \text{the other},
\end{cases}$$

and

$$f(z) = \int_{\mathbb{R}^n} |\hat{\varphi}(z, y)|^2 dy,$$

where $\hat{\varphi}(z, y)$ is the Fourier transform of $\varphi(x, y)$ in $x$. 
we have

\[ 0 \leq f \leq (2\pi)^{-n} V, \quad \int_{\mathbb{R}^n} f(z) dz = k, \]

\[ \int_{\mathbb{R}^n} |z|^2 f(z) dz = \sum_{i=1}^{k} \lambda_i. \]

Thus, from the lemma of Hörmander, proof of the theorem of Li and Yau is finished.
Contents

1. Introduction
   - A Dirichlet eigenvalue problem of Laplacian

2. Universal inequalities for eigenvalues
   - The case of a Euclidean space
   - The case of a unit sphere
   - The case of a hyperbolic space
   - The case of a complete Riemannian manifold

3. Lower bounds for eigenvalues
   - A conjecture of Pólya
   - A generalized conjecture of Pólya
For a complete Riemannian manifold $M$, eigenvalues of the Dirichlet eigenvalue problem (1.1) also satisfy the **Weyl’s asymptotic formula**. Hence, it is natural to ask

- Whether is it possible for one to consider the same problem as the conjecture of Pólya for a complete Riemannian manifold other than $\mathbb{R}^n$?

- First of all, a difficulty which we will encounter is that there is no the Fourier transform for a complete Riemannian manifold.

- In order to derive the result of Li and Yau, the lemma of Hörmander plays an important role which does not hold for a complete Riemannian manifold.
For a complete Riemannian manifold $M$, eigenvalues of the Dirichlet eigenvalue problem (1.1) also satisfy the Weyl’s asymptotic formula. Hence, it is natural to ask

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- First of all, a difficulty which we will encounter is that there is no the Fourier transform for a complete Riemannian manifold.

- In order to derive the result of Li and Yau, the lemma of Hörmander plays an important role which does not hold for a complete Riemannian manifold.
In order to consider the same problem as the conjecture of Pólya, we need to come over the following problems:

- What method will we use to replace the Fourier transform?
- What tool will we use to replace the lemma of Hörmander?
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- What method will we use to replace the Fourier transform?
- What tool will we use to replace the lemma of Hörmander?
From now, for a complete Riemannian manifold, let us consider the same problem as the conjecture of Pólya and derive a similar result to one of Li and Yau. First of all, we will propose the following version of the conjecture of Pólya for a complete Riemannian manifold.
Let $\Omega$ be a bounded domain in an $n$-dimensional complete Riemannian manifold $M$. Then, there exists a constant $c(M, \Omega)$, which only depends on $M$ and $\Omega$ such that eigenvalue $\lambda_i$'s of the Dirichlet eigenvalue problem (1.1) satisfy

$$\frac{1}{k} \sum_{i=1}^{k} \lambda_i + c(M, \Omega) \geq \frac{n}{n+2} \frac{4\pi^2}{(\omega_n \text{vol} \Omega)^{\frac{2}{n}}} k^{\frac{2}{n}}, \text{ for } k = 1, 2, \cdots,$$

$$\lambda_k + c(M, \Omega) \geq \frac{4\pi^2}{(\omega_n \text{vol} \Omega)^{\frac{2}{n}}} k^{\frac{2}{n}}, \text{ for } k = 1, 2, \cdots.$$
Remark

On the generalized conjecture of Pólya, we propose that

- if $M$ is the unit sphere $S^n(1)$, then $c(M, \Omega) = \frac{n^2}{4}$;
- if $M$ is the hyperbolic space $H^n(-1)$, then $c(M, \Omega) = -\frac{(n - 1)^2}{4}$;
- if $M$ is a complete minimal submanifold in $\mathbb{R}^N$, then $c(M, \Omega) = 0$. 
Remark

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Qing-Ming Cheng

Eigenvalues of Laplacian on Riemannian manifolds
Remark

On the generalized conjecture of Pólya, we propose that

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- if $M$ is the hyperbolic space $H^n(-1)$, then
  
  $$c(M, \Omega) = -\frac{(n-1)^2}{4};$$
- if $M$ is a complete minimal submanifold in $\mathbb{R}^N$, then $c(M, \Omega) = 0.$
Main theorem

**Theorem 3.2**

(Cheng and Yang). Let $\Omega$ be a bounded domain in an $n$-dimensional complete Riemannian manifold $M$. Then, there exists a constant $H^2_0$, which only depends on $M$ and $\Omega$ such that eigenvalue $\lambda_i$'s of the Dirichlet eigenvalue problem (1.1) satisfy

$$\frac{1}{k} \sum_{i=1}^{k} \lambda_i + \frac{n^2}{4} H^2_0 \geq \frac{n}{\sqrt{(n+2)(n+4)}} \frac{4\pi^2}{(\omega_n \text{vol} \Omega)^\frac{2}{n}} k^\frac{2}{n}, \text{ for } k = 1, 2, \cdots.$$

$$\lambda_k + \frac{n^2}{4} H^2_0 \geq \frac{n}{\sqrt{(n+2)(n+4)}} \frac{4\pi^2}{(\omega_n \text{vol} \Omega)^\frac{2}{n}} k^\frac{2}{n}, \text{ for } k = 1, 2, \cdots.$$

Qing-Ming Cheng

Eigenvalues of Laplacian on Riemannian manifolds
Corollary 3.1

Let $\Omega$ be a domain in the $n$-dimensional unit sphere $S^n(1)$. Then, eigenvalue $\lambda_i$'s of the Dirichlet eigenvalue problem (1.1) satisfy

$$\frac{1}{k} \sum_{i=1}^{k} \lambda_i + \frac{n^2}{4} \geq \frac{n}{\sqrt{(n+2)(n+4)}} \frac{4\pi^2}{(\omega_n \text{vol} \Omega)^{\frac{2}{n}}} k^{\frac{2}{n}}, \text{ for } k = 1, 2, \ldots$$
Corollary 3.2

Let $\Omega$ be a bounded domain in an $n$-dimensional complete minimal submanifold $M$ in a Euclidean space $\mathbb{R}^N$. Then, eigenvalue $\lambda_i$'s of the Dirichlet eigenvalue problem (1.1) satisfy

$$\frac{1}{k} \sum_{i=1}^{k} \lambda_i \geq \frac{n}{\sqrt{(n+2)(n+4)}} \frac{4\pi^2}{(\omega_n vol \Omega)^{\frac{2}{n}}} k^\frac{2}{n}, \text{ for } k = 1, 2, \ldots.$$
In order to prove our main theorem, we need to come over the problems, which we have told you:

- **What method will we use to replace the Fourier transform?** The answer is *Universal inequalities for eigenvalues.*
- **What tool will we use to replace the lemma of Hörmander?** The answer is a recursion formula of Cheng and Yang.
In order to prove our main theorem, we need to come over the problems, which we have told you:

- What method will we use to replace the Fourier transform? The answer is **Universal inequalities for eigenvalues**.
- What tool will we use to replace the lemma of Hörmander? The answer is **a recursion formula of Cheng and Yang**.
The recursion formula of Cheng and Yang

A recursion formula

(Cheng and Yang, Math. Ann. 2007)

Let $\mu_1 \leq \mu_2 \leq \ldots \leq \mu_{k+1}$ be any non-negative real numbers satisfying

$$\sum_{i=1}^{k} (\mu_{k+1} - \mu_i)^2 \leq \frac{4}{t} \sum_{i=1}^{k} \mu_i (\mu_{k+1} - \mu_i).$$

Define

$$G_k = \frac{1}{k} \sum_{i=1}^{k} \mu_i, \quad T_k = \frac{1}{k} \sum_{i=1}^{k} \mu_i^2, \quad F_k = \left(1 + \frac{2}{t}\right) G_k^2 - T_k.$$
Then, we have the following recursion formula

\[ F_{k+1} \leq C(t, k) \left( \frac{k + 1}{k} \right)^{\frac{4}{t}} F_k, \]

where \( t \) is any positive real number and

\[ C(t, k) = 1 - \frac{1}{3t} \left( \frac{k}{k + 1} \right)^{\frac{4}{t}} \left( 1 + \frac{2}{t} \right) \left( 1 + \frac{4}{t} \right) \frac{(k + 1)^3}{(k + 1)^3} < 1. \]

**Remark**

By making use of the recursion formula of Cheng and Yang, we can not only derive lower bounds for eigenvalues, but also derive upper bounds for eigenvalues.
Outline of proof of the theorem 3.2

**Theorem 3.2**

(Cheng and Yang). Let $\Omega$ be a bounded domain in an $n$-dimensional complete Riemannian manifold $M$. Then, there exists a constant $H_0^2$, which only depends on $M$ and $\Omega$ such that eigenvalue $\lambda_i$’s of the Dirichlet eigenvalue problem (1.1) satisfy

$$
\frac{1}{k} \sum_{i=1}^{k} \lambda_i + \frac{n^2}{4} H_0^2 \geq \frac{n}{\sqrt{(n+2)(n+4)}} \frac{4\pi^2}{(\omega_n \text{vol} \Omega)^{\frac{2}{n}}} k^{\frac{2}{n}}, \text{ for } k = 1, 2, \cdots.
$$

$$
\lambda_k + \frac{n^2}{4} H_0^2 \geq \frac{n}{\sqrt{(n+2)(n+4)}} \frac{4\pi^2}{(\omega_n \text{vol} \Omega)^{\frac{2}{n}}} k^{\frac{2}{n}}, \text{ for } k = 1, 2, \cdots.
$$
From the theorem 2.4, we have

\[
\sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)^2 \leq \frac{4}{n} \sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)(\lambda_i + \frac{n^2}{4} H_0^2).
\]

Letting \( \mu_i = \lambda_i + \frac{n^2}{4} H_0^2 \), we have

\[
\sum_{i=1}^{k} (\mu_{k+1} - \mu_i)^2 \leq \frac{4}{n} \sum_{i=1}^{k} (\mu_{k+1} - \mu_i)\mu_i.
\]
From the recursion formula of Cheng and Yang with $t = n$, we infer

$$\frac{F_{k+1}}{(k+1)^4} \leq \frac{F_k}{k^4}.$$  

According to the Weyl’s asymptotic formula, we derive, for any positive integer $k$,

$$\frac{F_k}{k^4} \geq \frac{2n}{(n+2)(n+4)} \frac{16\pi^4}{(\omega_n\text{vol}\Omega)^4}.$$  

Since

$$F_k \leq \frac{2}{n} G_k^2,$$

we infer

$$\frac{1}{k} \sum_{i=1}^{k} \lambda_i + \frac{n^2}{4} H_0^2 \geq \frac{n}{\sqrt{(n+2)(n+4)}} \frac{4\pi^2}{(\omega_n\text{vol}\Omega)^2} k^2.$$
Thank you!


Qing-Ming Cheng


